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共役点から導かれる協力ゲーム[1]

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A conjugate-set game induced from the conjugate point

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Abstract In some nonlinear diffusive phenomena, e.g., grain growth in annealing pure metal and segregation between biological species, the systems have three or more stable states, see the left picture of Fig.1. Sternberg and Zeimer(Ref. 1) established the existence of local minimizers to the problem of partitioning certain domain $\Omega \subset R^2$ into three subdomains having least interfacial area. Further, Ikota and Yanagida investigated stability for stationary curves with one triple junction in Ref. 2 and stability for stationary binary-tree type interfaces in Ref. 3. In this paper, we consider a static version of their diffusion problem, which is formulated as an unconstrained nonlinear programming problem. We consider second-order optimality conditions and discuss stability and instability for stationary curves with one or two triple junctions. The great difference between the previous researches and ours is that our main concern is not stability but instability. We give a new insight to this problem from the viewpoint of game theory.

1 Introduction

In this paper, we consider a static version of the diffusion problem, see the right picture of Fig.1. It is formulated as follows. Let Ω be a bounded domain in R^2 with a smooth boundary, $X_i = (x_i, y_i)$ $(i = 1, 2)$ be in the interior of Ω, and X_i $(i = 3, 4, 5, 6)$ be on the boundary $\partial\Omega$. Then our problem is to minimize the sum of the lengths of five line segments. We call this extremal problem the 4-phase partition problem.

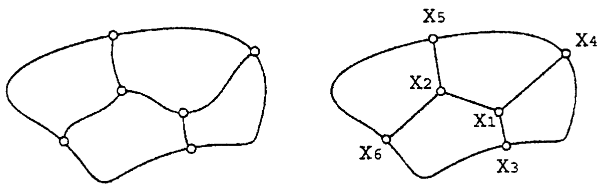

Figure 1: Dynamic version and static version.

By using arclength parameters, it can be formulated as an unconstrained nonlinear programming problem with eight variables. The main purposes of this paper are to discuss stability and instability of stationary solutions for the extremal problem in terms

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of the curvatures of the boundary and to give a new insight to the instable case from the viewpoint of the cooperative game.

This paper is organized as follows. In Section 2, we give a first-order optimality condition for the 4-phase partition problem. In Section 3, we analyze stability and instability of the stationary solution for the 4-phase partition problem. They are stated in terms of the curvature of the boundary. In Section 4, we define strict conjugate sets and a cooperative game that is called the conjugate-set game. In Section 5, we compute the Shapley value and the core for the 4-phase partition problem.

2 First-order optimality conditions

In this section, we give first-order necessary optimality conditions for the 4-phase partition problem. First, we deal with the nondegenerate case $X_1 \neq X_2$. Since X_i's on the boundary can be locally represented as $X_i(s_i) = (x_i(s_i), y_i(s_i))$, where s_i denotes an arclength parameter, the objective function is given by

$$\begin{aligned} & f(x_1, y_1, x_2, y_2, s_3, \ldots, s_6) \\ := \quad & ||X_2 - X_1|| + \sum_{i=3}^{4} ||X_i(s_i) - X_1|| + \sum_{i=5}^{6} ||X_i(s_i) - X_2||, \end{aligned} \tag{1}$$

where $|| \cdot ||$ denotes the Euclidean norm.

Theorem 2.1 *If $(X_1, \ldots, X_6)$ is a nondegenerate local minimizer of the 4-phase partition problem, then, (a) the angle between any two adjacent line segments equals $2\pi/3$ and (b) any line segment X_iX_j with $X_i \in$ int Ω and $X_j \in \partial\Omega$ orthogonally intersects the boundary.*

By using the following lemma, we see that $X_1 \neq X_2$ for any local minimizer $(X_1, \ldots, X_6)$.

Lemma 2.1 *Let $f^1(x_1, y_1, x_2, y_2) := \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}^{1/2}$. Then*

$$\partial f^1(x_1, y_1, x_1, y_1) = \{(-\xi, -\eta, \xi, \eta);\ \xi^2 + \eta^2 \leq 1\}.$$

3 Stability and instability

In this section, we discuss stability and instability of stationary solutions for the 4-phase partition problem. By virtue of Theorem 2.1, there is no loss of generality if we assume that the stationary solution is given by

$$\begin{aligned} & X_1 = (0, 0),\ X_2 = \ell_2(-1, 0),\ X_3 = \ell_3(1/2, -\sqrt{3}/2),\ X_4 = \ell_4(1/2, \sqrt{3}/2), \\ & X_5 = \ell_5(-1/2, \sqrt{3}/2) + X_2,\ X_6 = \ell_6(-1/2, -\sqrt{3}/2) + X_2. \end{aligned} \tag{2}$$

