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完備距離空間をなすファジィ数の集合に関するシャウダーの不動点定理に ついて
Schauder's Fixed Point Theorems Concerning Complete Metric Spaces of Fuzzy Numbers

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Abstract
Two aims of our study are follows: One is to prove that a complete metric space of fuzzy numbers becomes a Banach space under a condition that the metric has a homogeneous property. Another is to give sufficient conditions that a subset in the complete metric space and an into continuous mapping on the subset have at least one fixed point by applying Schauder's fixed point theorem.

1 Introduction

Fuzzy numbers are characterized by membership functions which have three properties: normality , compact convex support and upper semi-continuity. Membership functions are described by $\alpha-$cut sets, *i.e.*, level sets for $0 \leq \alpha \leq 1$, which are compact convex subsets in $\mathbf{R}^n$ under the above assumptions of membership functions hold. In [6] the author discussed an embedding theorem where metric spaces of compact convex sets are complete. There are so many results on completeness of metric spaces of various kinds of fuzzy numbers and metrics in [2, 4].

In analyzing qualitative properties of differential equations Schauder's fixed point in complete linear spaces is very useful, because it guarantees the existence of solutions for integral equations corresponding to the differential equations etc. Schauder's fixed point is as follows : let S be a bounded convex and closed subset in a Banach space. If an into mapping V on S is continuous and the closure $cl(V(S))$ is compact, then V has at least one fixed point in S. (See *e.g.*, [9]). It can be easily seen that Various sets of fuzzy numbers are complete metric spaces with suitable metrics, but it is not possible to discuss the qualitative properties of solutions in the complete metric spaces by applying Schauder's fixed point theorem rather than the contraction principle and the comparison method(*e.g.*, [2, 3, 4, 7, 8]).

In this paper we introduce a parametric representation of fuzzy numbers, which are strictly fuzzy convex, then the fuzzy numbers can be identified by bounded closed curves in the two-dimensional metric space. Moreover we show that the set of all the fuzzy numbers becomes a complete linear space and establish sufficient conditions for fixed points to exist in the complete metric space by applying Schauder's fixed point in the induced Banach space.

2 Complete Metric Space of Fuzzy Numbers

Denote $I = [0, 1]$. The following definition means that a fuzzy number can be identified with a membership function.

Definition 1 *Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by*

$$\mathcal{F}_b^{st} = \{\mu : \mathbf{R} \to I \text{ satisfying (i)-(iv) below}\}.$$

(i) μ *has a unique number* $m \in \mathbf{R}$ *such that* $\mu(m) = 1$ (*normality*);

(ii) $supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ *is bounded in* $\mathbf{R}$ (*bounded support*);

(iii) μ *is strictly fuzzy convex on* $supp(\mu)$ *as follows:*

(a) *if* $supp(\mu) \neq \{m\}$, *then*

$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$

for $\xi_1, \xi_2 \in supp(\mu)$ *with* $\xi_1 \neq \xi_2$ *and* $0 < \lambda < 1$;

(b) *if* $supp(\mu) = \{m\}$, *then* $\mu(m) = 1$ *and* $\mu(\xi) = 0$ *for* $\xi \neq m$;

(iv) μ *is upper semi-continuous on* $\mathbf{R}$ *(upper semi-continuity).*

It follows that $\mathbf{R} \subset \mathcal{F}_b^{st}$. Because m has a membership function as follows:

$$\mu(m) = 1\ ; \quad \mu(\xi) = 0\ (\xi \neq m) \tag{2.1}$$

Then μ satisfies the above (i)-(iv).

In usual case a fuzzy number x satisfies *fuzzy convex on* $\mathbf{R}$, *i.e.*,

$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu(\xi_1), \mu(\xi_2)] \tag{2.2}$$

for $0 \leq \lambda \leq 1$ and $\xi_1, \xi_2 \in \mathbf{R}$. Denote $\alpha-$cut sets by

$$L_\alpha(\mu) = \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}$$

for $\alpha \in I$. When the membership function is fuzzy convex, then we have the following remarks.

Remark 1 *The following statements* (1) - (4) *are equivalent each other, provided with* (i) *of Definition* 1.

(1) (2.2) *holds;*

(2) $L_\alpha(\mu)$ *is convex with respect to* $\alpha \in I$;

(3) μ *is non-decreasing in* $\xi \in (-\infty, m)$, *non-increasing in* $\xi \in [m, +\infty)$, *respectively;*

(4) $L_\alpha(\mu) \subset L_\beta(\mu)$ *for* $\alpha > \beta$.

Remark 2 *The above condition* (iiia) *is stronger than* (2.2). *From* (iiia) *it follows that* $\mu(\xi)$ *is strictly monotonously increasing in* $\xi \in [\min supp(\mu), m]$. *Suppose that* $\mu(\xi_1) \geq \mu(\xi_2)$ *for* $\xi_1 < \xi_2 \leq m$. *From Remark* 1(3), *it follows that* $\mu(\xi_1) = \mu_1(\xi_2)$ *for some* $\xi_1 < \xi_2$, *so we get* $\mu(\xi) = \mu(\xi_1) = \mu(\xi_2)$ *for* $\xi \in [\xi_1, \xi_2]$. *This contradicts with Definition* 1 (iiia). *Thus* μ *is strictly monotonously increasing. In the similar way* μ *is strictly monotonously decreasing in* $\xi \in [m, \max supp(\mu)]$. *This condition plays an important role in Theorem* 1.

