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Complex dynamical systems of the quartic polynomials

防衛大学校 数学教育室　藤村 雅代 (Masayo FUJIMURA)
Department of Mathematics
National Defense Academy

Abstract

The space $M_4(\mathbb{C})$ is the space of all affine conjugacy classes of quartic polynomials. We define a projection Ψ_4 from this space to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In [2], we show the projection is not surjective. The image of $M_4(\mathbb{C})$ under Ψ_4 is denoted by $\Sigma(4)$. The complement $\mathbb{C}^3\setminus\Sigma(4)$ is called the exceptional set. By analyzing the dynamics on the section $\{(4,\sigma_2,\sigma_4)\}$, we verify that quartic polynomial degenerates into "twins" of quadratic polynomials on the exceptional set.

1 Introduction

Let $\mathrm{Poly}_4(\mathbb{C})$ be the space of all quartic polynomials, and $M_4(\mathbb{C})$ be the space of all affine conjugacy classes of quartic polynomials. We define a projection Ψ_4 from $M_4(\mathbb{C})$ to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In [2], we show the projection is not surjective. The image of $M_4(\mathbb{C})$ under Ψ_4 is denoted by $\Sigma(4)$. The complement $\mathbb{C}^3\setminus\Sigma(4)$ is denoted by $\mathcal{E}(4)$, and called the exceptional set. For the cubic (resp. quadratic) polynomials, the exceptional set is empty.

As a Corollary of Theorem 1 in [3] we have: *If n given values $m_1, m_2, \cdots, m_n$ satisfy $\sum_{i=1}^{n}\frac{1}{1-m_i}=0$ and if $\sum_{j=1}^{k}\frac{1}{1-m_{i_j}}\neq 0$ for any choice of $\{i_j\}_{j=1}^{k}, 1\leqq i_1<i_2<\cdots<i_k\leqq n$, then there exists a polynomial of degree exactly n having the fixed points of the multipliers $m_1, m_2, \cdots, m_n$.*

We define an algebraic variety, $G(c)$ defined in Section 2, that indicates essential property of the projection Ψ_4, and as Theorem 1 we have a defining equation of the exceptional set and of the branch locus.

According to Theorem 1, we will need to consider the following:

- Why the exceptional set is non empty?
- Find a relation between dynamics of conjugacy classes in $\Psi_4^{-1}(s),\ s\in\mathbb{C}^3$.

In this paper, we examine dynamical behavior on the parameter space $\Sigma(4)\cup\mathcal{E}(4)$ (disjoint union), and we have the following conjectures by constructing of two suitable polynomial-like maps.

Conjecture On the exceptional set, a quartic polynomial degenerates into "twins" of quadratic polynomials conjugate to z^2+c for some c.

Conjecture None of quartic polynomial p has two disjoint quadratic-like restrictions of p such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial $z^2+c,\ c \in \mathrm{M} \setminus \{\frac{1}{4}\}$, where M is Mandelbrot set.

These conjectures give the reason why the exceptional set is not empty. The following theorem gives a support for these conjectures.

Theorem There is a component $D \subset \Sigma(4)$ such that two polynomial-like maps $(U, V, p) \sim_{hb} z^2+c$ and $(\tilde{U}, \tilde{V}, p) \sim_{hb} z^2+\bar{c}$ are constructed for any $\langle p \rangle \in D$, and the imaginary part of c converges to zero as $\langle p \rangle \to \mathcal{E}(4)$.

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2 Definitions

2.1 Definitions and Notations

Let $\mathrm{Poly}_4(\mathbb{C})$ be the space of all polynomials of the form

$$p : \mathbb{C} \to \mathbb{C},$$
$$p(z) = a_4 z^4 + a_3 z^3 + a_2 z^3 + a_1 z + a_0 \quad (a_4 \neq 0).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_4(\mathbb{C})$ are *holomorphically conjugate*, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathfrak{A}(\mathbb{C})$ with $g \circ p_1 \circ g^{-1} = p_2$, where $\mathfrak{A}(\mathbb{C})$ is the group of all affine transformations.

The space, $\mathrm{Poly}_4(\mathbb{C})/_\sim$, of holomorphic conjugacy classes $\langle p \rangle$ of quartic polynomials is denoted by $\mathrm{M}_4(\mathbb{C})$.