Hence, from the transversality condition (b) of Theorem 2.1, we may assume that the tangent vectors at X_k's, say X'_k, are given by

$$X'_3 = (\sqrt{3}/2, 1/2), \quad X'_4 = (-\sqrt{3}/2, 1/2),$$

$$X_5' = (-\sqrt{3}/2, -1/2), \quad X_6' = (\sqrt{3}/2, -1/2).$$

As is well known, if the Hesse matrix of the objective function $A := f''(x_1, \ldots, s_6)$ is positive definite, then $(X_1, \ldots, X_6)$ is a local minimizer. According to Sylvester's criterion, A is positive definite if and only if its leading principal minors are all positive. Let A_k denote the k-th leading principal submatrix of A. Then the first four leading principal minors are always positive, and

$$|A_5| = 9(1 + h_3\ell)/16L, \quad |A_6| = 9(h_3 + h_4 + h_3h_4\ell)/16L,$$

$$|A_7| = 9(h_4h_5 + h_3h_5 + h_3h_4 + h_3h_4h_5\ell)/16L,$$

$$|f''| = |A_8| = 9(h_4h_5h_6 + h_3h_5h_6 + h_3h_4h_6 + h_3h_4h_5 + h_3h_4h_5h_6\ell)/16L,$$

where

$$\ell_i := \begin{cases} ||X_i - X_1||, \ i = 2,3,4, \\ ||X_i - X_2||, \ i = 1,5,6, \end{cases},$$

$h_k := X_k^T X_k''/||X_k||$ denotes the curvature of the boundary at X_k, $\ell := \ell_2 + \ell_3 + \ldots + \ell_6$ and $L := \ell_2\ell_3 \cdots \ell_6$.

Theorem 3.1 *The Hesse matrix of f at a stationary solution is positive definite if and only if $D_5 := 1 + h_3\ell$, $D_6 := h_3 + h_4 + h_3h_4\ell$, $D_7 := h_4h_5 + h_3h_5 + h_3h_4 + h_3h_4h_5\ell$, and $D_8 := h_4h_5h_6 + h_3h_5h_6 + h_3h_4h_6 + h_3h_4h_5 + h_3h_4h_5h_6\ell$ are positive.*

Theorem 3.2 *If at least two of h_k's are negative, then the stationary solution is instable, that is, at least one of the leading principal minors is negative.*

Theorem 3.3 *When just one of h_k's is zero, f'' is positive definite if and only if three other h_k's are positive.*

Theorem 3.4 *When one h_k is negative and others are positive, f'' is positive definite if and only if*

$$h_k > -\prod_{j\neq k} h_j \Big/ \left(\sum_{i\neq k,\ j\neq k} h_i h_j + \ell \prod_{j\neq k} h_j \right). \tag{3}$$

Theorem 3.5 *When at least two of h_k's are zero and others are positive, f'' is nonnegative definite.*

Table 1 summarizes Theorem 3.1- Theorem 3.5. The numbers $0, 1, \ldots, 4$ in the first column (row) stand for the number of negative (positve) curvatures, respectively. "S" means stable, that is, f'' is positive definite. "IS" means instable, that is, a certain principal minor of f'' is negative. "+0" means f'' is nonnegative definite. $*$ corresponds to Theorem 3.4.

$\sharp\{h_k < 0\} \setminus \sharp\{h_k > 0\}$	0	1	2	3	4
0	+0	+0	+0	S	S
1	IS	IS	IS	*	×
2	IS	IS	IS	×	×
3	IS	IS	×	×	×
4	IS	×	×	×	×

Table 1: Stability and instability of the 4-phase partition problem.

4 Conjugate-set game

Stability of stationary solutions is deeply related to the conjugate point. The conjugate point was defined by Jacobi for a variational problem to give sufficient optimality conditions, see e.g. Gelfand and Fomin (Ref. 5). Recently, the author defined the conjugate point for an extremal problem of a smooth function $f(x)$ with n variables and described stability in terms of the conjugate point, see Kawasaki (Refs. 6-8). As we have mentioned in Section 3, an $n \times n$ symmetric matrix $A = (a_{ij})_{1 \le i,j \le n}$ is positive definite if and only if the leading principal minors $|A_k| = |(a_{ij})_{1 \le i,j \le k}|$ $(k = 1, \ldots, n)$ are positive. We say that a number $1 \le k \le n$ is conjugate to 1 if $|A_1| > 0, |A_2| > 0, \ldots, |A_{k-1}| > 0$ and $|A_k| \le 0$. On the other hand, we have to check all principal minors to test whether A is nonnegative definite or not, see e.g. Strang (Ref. 9). So, A is not nonnegative definite if and only if there exists a subset T of $N := \{1, \ldots, n\}$ such that $|A_T| := |(a_{ij})_{i,j \in T}|$ is negative. When a subset S of N contains such a subset T, we call S a strict conjugate set.