We introduce the following parametric representation of $\mu \in \mathcal{F}_b^{st}$ as

$$x_1(\alpha) = \min L_\alpha(\mu),$$
$$x_2(\alpha) = \max L_\alpha(\mu)$$

for $0 < \alpha \leq 1$ and

$$x_1(0) = \min supp(\mu),$$
$$x_2(0) = \max supp(\mu).$$

In the following example we illustrate typical types of fuzzy numbers.

Example 1 *Consider the following* $L-R$ *fuzzy number* $x \in \mathcal{F}_b^{st}$ *with a membership function as follows:*

$$\mu(\xi) = \begin{cases} L(\frac{|m-\xi|}{\ell})_+ & (\xi \leq m) \\ R(\frac{|\xi-m|}{r})_+ & (\xi > m) \end{cases}$$

Here it is said that $m \in \mathbf{R}$ *is a center and* $\ell > 0, r > 0$ *are spreads.* L, R *are* $I-$*valued functions. Let* $L(\xi)_+ = \max(L(|\xi|), 0)$ *etc. We identify* μ *with* $x = (x_1, x_2)$. *As long as there exist* L^{-1} *and* R^{-1}, *we have* $x_1(\alpha) = m - L^{-1}(\alpha)\ell$ *and* $x_2(\alpha) = m + R^{-1}(\alpha)r$.

Let $L(\xi) = -c_1\xi + 1$, *where* $c_1 > 0$ *and* $|x_1 - m| \leq \ell$. *We illustrate the following cases* (i)-(iv).

(i) *Let* $R(\xi) = -c_2\xi + 1$, *where* $c_2 > 0$. *Then* $c_2\ell(x_2 - m) = c_1 r(m - x_1)$.

(ii) *Let* $R(\xi) = -c_2\sqrt{\xi} + 1$, *where* $c_2 > 0$. *Then* $c_2\ell(x_2 - m)^2 = c_1 r^2(m - x_1)$.

(iii) *Let* $R(\xi) = -c_2\xi^2 + 1$, *where* $c_2 > 0$. *Then* $c_2^2\ell^2(x_2 - m) = c_1^2 r(x_1 - m)^2$.

(iv) *Let* c *be a real number such that* $0 < c < 1$. *Denote*

$$L(\xi) = \begin{cases} 1 & (\xi = 0) \\ -c\xi + c & (0 < \xi \leq 1) \end{cases}$$

and let $R(\xi) = L(\xi)$. *Then we have* $\ell(x_2 - m) = r(m - x_1)$ *for* $|x_1 - m| \leq \ell$. *The representation of* $x = (x_1, x_2)$ *is as follows:*

$$x_1(\alpha) = m - (1 - \frac{\alpha}{c})\ell,$$
$$x_2(\alpha) = m + (1 - \frac{\alpha}{c})r \quad (0 \leq \alpha < c)$$
$$x_1(\alpha) = x_2(\alpha) = m \quad (c \leq \alpha \leq 1)$$

The membership function is given by as follows:

$$\mu(\xi) = \begin{cases} 0 & (\xi < x_1(0), \xi > x_2(0)) \\ x_1^{-1}(\xi) & (x_1(0) \leq \xi < m) \\ 1 & (\xi = m) \\ x_2^{-1}(\xi) & (m < \xi \leq x_2(0)) \end{cases}$$

Denote by $C(I)$ the set of all the continuous functions on I to $\mathbf{R}$. The following theorem shows a membership function is characterized by x_1, x_2.

Theorem 1 *Denote the left-, right-end points of the $\alpha-$cut set of $\mu \in \mathcal{F}_b^{st}$ by $x_1(\alpha), x_2(\alpha)$, respectively. Here $x_1, x_2 : I \to \mathbf{R}$. The following properties* (i)-(iii) *hold.*

(i) $x_1, x_2 \in C(I)$;

(ii) $\max_{\alpha \in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha \in I} x_2(\alpha) = x_2(1)$;

(iii) *x_1, x_2 are non-decreasing, non-increasing on I, respectively, as follows :*

- (a) *there exists a positive number $c \leq 1$ such that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in [0, c)$ and that $x_1(\alpha) = m = x_2(\alpha)$ for $\alpha \in [c, 1]$;*
- (b) *$x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in I$;*

Conversely, under the above conditions (i) -(iii), *if we denote*

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \qquad (2.3)$$

for $\xi \in \mathbf{R}$, then $\mu \in \mathcal{F}_b^{st}$.