For each $p(z) \in \mathrm{Poly}_4(\mathbb{C})$, let $z_1, \cdots, z_4, z_5 = \infty$ be the fixed points of p, and $\mu_1, \cdots, \mu_4, \mu_5 = 0$ the multipliers of z_i (i.e. $\mu_i = p'(z_i)$).
Let $\sigma_1, \sigma_2, \cdots, \sigma_5$ be the elementary symmetric functions of these multipliers

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3 + \mu_4,$$
$$\sigma_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4,$$
$$\sigma_3 = \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4,$$
$$\sigma_4 = \mu_1\mu_2\mu_3\mu_4,$$
$$\sigma_5 = 0.$$

These multipliers are *invariant* under the action of (conjugation) $\mathfrak{A}(\mathbb{C})$.

The holomorphic index of a rational function f at a fixed point $\zeta \in \mathbb{C}$ is defined to be the complex number

$$\iota(f, \zeta) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)},$$

where we integrate in a small loop in the positive direction around ζ.

The following results are well known as "Fatou's index theorem":

- If a multiplier $\mu \neq 1$, then $\iota(f,\zeta) = \frac{1}{1-\mu}$.
- For any polynomial p which is not the identity map,
$$\sum_{\zeta\in\mathbb{C}} \iota(p,\zeta) = 0, \tag{1}$$
where this summation is over all fixed points of p.

A *polynomial-like* map of degree d is a triple (U,V,f) where U and V are topological disks, with V relatively compact in U, and $f : V \to U$ is analytic, proper of degree d.

The filled-in Julia set K_f of a polynomial-like map (U,V,f) is defined by

$$K_f = \bigcap_{n\geq 0} f^{-n}(V).$$

Polynomial-like maps (U,V,f) and $(\widetilde{U},\widetilde{V},\tilde{f})$ are *hybrid equivalent*, $f \sim_{hb} \tilde{f}$, if there exists a quasi-conformal homeomorphism h from a neighborhood of K_f to a neighborhood of $K_{\tilde{f}}$, such that $h\circ f = \tilde{f}\circ h$ near K_f and $\bar{\partial}h = 0$ almost everywhere on K_f.

From Straightening Theorem in [1], every polynomial-like map (U,V,f) of degree d is hybrid equivalent to a polynomial P of degree d. If K_f is connected then P is unique up to conjugation by an affine map.

2.2 Transformation formula

The following relation is obtained by Fatou's index theorem.

Lemma 1 (Theorem 1 in [2]) Among $\sigma_i's$, there is a linear relation

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0.$$

For a monic and centered quartic polynomial $z^4+c_2z^2+c_1z+c_0$, the three values σ_1, σ_2, σ_4 are given by **Transformation formula**:

$$\begin{aligned}
\sigma_1 &= -8c_1 + 12,\\
\sigma_2 &= 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48,\\
\sigma_4 &= 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + (144c_0c_1^2 - 288c_0c_1 + 128c_0)c_2\\
&\quad -27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3.
\end{aligned}$$

To remove an affine ambiguity from Transformation formula, we consider the following:

1. for a point $\langle p\rangle \in \mathrm{M}_4(\mathbb{C})$, choose a monic and centered representative $z^4+c_2z^2+c_1z+c_0$.
2. getting rid of the affine ambiguity on "Transformation formula", set $c := c_2^3$ (if $c_2 = 0$, set $\tilde{c} := c_0^3$), and
3. rebuild Transformation formula of $\sigma_1, \sigma_2, \sigma_4, c, c_0, c_1$ variables.

4. remove two variables c_0, c_1, from the above formula.

After these procedure, we obtain a parametrized algebraic variety.

Definition We define an algebraic variety in $\mathbb{C}^3$ with a parameter $c \in \mathbb{C}$,

$$G(c): \quad 262144(\sigma_1-4)^2c^2 + 1024(27\sigma_1^4 + (-144\sigma_2 - 576)\sigma_1^2 + (384\sigma_2 + 1280)\sigma_1 + 128\sigma_2^2 \\ -256\sigma_2 - 512\sigma_4 - 768)c + (9\sigma_1^2 + 24\sigma_1 - 32\sigma_2 - 48)^3 = 0.$$

$G(c)$ implies the following: For any point $(\sigma_1, \sigma_2, \sigma_4) \in \mathbb{C}^3$, on $G(c)$, the number of parameter values is equal to the number of conjugacy classes corresponds to the point $(\sigma_1, \sigma_2, \sigma_4)$.