Definition 4.1 *(Kawasake, Ref. 10) Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix, and S a subset of $N = \{1, \ldots, n\}$. If a submatrix $(a_{ij})_{i,j \in S}$ of A has a negative principal minor, then we call S a strict conjugate set. For the sake of convenience, we call the corresponding set of variables $\{x_k\}_{k \in S}$ a strict conjugate set. When any proper subset T of a strict conjugate set S is not a strict conjugate set, we call S (and $\{x_k\}_{k \in S}$) a minimal strict conjugate set.*

It is clear that if there exists a strict conjugate set S, then we can improve a solution by suitably changing the variables of S. This leads us to a cooperative game.

Definition 4.2 *(Ref. 10) For any subset S of N, we define a characteristic function $v(S)$ by the maximum number $0 \le k \le n$ of disjoint strict conjugate sets contained in S. Let X denote the set of all variables $\{x_1, \ldots, x_n\}$. For any subset $X_S := \{x_i \mid i \in S\}$ of X, we define $v(X_S) := v(S)$. We call this cooperative game the conjugate-set game.*

Then it is easily seen that v is superadditive, that is, $v(S) + v(T) \le v(S \cup T)$ whenever $S \cap T = \phi$.

Game theory provides mathematical tools to analyze cooperative games. In this section, we use the Shapley value and the core to evaluate the contribution of each variable to decreasing the objective function. Namely, they answer the question: *How much does each variable of the strict conjugate set contribute to decrease the objective function?*

The Shapley value of player i for the cooperative game with the characteristic function v is defined by

$$\phi_i(v) = \sum_{i \in S} \{v(S) - v(S - \{i\})\} n!/(n-s)!(s-1)!, \tag{4}$$

where s denotes the cardinal number of S. Then it holds that $\phi_1 + \cdots + \phi_n = v(N)$, see e.g. Aumann and Hart (Ref. 11). Since the conjugate-set game is superadditive, the core is defined as follows.

$$\{x \in R^n \mid \sum_{i \in S} x_i \geq v(S) \ \forall S \subset N, \ x \geq 0\}. \tag{5}$$

In the next two sections, we compute the Shapley value and the core for the 4-phase and 3-phase partition problems.

5 Game theoretic aspect of the 4-phase partition problem

In this section, we consider a special case of the 4-phase partition problem that $l_k \equiv 1$ and $h := h_3 = h_4 = h_5 = h_6 < 0$. We first list off the strict conjugate sets. Next, we compute the Shapley value (4) and the core (5). In the following examples, for a vector $(v_1, \ldots, v_m)$ and a constant $c \neq 0$, we simply denote $(v_1/c, \ldots, v_m/c)$ by $(v_1, \ldots, v_m)/c$.

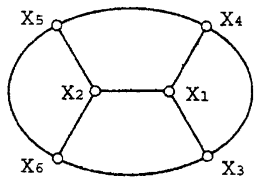

Figure 2: l_k's and h_k's are constant.

Although this problem has eight variables x_1, $y_1, \ldots s_5$ and s_6, we regard two variables x_k and y_k $(k = 1, 2)$ as a pair X_k in this section. Because, the conclusion delivered from the former way depends on the coordinates. Then, the (instable) stationary point (2) reduces to

$$X_1 = (0,0),\ X_2 = (-1,0),\ X_3 = (1/2, -\sqrt{3}/2),$$
$$X_4 = (1/2, \sqrt{3}/2),\ X_5 = (-3/2, \sqrt{3}/2),\ X_6 = (-3/2, -\sqrt{3}/2).$$

Example 5.1 *We consider the case of $h = -1/4$. Then, since the principal minors of the Hesse matrix corresponding to two points X_i and X_j $(i \neq j)$ are positive, any subset $\{X_i, X_j\}$ is not strict conjugate set. It is easily seen by computing principal minors that minimal strict conjugate sets have forms of $\{X_1, X_2, X_3\}$ or $\{X_2, X_5, X_6\}$ up to symmetry. Then the Shapley value is given by*

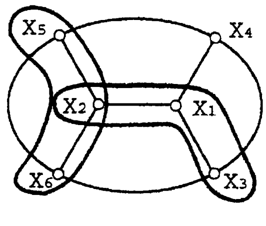

Figure 3: Minimal strict conjugate sets when $h = -1/4$.

$$\phi := (\phi_{X_1}, \phi_{X_2}, \phi_{X_3}, \phi_{X_4}, \phi_{X_5}, \phi_{X_6}) = (7, 7, 4, 4, 4, 4)/15. \tag{6}$$