Remark 3 *From the above Condition* (i) *a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve over $\mathbf{R}^2$ and $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$.*

Proof. (i) Let $x = (x_1, x_2) \notin \mathbf{R}$. Let $\lim_{n\to\infty} \alpha_n = \alpha_0$ for $\alpha_0 \in I$. Denote $A_1 = \liminf_{n\to\infty} x_1(\alpha_n)$. We shall prove that $A_1 \geq x_1(\alpha_0)$. Suppose that $A_1 < x_1(\alpha_0)$. Then for any sufficiently small $\varepsilon > 0$ there exist a number ℓ such that $A_1 - \varepsilon < x_1(\alpha_\ell) < A_1 + \varepsilon < x_1(\alpha_0)$. Denote

$$M = \{\alpha \in I : x_1(\alpha) = x_2(\alpha) = m\},$$
$$S(c) = \{\alpha \in I : x_1(\alpha) = c \text{ on some interval}\} \text{ for } c \in \mathbf{R}.$$

There are the three cases as follows;

(a) $\alpha_0 \in M$; (b) $\alpha_0 \in S(c)$ for some c ; (c) $\alpha_0 \notin M \cup S(c)$ for any c.

In case of (a) we consider two cases: (a1) α_0 is an interior point of M, *i.e.*, there exists a sufficiently small number $\delta >$ such that the neighborhood $U_\delta(\alpha_0) \subset M$; (a2) α_0 is a isolated point. In (a1) it follows that $m < A_1 + \varepsilon < m$, which leads o a contradiction. In (a2) there exist two integers $p < q$ such that

$$|x_1(\alpha_q) - A_1| < 1/q < |x_1(\alpha_p) - A_1| < 1/p.$$

Then $\min L_{\alpha_q}(\mu) = x_1(\alpha_q) < x_1(\alpha_p) = \min L_{\alpha_p}(\mu) < m$ and this means that $L_{\alpha_p}(\mu) \subset L_{\alpha_q}(\mu)$ and $L_{\alpha_p}(\mu) \neq L_{\alpha_q}(\mu)$. On the other hand $L_{\alpha_p}(\mu) \supset L_{\alpha_q}(\mu)$ because $\alpha_p < \alpha_q < 1$. This leads to a contradiction.

In case of (b) the point α_0 is an interior point of $S(c)$, *i.e.*, there exists a sufficiently small number $\delta >$ such that the neighborhood $U_\delta(\alpha_0) \subset S(c)$. Then $c = x_1(\alpha_\ell) < A_1 + \varepsilon < c$, which means a contradiction.

In case of (c), by Relation (3) of Remark1, $x_1(\alpha)$ is strictly monotonously incerasing in α. Consider a sequence $\{\varepsilon_n > 0\}$ such that $\varepsilon_n > \varepsilon_{n+1} > 0$ and that $\varepsilon_n \to +0$ as $n \to \infty$. Then

$$\alpha_\ell = \mu(x_1(\alpha_\ell)) < \mu(A_1 + \varepsilon_1) < \mu(x_1(\alpha_0)) = \alpha_0,$$

which contradicts with $\lim_{n\to\infty} \alpha_n = \alpha_0$. Therefore $A_1 \geq x_1(\alpha_0)$ and x_1 is lower semi-continuous. In the same way x_1 is upper semi-continuous and x_1 is continuous on I. It can be seen that $x_2(\alpha)$ is continuous on I by the same discussion.

(ii) It is clear that the uniquess of m and that $x_1(1) = m = x_2(1)$. Since the membership is fuzzy convex, it follows that $x_1(\alpha) \leq m \leq x_2(\alpha)$ for $\alpha \in I$.

(iii) Let M be defined in (i). In case that $M = (0, 1]$, we have $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in (0, 1]$. This means that (iiib) holds. In case that $M \neq (0, 1]$, because of the continuity of x_1, x_2, denoting $c = \inf M$, it follows that $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in [c, 1]$ and that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in (0, c)$, which means that (iiia) holds.

Conversely (2.3) means that the upper level set $L_\beta(\mu)$ satisfies $L_\beta(\mu) = [x_1(\beta), x_2(\beta)] \subset \mathbf{R}$ for $\beta \in I$. From (2.3) it follows that if $\xi \in [x_1(\alpha), x_2(\alpha)]$ then $\mu(\xi) \geq \alpha$ and that $\xi \notin [x_1(\mu(\xi) + \varepsilon), x_2(\mu(\xi) + \varepsilon)]$ for each $\varepsilon > 0$. Then it can be seen that $[x_1(\beta), x_2(\beta)] \subset L_\beta(\mu)$. When $\mu(\xi) = \beta$, from (2.3), it follows that a $\xi \in [x_1(\beta), x_2(\beta)]$. When $\mu(\xi) > \beta$, then there exists an $\alpha \in I$ such that $\xi \in [x_1(\alpha), x_2(\alpha)]$ and $\alpha \geq \beta$, which means that $\xi \in [x_1(\alpha), x_2(\alpha)] \subset [x_1(\beta), x_2(\beta)]$. Therefore we have $L_\beta(\mu) = [x_1(\beta), x_2(\beta)]$.