Hence, there is a natural projection

$$\begin{array}{cccc} \Psi_4: & \mathrm{M}_4(\mathbb{C}) & \longrightarrow & \Sigma(4) \\ & \cup & & \cup \\ & \langle p \rangle & \longmapsto & (\sigma_1, \sigma_2, \sigma_4), \end{array}$$

where $\Sigma(4)$ is the image of $\mathrm{M}_4(\mathbb{C})$ under Ψ_4. The complement $\mathbb{C}^3 \setminus \Sigma(4)$ is denoted by $\mathcal{E}(4)$, and called the *exceptional set*.

The algebraic variety $G(c)$ perfectly exhibits phenomena induced by $\Psi_4 : \mathrm{M}_4(\mathbb{C}) \to \Sigma(4)$. Therefore we have the following Theorem.

Theorem 1 For $(\sigma_1, \sigma_2, \sigma_4) \in \mathbb{C}^3$, number of the elements of set $\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4)$ are ∞, 0, 1 or 2.

Case 1 $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = \infty$ if and only if $(\sigma_1, \sigma_2, \sigma_4) = (4, 6, 1)$.

$$\Psi_4^{-1}(4,6,1) = \{p_a(z) = (z^2-a)^2 + z\}_{a \in \mathbb{C}} \quad (\text{note } p_a \sim p_{\pm\omega a} \text{ by } z \mapsto \pm\omega z)$$

Case 2 $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = 0$ if and only if the point $(\sigma_1, \sigma_2, \sigma_4)$ cannot belong to $G(c)$ for any c.

$$(\sigma_1, \sigma_2, \sigma_4) = \left(4,\ s,\ \frac{(s-4)^2}{4}\right), \quad s \neq 6. \ \textbf{(the exceptional set)}$$

Case 3 $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = 1$ if and only if discriminant of the defining equation of $G(c)$ vanishes or $\sigma_1 = 4$ (**the branch locus**).

Case 4 $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = 2$, for the remains of the above.

Theorem 1 leads immediately to the following two corollaries.

Corollary 1 The exceptional set $\mathcal{E}(4)$ is contained in the plane $\{(4, \sigma_2, \sigma_4)\} \cong \mathbb{C}^2$.

Corollary 2 There is not a quartic polynomial having the fixed points of the multipliers $\mu, \mu, 2-\mu, 2-\mu$, $(\mu \neq 1)$.

3 Loci $\mathrm{Per}_1(\mu)$ on the space $\{(4, s_2, s_4)\}$

In this section, we consider dynamical behavior on the real section $\mathbb{R}^2 \cong \{(4, s_2, s_4)\}$, by Theorem 1, and show some figures supporting the conjectures.

The locus $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle p\rangle$ of maps p having a fixed point of multiplier μ.

Proposition 1 For each $\mu \in \mathbb{C}$, $\mathrm{Per}_1(\mu)$ is a straight line with the following defining equation:

$$\mathrm{Per}_1(\mu) : \ \sigma_4 - (2\mu - \mu^2)\sigma_2 + \mu^4 - 4\mu^3 + 8\mu = 0.$$

Proof. The multipliers at the fixed points are the roots of the equation,

$$\mu^4 - \sigma_1\mu^3 + \sigma_2\mu^2 - \sigma_3\mu + \sigma_4 = 0.$$

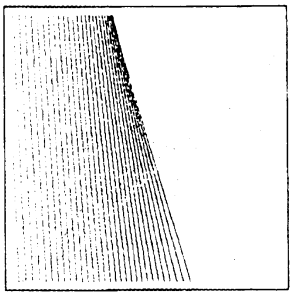
From the linear relation of Lemma 1, we have the defining equation of $\mathrm{Per}_1(\mu)$. ∎

We remark that the cases of the multipliers of a quartic polynomial on the real plane $\{(4, \sigma_2, \sigma_4)\}$ are 'four real values', 'two real and a pair of complex conjugates', or 'two pair of complex conjugates'.

3.1 $\mathrm{Per}_1(\mu)$ $(\mu \in \mathbb{R})$

At first we consider $\mu \in \mathbb{R}$. In this case we can illustrate the figure of $\mathrm{Per}_1(\mu)$. (See Figure 1.) The following results are easily verified.