Next, let us compute the core. Following the standard notation on the core, we denote by x_k the imputation of player X_k $(k = 1, \ldots, 6)$. Since $x_1 + \cdots + x_6 = v(\{X_1, \ldots, X_6\}) = 2$, $x_1 + x_3 + x_4 \geq 1$, and $x_2 + x_5 + x_6 \geq 1$, we have $x_1 + x_3 + x_4 = x_2 + x_5 + x_6 = 1$. On the other hand, since $x_1 + x_2 + x_3 \geq 1$, we get $x_2 \geq x_4$. Similarly, we have $x_2 \geq x_3$ and $x_1 \geq \max\{x_5, x_6\}$. So the core is given by

$$\{x \geq 0 | x_1 + x_3 + x_4 = x_2 + x_5 + x_6 = 1,\ x_1 \geq x_5 \vee x_6,\ x_2 \geq x_3 \vee x_4\},$$

where $a \vee b := \max\{a, b\}$, and the Shapley value ϕ belongs to the core.

6 3-phase partition problem

It is easy to discuss stability and instability of stationary solutions for the 3-phase partition problem, see Fig. 4. So, we only describe the conclusion.

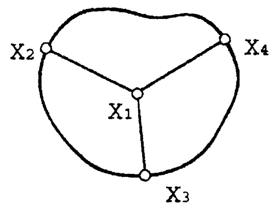

Figure 4: 3-phase partition problem.

In the following, let $f := \sum_{k=2}^{4} ||X_k - X_1||$, $\ell_k := ||X_k - X_1||$, and $\ell := \ell_2 + \ell_3 + \ell_4$.

Theorem 6.1 *(Kawasaki, Ref. 12) If* $(X_1, \ldots, X_4)$ *is a local minimizer of the 3-phase partition problem, then (a) the angle between any two adjacent line segments equals* $2\pi/3$ *and (b) any line segment* X_1X_j *with* $X_j \in \partial\Omega$ *orthogonally intersects the boundary.*

Theorem 6.2 *(a) The Hesse matrix of* f *at a stationary solution is positive definite if and only if* $1 + h_2\ell$, $h_2 + h_3 + h_2h_3\ell$, *and* $h_3h_4 + h_2h_4 + h_2h_3 + h_2h_3h_4\ell$ *are positive. (b) If at least two of* h_k*'s are negative, then the stationary solution is instable. (c) When just one of* h_k*'s is zero,* f'' *is positive definite if and only if other* h_k*'s are positive. (d) When* h_k *is negative, and* h_i *and* h_j *are positive,* f'' *is positive definite if and only if*

$$h_k > -h_ih_j/(h_i + h_j + h_ih_j\ell). \tag{7}$$

(e) When at least two of h_k*'s are zero and the other is positive,* f'' *is nonnegative definite.*

References

1. Sternberg, P., and Zeimer W. P., *Local minimizers of a three-phase partition problem with triple junctions*, Proceedings of the Royal Society of Edinburgh, Vol. 124A, pp. 1059–1073, 1994.
2. Ikota, R., and Yanagida, E., *A stability criterion for stationary curves to the curvature-driven motion with a triple junction*, Differential and Integral Equations, Vol. 16, pp. 707–726, 2003.
3. Ikota, R., and Yanagida, E., *Stability of stationary interfaces of binary-tree type*, to appear in Calculus of Variations and Partial Differential Equations.
4. Rockafellar, R. T., *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
5. Gelfand, I. M., and Fomin, S. V., *Calculus of variations.* Prentice Hall, 1963.
6. Kawasaki, H., *Conjugate points for a nonlinear programming problem with constraints*, Journal of Nonlinear and Convex Analysis, Vol. 1, pp. 287–293, 2000.
7. Kawasaki, H., *A conjugate points theory for a nonlinear programming problem*, SIAM Journal Control and Optimization, Vol. 40, pp. 54–63, 2001.
8. Kawasaki, H., *Analysis of conjugate points for constant tridiagonal Hesse matrices of a class of extremal problems*, Optimization Methods and Software, Vol. 18, pp. 197–205, 2003.
9. Strang, G., *Linear Algebra and its Applications.* Academic Press, New York, 1976.
10. Kawasaki, H., *A game-theoretic aspect of conjugate sets for a nonlinear programming problem*, to appear in Proceedings of the third International Conference on Nonlinear Analysis and Convex Analysis, Edited by W. Takahashi and T. Tanaka, Yokohama Publishers, Yokohama.
11. Aumann, R. J., and Hart, S., Editors, *Handbook of game theory with Economic Applications Vol. 1*, Elsevier, Amsterdam, 1992.
12. Kawasaki, H., *Conjugate-set game for a nonlinear programming problem*, to appear in Game theory and applications 10, Edited by L. A. Petrosyan, Nova Science Publications, New York, USA.