From (2.3) it is immediately seen that (i) and (ii) of Definition1 hold. The $\alpha-$ cut set $L_\alpha(\mu)$ is closed for $\alpha \in I$, i.e., the function μ is upper semi-continuous on $\mathbf{R}$. For $\alpha \in I$, $L_\alpha(\mu)$ is convex, i.e., the function μ is fuzzy convex on $\mathbf{R}$ See, e.g., [10].

From (2.1), $\mu(\xi) = \overline{\alpha}$ means that $\xi = a(\overline{\alpha})$ or $\xi = b(\overline{\alpha})$. If suppose that $a(\overline{\alpha}) < \xi < b(\overline{\alpha})$, which means that $\mu(\xi) > \overline{\alpha}$. Suppose that there exist $\xi_1, \xi_2 \in J$ and λ such that $\xi_1 \neq \xi_2, 0 < \lambda < 1$ and $\mu(\xi_3) = \mu(\overline{\xi})$, where $\xi_3 = \lambda\xi_1 + (1-\lambda)\xi_2$ and $\mu(\overline{\xi}) = \min[\mu(\xi_1), \mu(\xi_2)]$. Then we have $\xi_3 \neq \overline{\xi}$ and $\xi_3 = a(\mu(\overline{\xi}))$ or $\xi_3 = b(\mu(\overline{\xi}))$, i.e., $a^{-1}(\xi_3) = \mu(\overline{\xi})$ or $b^{-1}(\xi_3) = \mu(\overline{\xi})$. Thus we get, from (2.1), $\overline{\xi} = a(\mu(\overline{\xi})) = a(a^{-1}(\overline{\xi})) = \xi_3$ or $\xi_3 =$

$b(b^{-1}(\bar{\xi})) = \bar{\xi}$. This leads to a contradiction. Therefor μ_x is strictly fuzzy convex.

Q.E.D.

In what follows we denote $\mu = (x_1, x_2)$ for $\mu \in \mathcal{F}_b^{st}$. The parametric representation of μ is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be an $\mathbf{R}$-valued function. The corresponding binary operation of two fuzzy numbers $x, y \in \mathcal{F}_b^{st}$ to $g(x,y) : \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} \to \mathcal{F}_b^{st}$ is calculated by the extension principle of Zadeh. The membership function $\mu_{g(x,y)}$ of g is as follows:

$$\mu_{g(x,y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_x(\xi_1), \mu_y(\xi_2))$$

Here $\xi, \xi_1, \xi_2 \in \mathbf{R}$ and μ_x, μ_y are membership functions of x, y, respectively. From the extension principle, it follows that, in case where $g(x,y) = x + y$,

$$\begin{aligned} &\mu_{x+y}(\xi) \\ &= \max_{\xi=\xi_1+\xi_2} \min_{i=1,2}(\mu_i(\xi_i)) \\ &= \max\{\alpha \in I : \xi = \xi_1 + \xi_2,\ \xi_i \in L_\alpha(\mu_i), i = 1,2\} \\ &= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}. \end{aligned}$$

Thus we get $x + y = (x_1 + y_1, x_2 + y_2)$. In the similar way $x - y = (x_1 - y_2, x_2 - y_1)$.

Denote a metric by

$$d_\infty(x,y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)$$

for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_b^{st}$.

Theorem 2 *$\mathcal{F}_b^{st}$ is a complete metric space in $C(I)^2$.*

Proof. Let a Cauchy sequence $\{x_k = (x_1^{(k)}, x_2^{(k)}) \in \mathcal{F}_b^{st} : k = 1, 2, \cdots\}$. It suffieces that there an fuzzy number $x_0 \in \mathcal{F}_b^{st}$ such that $\lim_{n\to\infty} d(x_n, x_0) = 0$. Since $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, from the well-known the Cauchy's theorem in Calculus, there exists an limit $x_0 = (x_1^{(0)}, x_2^{(0)}) \in C(I) \times C(I)$ such that the following properties(i)-(iv) hold.

(i) $\lim_{k\to\infty} d(x_k, x_0) = 0$;

(ii) $x_1^{(0)}$ and $x_2^{(0)}$ are non-decreasing, non-increasing on I, respectively;

(iii) $x_1^{(0)}(\alpha) \le m \le x_2^{(0)}(\alpha)$ for $\alpha \in I$ and $x_1^{(0)}(1) = m = x_2^{(0)}(1)$.

Suppose that there exists a number $n \neq m$ such that $x_1(1) = x_2(1) = n$. This contraicts with the uniform convergence of the Cauchy's squence. Thus a unique $m \in \mathbf{R}$ satisfies Theorem1(ii). Denote $C = \{\alpha \in I : x_1^{(0)}(\alpha) = x_2^{(0)}(\alpha) = m \text{ and } \alpha > 0\}$. In case when $C = (0,1]$, we get $x_1^{(0)}(\alpha) = x_2^{(0)}(\alpha) = m$ for $0 < \alpha \le 1$, which means that Theorem1(iiia) holds. In case $C \neq (0,1]$, by the continuity of x_1, x_2, there exists a real number c such that $0 < c \le 1$ and that c satisfies the following statements (1) and (2) hold.