Proposition 2 For $\langle p\rangle \in \{(4, \sigma_2, \sigma_4)\} \cap \Sigma(4)$, the corresponding multipliers of p are $\mu, 2 - \mu, \lambda, 2 - \lambda$.



The left figure shows $\mathrm{Per}_1(\mu)$ $(-10 < \mu < 1)$:
$-20 < s_2, s_4 < 20$,
Gray lines mean $\mathrm{Per}_1(\mu)$ $(|\mu| \geq 1)$ and
black lines mean $\mathrm{Per}_1(\mu)$ $(|\mu| < 1)$.

Figure 1:

Corollary 3

- If p has a attracting fixed point then p has a repelling fixed point with positive multiplier.
- If p has a repelling fixed point with negative multiplier then p has a repelling fixed point with positive multiplier.

Namely, each line of Figure 1 is overlapped by a line $\mathrm{Per}_1(\mu)$ for some $\mu > 1$, and p cannot have three attracting fixed points.

3.2 $\mathrm{Per}_1(\mu)$ and $\mathrm{Per}_1(\bar{\mu})$

Next, we consider the multipliers of a quartic polynomial are 'two real and a pair of complex conjugates'. In this case, the multipliers are $1 \pm i\beta$, λ, and $2 - \lambda$ from Proposition 2. Then we have the following from Proposition 1.

Proposition 3 For each $\beta \in \mathbb{R}$, $\mathrm{Per}_1(1 \pm i\beta)$ is a straight line with the following defining equation:

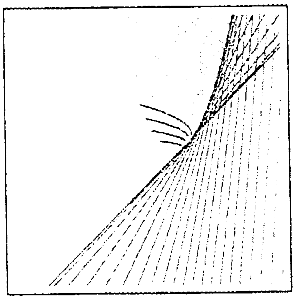
$$\mathrm{Per}_1(1 \pm i\beta): \quad \sigma_4 = (1 + \beta^2)\sigma_2 - (1 + \beta^2)(5 + \beta^2).$$

Proof. Removing λ from two equations $\sigma_2 = 5 + \beta^2 + \lambda(2 - \lambda)$ and $\sigma_4 = (1 + \beta^2)\lambda(2 - \lambda)$, we have the above defining equation of $\mathrm{Per}_1(1 \pm i\beta)$. ∎

Note that these loci are corresponds to repelling fixed points.

Now, we consider the last case: multipliers of a quartic polynomial are 'two pair of complex conjugates'. In this case, the multipliers are $a \pm ib$ and $2 - a \pm ib$ from Proposition 2. Because defining equation of $\mathrm{Per}_1(\mu)$ can express a line on the real plane no longer, we need a new device $\widetilde{\mathrm{Per}_1}(t)$ for illustrating figures of $\mathrm{Per}_1(\mu)$. (See Figure 2.)

The locus $\widetilde{\mathrm{Per}_1}(t)$ be the set of all conjugacy classes $\langle p \rangle$ of maps p having a fixed point of multiplier μ with $t = \mu\bar{\mu}$.



The left figure shows $\mathrm{Per}_1(1 \pm i\beta)$ and $\widetilde{\mathrm{Per}_1}(t)$.
$-20 < s_2, s_4 < 20$,
Dark gray lines mean $\mathrm{Per}_1(1 \pm i\beta)$,
gray curves mean $\widetilde{\mathrm{Per}_1}(t)$, $t \geq 1$ and
black curves mean $\widetilde{\mathrm{Per}_1}(t)$, $t < 1$.

Figure 2:

Proposition 4 In the case that the multipliers are $a \pm ib$ and $2 - a \pm ib$, we have a defining equation of $\widetilde{\mathrm{Per}_1}(t)$.

$$\widetilde{\mathrm{Per}_1}(t): \quad \sigma_4^2 - 2(t^2 + 2t)\sigma_4 + t^4 - 4t^3 + (\sigma_2 - 16)t^2 = 0,$$

where $t = a^2 + b^2$.