(1) $x_1(\alpha) = x_2(\alpha)$ for $\alpha \in [c,1]$; (2) $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in (0,c)$.

This means that Theorem1(iiib) holds. Therefore, $x_0 \in \mathcal{F}_b^{st}$ and the metric space $(\mathcal{F}_b^{st}, d)$ is complete. **Q.E.D.**

3 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions μ_x, μ_y of $x, y \in \mathcal{F}_b^{st}$ and $\lambda \in \mathbf{R}$, the following addition and a scalar product are given as follows :

$$\begin{aligned} \mu_{x+y}(\xi) &= \sup\{\alpha \in [0,1] : \\ &\quad \xi = \xi_1 + \xi_2,\ \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\}; \\ \mu_{\lambda x}(\xi) &= \begin{cases} \mu_x(\xi/\lambda) & (\lambda \neq 0) \\ 0 & (\lambda = 0,\ \xi \neq 0) \\ \sup_{\eta \in \mathbf{R}} \mu_x(\eta) & (\lambda = 0,\ \xi = 0) \end{cases} \end{aligned}$$

In [5] they introduced the following equivalence relation $(x,y) \sim (u,v)$ for $(x,y), (u,v) \in \mathcal{F}_b^{st} \times \mathcal{F}_b^{st}$, *i.e.*,

$$(x,y) \sim (u,v) \iff x + v = u + y. \tag{3.4}$$

Putting $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$ by the parametric representation, the relation (3.4) means that the following equations hold.

$$x_i + v_i = u_i + y_i \quad (i = 1,2)$$

Denote an equivalence class by $[x,y] = \{(u,v) \in \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} : (u,v) \sim (x,y)\}$ for $x, y \in \mathcal{F}_b^{st}$ and the set of equivalence classes by

$$\mathcal{F}_b^{st}/\sim = \{[x,y] : x, y \in \mathcal{F}_b^{st}\}$$

such that one of the following cases (i) and (ii) hold:

(i) if $(x,y) \sim (u,v)$, then $[x,y] = [u,v]$;

(ii) if $(x,y) \not\sim (u,v)$, then $[x,y] \cap [u,v] = \emptyset$.

Then $\mathcal{F}_{\mathrm{b}}^{st}/\sim$ is a linear space with the following addition and scalar product

$$[x,y]+[u,v]=[x+u,y+v] \quad (3.5)$$

$$\lambda[x,y]=\begin{cases}[(\lambda x,\lambda y)] & (\lambda\geq 0)\\ [((-\lambda)y,(-\lambda)x)] & (\lambda<0)\end{cases} \quad (3.6)$$

for $\lambda\in\mathbf{R}$ and $[x,y],[u,v]\in\mathcal{F}_{\mathrm{b}}^{st}/\sim$. They denote a norm in $\mathcal{F}_{\mathrm{b}}^{st}/\sim$ by

$$\| [x,y] \| = \sup_{\alpha\in I} d_H(L_\alpha(\mu_x),L_\alpha(\mu_y)).$$

Here d_H is the Hausdorff metric is as follows:

$$\begin{aligned} & d_H(L_\alpha(\mu_x),L_\alpha(\mu_y))\\ &=\max(\sup_{\xi\in L_\alpha(\mu_x)}\inf_{\eta\in L_\alpha(\mu_y)}|\xi-\eta|,\\ &\quad \sup_{\eta\in L_\alpha(\mu_x)}\inf_{\xi\in L_\alpha(\mu_y)}|\xi-\eta|)\end{aligned}$$

It can be easily seen that $\| [x,y] \| = d_\infty(x,y)$.

Note that $\| [x,y] \| = 0$ in $\mathcal{F}_{\mathrm{b}}^{st}/\sim$ if and only if $x=y$ in $\mathcal{F}_{\mathrm{b}}^{st}$.

4 Schauder's Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space $\mathcal{F}_{\mathrm{b}}^{st}$ has an induced Banach space.

Theorem 3 *Let S be a bounded closed subset in $\mathcal{F}_{\mathrm{b}}^{st}$. Assume that S contains any segments of $x,y\in S$, i.e., $\lambda x+(1-\lambda)y\in S$ for $\lambda\in I$. Let V be an into continuous mapping on S. Assume that the closure $cl(V(S))$ is compact in $\mathcal{F}_{\mathrm{b}}^{st}$. Then V has at least one fixed point x in S, i.e., $V(x)=x$.*

Proof. Denote $X=\{[x,0]\in\mathcal{F}_{\mathrm{b}}^{st}/\sim : x\in\mathcal{F}_{\mathrm{b}}^{st}\}$. We shall prove that X is a Banach space. Let $\{[x,0]_n : i=1,2,\cdots\}$ be a Cauchy sequence in X. Without loss of generality we denote $[x,0]_n=[x_n,0]$ for $x_n\in\mathcal{F}_{\mathrm{b}}^{st}$ with $\| [x_n,0]-[x_m,0] \| \to 0$ as $n,m\to\infty$, which means that $\lim_{n,m\to\infty} d_\infty(x_n,x_m)=0$. By the completeness of $\mathcal{F}_{\mathrm{b}}^{st}$, there exists an element $x_0\in\mathcal{F}_{\mathrm{b}}^{st}$ such that $\lim_{n\to\infty} d_\infty(x_n,x_0)=0$ from Theorem 2. This measn that $\lim_{n\to\infty} \| [x_n,0]-[x_0,0] \| \to 0$ and X is a Banach space.

Put a subset $S_1=\{[x,0]\in X : x\in S\}$. Then S_1 is clearly bounded in X. Denote a mapping on S_1 by $V_1([x,0])=[V(x),0]$ for $[x,0]\in S_1$. It follows that for $[u,0]=[v,0]\in X$ we have $u=v$ and $V_1([u,0])=[V(u),0]=[V(v),0]=V_1([v,0])$. We get V_1 is an into mapping on S_1.

Let $x\in\mathcal{F}_{\mathrm{b}}^{st}$ be a limit of a sequence $\{y_n\in S\}$ such that $d(y_n,x)\to 0$ as $n\to\infty$. From the closedness of S, it follows that $x\in S$ and as long as $d(y_n,x)=\| [y_n,0]-[x,0] \| \to 0$ $(n\to\infty)$ with $[y_n,0]\in S_1$ and $[x,0]\in X$, we have $[x,0]\in S_1$. Thus S_1 is closed. For $x,y\in S$ it follows that $\lambda x+(1-\lambda)y\in S$ and

$$\lambda[x,0]+(1-\lambda)[y,0]=[\lambda x+(1-\lambda)y,0]\in S_1.$$

Therefore S_1 is convex in the Banach space X. When $y\to x$ in $\mathcal{F}_{\mathrm{b}}^{st}$, by the continuity of V, we have

$$\begin{aligned}\| V_1([y,0])-V_1([x,0]) \| &= \| [V(y),0]-[V(x),0] \|\\ &= \| [V(y),V(x)] \|\\ &= d_\infty(V(x),V(y))\to 0.\end{aligned}$$

Thus V_1 is continuous on S_1.

Finally, we shall prove the relative compactness of $V_1(S_1)$. Let $\{V_1([x_n,0]) : n=1,2,\cdots\}$ be a sequence in S_1. Because of the relative compactness of $V(S)$, there exists a subsequence $\{V_1([\overline{x_m},0])\}\subset\{V_1([x_m,0])\}$ such that $\lim_{m\to\infty} d_\infty(V(\overline{x_m}),y)=0$, where $y\in cl(V(S))$. Since

$$\begin{aligned} d_\infty(V(\overline{x_m}),y) &= \| [V(\overline{x_m}),y] \|\\ &= \| [V(\overline{x_m}),0]-[y,0] \|,\end{aligned}$$

we have $[y,0]\in cl(V_1(S_1))$. Thus $cl(V_1(S_1))$ is compact in X.

Therefore the mapping $V_1 : S_1\to S_1$, where S_1 is a bounded closed and convex subset in the Banach space X, is continuous. Here $cl(V_1(S_1))$ is relatively compact in X. By Schauder's fixed point theorem in Banach spaces, there exists a fixed point of V_1 in S_1, i.e., $[V(x),0]=[x,0]$, which means that $V(x)=x$ in S.

Q.E.D.

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

Theorem 4 *Let $\mathcal{F}$ be a complete metric space with a metric d. Assume that $\mathcal{F}$ is closed under addition and scalar product, and that $d(\lambda x,0)=|\lambda|d(x,0)$ for the scalar product λx and $\lambda\in\mathbf{R}, x\in\mathcal{F}$. Denote $X=\{[x,0] : x,0\in\mathcal{F}\}$. Here $[x,y]$ for $x,y\in\mathcal{F}$ are equivalence classes of* (3.4) *and 0 is the origin. Then X is a Banach space concerning addition* (3.5)*, scalar product* (3.6) *and norm $\| [x,0] \| = d(x,0)$ for $[x,0]\in X$.*

Moreover let S be a bounded closed subset in $\mathcal{F}$. Assume that S contains any segments of $x,y\in S$ in the same meaning of Theorem 3*. Let V be an into continuous mapping on S. Assume that the closure $cl(V(S))$*

is compact in $\mathcal{F}$. Then V has at least one fixed point in S.

Proof. It can be seen that X is a linear space. $\| [x,0] \|$ is a norm in X. For $[x,0],[y,0] \in X$ it follows that

$$\begin{aligned} \| [x,0]+[y,0] \| &= \| [x+y,0] \| \\ &= d(x+y,0) \\ &\leq d(x+y,y)+d(y,0) \\ &= d(x,0)+d(y,0) \\ &= \| [x,0] \| + \| [y,0] \|, \end{aligned}$$

since we have $[x+y,y]=[x,0]$and $d(x+y,y)=d(x,0)$ for $x,y \in \mathcal{F}$. It is clearly that $\| [x,0] \|$ is positive definite and for $\lambda \in \mathbf{R}$

$$\| [\lambda x,0] \| = d(\lambda x,0) = |\lambda| \, \| [x,0] \| .$$

In the same way as the discussion of Theorem 3, X is complete.