Proof. In this case the multipliers are $a \pm ib$ and $2 - a \pm ib$. By setting $t = a^2 + b^2$ for two equations $\sigma_2 = -2a^2 + 4a + 4 + 2b^2$ and $\sigma_4 = (a^2 + b^2)((2 - a)^2 + b^2)$, we have

$$\sigma_2 = -4a^2 + 4a + 4 + 2t, \qquad \sigma_4 = t(t - 4a + 4). \tag{2}$$

Removing a from the above two equations, we have a defining equation of $\widetilde{\mathrm{Per}_1}(t)$. ∎

Remark If $0 \le t < 1$, $\widetilde{\mathrm{Per}_1}(t)$ corresponds to polynomials having two attracting fixed points of multiplier $a+ib$ and $a-ib$. As $a, b \in \mathbb{R}$, the discriminant $4+4(4+2t-\sigma_2)$ of (2) must be positive. Therefore, on a region $\{(4,\sigma_2,\sigma_4)\,|\,\sigma_2 < -\frac{1}{4}(\sigma_4^2 - 6\sigma_4 - 19),\ \sigma_4 < \frac{(2-\sigma_2)^2}{4}\}$, corresponding polynomial p have two attracting fixed points of multipliers $a \pm ib$.

4 The exceptional set

The lines $\{\mathrm{Per}_1(\mu)\}$ have a close relation with the exceptional set. As an example, we give the following results directly obtained by the results in the section 3.1 and 3.2.

- On the plane $\{(4, s_2, s_4)\} \cong \mathbb{R}^2$, the envelopes of the lines $\{\mathrm{Per}_1(\mu)\}_{\mu\in\mathbb{R}}$ and of $\{\mathrm{Per}_1(1 \pm i\beta)\}_{\beta\in\mathbb{R}}$ coincides with the exceptional set. (See Figure 1, 2 and 3.)
- On the region $\left\{(4,\sigma_2,\sigma_4)\,|\,\sigma_4 < \frac{(2-\sigma_2)^2}{4}\right\}$ that bounded by the exceptional set, corresponding quartic polynomial has the fixed points of the multiplier with two pair of complex conjugates.

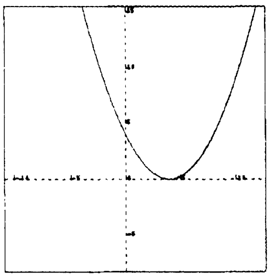


Figure 3:

The left figure shows the real section of the exceptional set

$$\mathcal{E}(4): \left(4,\ s,\ \frac{(s-4)^2}{4}\right), \quad (s \neq 6).$$

Conjecture On the exceptional set, a quartic polynomial degenerates into "twins" of quadratic polynomials conjugate to z^2+c for some c.

Theorem 2 There is a component $D \subset \Sigma(4)$ such that two polynomial-like maps $(U,V,p) \sim_{hb} z^2+c$ and $(\tilde{U},\tilde{V},p) \sim_{hb} z^2+\bar{c}$ are constructed for any $\langle p\rangle \in D$, and the imaginary part of c converges to zero as $\langle p\rangle \to \mathcal{E}(4)$.

Proof. On a region $\{(4,\sigma_2,\sigma_4)\,|\,\sigma_2 < -\frac{1}{4}(\sigma_4^2 - 6\sigma_4 - 19),\ \sigma_4 < \frac{(2-\sigma_2)^2}{4}\}$, any corresponding polynomial $p(z)$ has two attracting fixed points of multiplier μ, $\overline{\mu}$. Dynamics of $p(z)$ are symmetry for the real axis. (See Figure 4.) Therefore we can choose suitable topological disk U, $\widetilde{U}$ bounded by equipotential curves such that (U,V,p) and $(\widetilde{U},\widetilde{V},p)$ $(U \cap \widetilde{U} = \emptyset)$ are quadratic-like maps hybrid equivalent to z^2+c and $z^2+\bar{c}$ respectively. (See Figure 6 and 7.) ∎

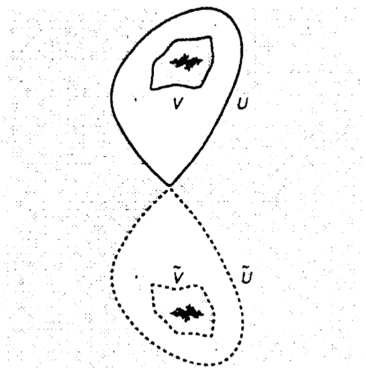


Figure 4: $(4, -1.7696160, 8.8480801)$, Julia set of $p(z) = z^4 + 3.8199z^2 + z + 3.775218$, $-2 < \Re z,\ \Im z < 2$