Denote a subset $S_1 = \{[x,0] \in X : x \in S\}$ and a mapping V_1 such that $V_1([x,0]) = [V(x),0]$ for $[x,0] \in S_1$. The following properties (i)-(iii) can be proved in the similar way in the proof of Theorem 3.

(i) S_1 is bounded closed and convex in X;

(ii) V_1 is an into continuous mapping on S_1;

(iii) $cl(V_1(S_1))$ is relatively compact in X.

Then, by Schauder's fixed point theorem, there exists at least one fixed point $[x_0,0]$ of V_1 in S_1, *i.e.*, $V(x_0) = x_0$ in S.

Q.E.D.

Example 2 (1) *Let $(\mathbf{R},d)$ be the discrete metric space with*

$$d(x,y)=0\ (x=y) \quad ; d(x,y)=1\ (x \neq y).$$

It follows that $d(\lambda x,0)=1 \neq |\lambda| d(x,0) = |\lambda|$ for $x \neq 0, |\lambda| \neq 0,1$. Then $X = \{[x,0] : x,0 \in \mathbf{R}\}$ cannot be a normed space concerning $\| [x,0] \| = d(x,0)$ for $x \in \mathbf{R}$, because $\| [x,0] \|$ is not homogeneous.

(2) *Let $K_C(\mathbf{R}^n)$ be the set of all compact convex subsets in $\mathbf{R}^n$. Assume that d_H is the Hausdorff metric in $\mathbf{R}^n$ as follows:*

$$d_H(A,B) = \max(\sup_{\xi \in A} \inf_{\eta \in B} \| \xi - \eta \|, \sup_{\eta \in B} \inf_{\xi \in A} \| \xi - \eta \|)$$

Here $A,B \in K_C(\mathbf{R}^n)$ and $\| \cdot \|$ is a norm in $\mathbf{R}^n$. Then we have $d_H(\lambda A,\emptyset) = |\lambda| d_H(A,\emptyset)$ for $A \in K_C(\mathbf{R}^n), \lambda \in \mathbf{R}$ where $\lambda A = \{\lambda a : a \in K_C(\mathbf{R}^n)\}$. By Theorem 4 it follows that the set of equivalence classes $X = \{[A,\emptyset] \in K_C(\mathbf{R}^n)/\sim : A \in K_C(\mathbf{R}^n)\}$ is a linear space with a norm $\| [A,\emptyset] \| = d_H(A,\emptyset)$. Here the equivalence relation $\sim$ is given in (3.4). It can be seen that X is a Banach space by the embedding theorem in [6].

Let S be a bounded closed subset in $K_C(\mathbf{R}^n)$. Assume that S contains any segments of $A,B \in S$ in the same meaning of Theorem 4. Let V be an into continuous set-valued mapping on S. Assume that the closure $cl(V(S))$ is compact in $K_C(\mathbf{R}^n)$. Then V has at least one fixed point $A_0 \in S$, i.e., $V(A_0) = A_0$.

5 Applications to FBVP

Consider the following boundary value problems of fuzzy differential equations

$$x''(t) = f(t,x,x'), \quad x(a)=A, x(b)=B. \qquad (5.7)$$

Here $t \in J = [a,b] \subset \mathbf{R} = (-\infty,+\infty)$ and fuzzy numbers $A,B \in \mathcal{F}_b^{st}$,which is a set of fuzzy numbers with compact supports and strict quasi-concavity, and $f : J \times \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} \to \mathcal{F}_b^{st}$ is a continuous function.

In order to discuss the qualitative properties of solutions to (5.7) we consider the following Fredholm equation

$$x(t) = w(t) + \int_a^b G(t,s) f(s,x(s),x'(s))ds$$

for $t \in J$. Let $A,B \in \mathcal{F}_b^{st}$ be in the same fuzzy numbers of (5.7). Here a fuzzy function $w \in C(J;\mathcal{F}_b^{st})$ and an $\mathbf{R}$-valued function $G \in C(\mathbf{R}^2;\mathbf{R})$ with $G(t,s) \geq 0$ such that

$$w(t) = \frac{A(b-t)+B(t-a)}{b-a}, \qquad (5.8)$$

$$G(t,s) = \begin{cases} \frac{(b-t)(s-a)}{b-a} & (a \leq t \leq s \leq b) \\ \frac{(b-s)(t-a)}{b-a} & (a \leq s < t \leq b) \end{cases} \qquad (5.9)$$

In the same way as in the discussion concerning boundary value problems of ordinary differential equation the following lemma is shown immediately.

Lemma 1 *A fuzzy function x is a continuously differentiable solution of (5.7) if and only if x is a fixed point of $T : C^1(J;\mathcal{F}_b^{st}) \to C^2(J;\mathcal{F}_b^{st})$ such that*

$$[T(x)](t) = w(t) + \int_a^b G(t,s) f(s,x(s),x'(s))ds.$$

Assume that the following properties (i) -(iii) hold.