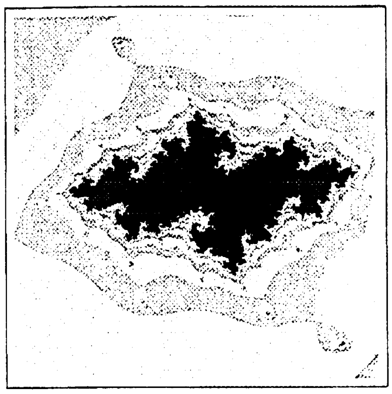


Figure 5: Julia set of $p(z) = z^4 + 3.8199z^2 + z + 3.775218$, $-0.2 < \Re z < 0.28$, $1.137 < \Im z < 1.617$

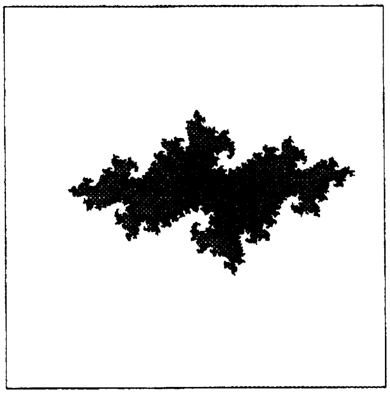


Figure 6: Julia set of quadratic-like map $-0.2 < \Re z < 0.28$, $1.137 < \Im z < 1.617$

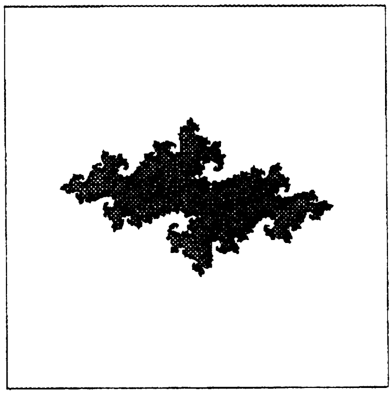


Figure 7: Julia set of $p_c(z) = z^2 + (-0.726 + 0.183i)$, $-2 < \Re z,\ \Im z < 2$.

5 On the point $(4, 6, 1) \in \Sigma(4)$

One parameter family $\{p_a(z) = (z^2 - a)^2 + a\}_{a \in \mathbb{C}}$ (note $p_a \sim p_{\pm\omega a}$ by $z \mapsto \pm\omega z$) corresponds to the point $(4, 6, 1)$. (See Figure 8 and 9.) There is a map p in this family such that p has two disjoint quadratic-like restriction hybrid equivalent to common quadratic map $z^2 + \frac{1}{4}$. (See Figure 8.)

Conjecture None of quartic polynomial p have two disjoint quadratic-like restrictions of p such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial $z^2 + c$, $c \in \mathrm{M} \setminus \{\frac{1}{4}\}$, where M is Mandelbrot set.

This conjecture gives a reason why the exceptional set is not empty.

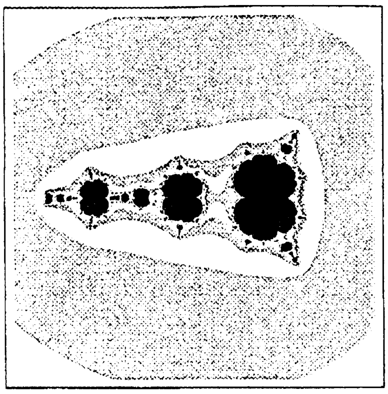


Figure 8: Julia set of $p(z) = z^4 - 2z^2 + z + 1$, $-2 < \Re z,\ \Im z < 2$. $(4,6,1) \in \Sigma(4)$

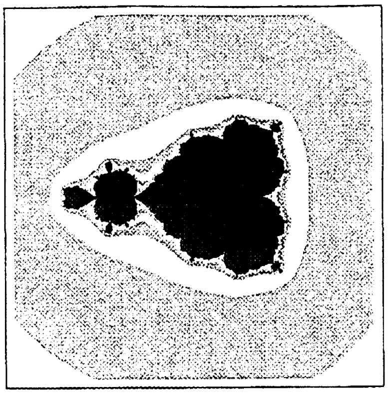


Figure 9: Julia set of $p(z) = z^4 - z^2 + z + 0.25$, $-2 < \Re z,\ \Im z < 2$. $(4,6,1) \in \Sigma(4)$

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