(i) A function $f = (f_1, f_2) : J \times \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_{\mathbf{b}}^{st}$ is continuous. Here (f_1, f_2) is the parametric representation of f.

(ii) Let $r_i > 0$ for $i = 1, 2$. Then there exists a function $h_i : [0, \infty) \to [0, \infty)$ such that

$$|f_i(t, x, y, \alpha)| \leq h_i(|y_i(\alpha)|)$$

for $t \in J, \alpha \in I, i = 1, 2$, and $|x_i(\alpha)| \leq r_i$, $y = (y_1, y_2) \in \mathcal{F}_{\mathbf{b}}^{st}$. Here $x = (x_1, x_2), y = (y_1, y_2)$ are the parametric representation of x, y, respectively.

(iii) Assume that h_i, $i = 1, 2$, satisfy

$$\int_{+0}^{\infty} \frac{\eta d\eta}{h_i(\eta)} > 2r_i.$$

We say that the above conditions (i) -(iii) are a fuzzy type of Nagumo's conditions and they are applied to the fuzzy boundary value problem (5.7) in the same way as [1].

Lemma 2 *Assume that $f = (f_1, f_2)$ satisfies fuzzy type of Nagumo's conditions. Let $r_i > 0$, $i = 1, 2$, be in fuzzy type of Nagumo's conditions and a solution $x = (x_1, x_2) \in C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$ of (5.7) satisfy $|x_i(t, \alpha)| \leq r_i$ for $i = 1, 2, t \in J, \alpha \in I$.*

Then, there exist numbers $N_i > 0, i = 1, 2$ such that $|x_i^{'}(t, \alpha)| \leq N_i$ for $t \in J, \alpha \in I$.

Proof.

From the above lemmas we get the following existence theorem on the fredholm equation by the Schauder's fixed point theorem in Section 4.

Theorem 5 *Assume that the same conditions of Lemma 2 hold. Let*

$$|f_i(t, x, y, \alpha)| \leq \min(\frac{2N_i}{b-a}, \frac{8r_i}{(b-a)^2})$$

for $t \in J, (x, y) \in S_w(r, N), i = 1, 2, \alpha \in I$.

Then (5.7) has at least one solution x such that $(x(t), x^{'}(t)) \in S_w(r, N)$ for $t \in J$ and any $A, B \in \mathcal{F}_{\mathbf{b}}^{st}$.

The above theorem is proved in [7].

In case where (5.7) is redued to the following Volterra eqauton

$$\begin{aligned} z_u(t) &= X(t)U^{-1}(c - \mathcal{L}(q_u)) + q_u(t) \\ &= X(a)U^{-1}(c - \mathcal{L}(q_u)) \\ &\quad + \int_a^t Mz_u(s)ds + \int_a^t F(s, u(s))ds, \end{aligned}$$

we have an existence theorem of (5.7) by the Schauder's fixed point theorem in Section 4. Here $z = (x_1, x_2, x_1^{'}, x_2^{'})^T$, $u \in C^1(J; \mathbf{R})^2 \times C(J; \mathbf{R})^2, c = (A_1, A_2, B_1, B_2)^T \in C(J; \mathbf{R})^4$,

$$X(t) = e^{tM} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } X(0) = E,$$

where E is the identity matrix, $M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

and $\mathcal{L}$ denotes a bounded linear operator from $C^1(J; \mathbf{R})^2 \times C(J; \mathbf{R})^2$ to $C(J; \mathbf{R})^4$ by

$\mathcal{L}(z) = (x_1(a), x_2(a), x_1(b), x_2(b))^T$. Let U satisfy $\mathcal{L}(X(\cdot)v_0) = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ 1 & 0 & b & 0 \\ 0 & 1 & 0 & b \end{pmatrix} v_0 = Uv_0$ for $v_0 \in \mathbf{R}^4$. Putting $q_z(t) = \int_a^t X(t)X^{-1}(s)F(s, z(s))ds$ and $F(t, z) = \begin{pmatrix} 0 \\ 0 \\ f_1(t, z) \\ f_2(t, z) \end{pmatrix}$.

Then, in [8], we get the existence theorem on the Volterra type of (5.7) as follows.

Theorem 6 *Assume that positive numbers R, r satisfy $R < e^{-(b-a)}$ and $r > \frac{Q\|\mathcal{L}\|(b+1)\|U^{-1}\|}{e^{-(b-a)} - R}$. Let f satisfy $\int_a^b \max_{d(z,0) \leq r} d(f, 0)ds \leq rR$. If $A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ satisfy $d(A, 0) + d(B, 0) \leq \frac{r(e^{-(b-a)} - R)}{(b+1)\|U^{-1}\|} - \| \mathcal{L} \| Q$, then (5.7) has at least one solution in S. Here $Q = \int_a^b \max_{d(z,0) \leq r}(b - s + 1)d(f(s, z), 0)ds$.*

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