数理解析研究所講究録 1415 巻 2005 年 194-197

A relation between multidimensional data compression and Hilbert's 13th problem

Masahiro Yamada and Shigeo Akashi

Department of Information Sciences
Faculty of Science and Technology
Tokyo University of Science

2641 Yamazaki Noda-shi Chiba-ken,
278-8510 JAPAN

Abstract

Classification of superposition representation problems, which is based on Hilbert's 13th problem, is given. This result is applied to the problem asking how efficiently we can compress a finite dimensional numerical datum if the datum can be approximately represented by a superposition of several fewer dimensional numerical data.

1 Superposition representability and superposition irrepresentability

Let $f(\cdot,\cdot,\cdot)$ be the function of three variable defined as

$$f(x,y,z) = xy+yz+zx, \quad x,y,z\in\mathbb{C}.$$

Then, we can easily prove that there do not exist functions of two variables $g(\cdot,\cdot)$, $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ satisfying the following equality:

$$f(x,y,z)=g(u(x,y),v(x,z)), \quad x,y,z\in\mathbb{C}.$$

This result shows us that f cannot be represented any 1-time nested superposition constructed from three complex-valued functions of two variables. But it is clear that the following equality holds:

$$f(x,y,z)=x(y+z)+(yz), \quad x,y,z\in\mathbb{C}.$$

This result shows us that f can be represented as a 2-time nested superposition.

In 1957, Kolmogorov and Arnold solved Hilbert's 13th problem asking whether all continuous real-valued functions of several real variables can be represented as superpositions of functions of fewer variables or not. If their original result is applied to the set of all real-valued functions of three variables, for any continuous real-valued function f of three real variables, we can choose a family of seven continuous real-valued functions of one variable $\{g_i^f; 0 \leq i \leq 6\}$, which is dependent on f, and a family of twenty one continuous real-valued functions of one real variable $\{\phi_{ij}; 0 \leq i \leq 6, 1 \leq j \leq 3\}$, which is independent of f, satisfying

$$f(x_1, x_2, x_3) = \sum_{i=0}^{6} g_i \left(\sum_{j=1}^{3} \phi_{ij}(x_j) \right).$$

This result, which is called Kolmogorov-Arnold representation theorem [**1**], immediately implies that any continuous real-valued function of three real variables can be represented as a 10-time nested superposition of continuous real-valued functions of two real variables, because the following equality holds:

$$\begin{aligned} f(z_1, z_2, z_3) = & \Big(\Big(\Big(\Big(\Big(\Big(g_0((\phi_{01}(z_1) + \phi_{02}(z_2)) + \phi_{03}(z_3)) \\ & + g_1((\phi_{11}(z_1) + \phi_{12}(z_2)) + \phi_{13}(z_3))\Big) \\ & + g_2((\phi_{21}(z_1) + \phi_{22}(z_2)) + \phi_{23}(z_3))\Big) \\ & + g_3((\phi_{31}(z_1) + \phi_{32}(z_2)) + \phi_{33}(z_3))\Big) \\ & + g_4((\phi_{41}(z_1) + \phi_{42}(z_2)) + \phi_{43}(z_3))\Big) \\ & + g_5((\phi_{51}(z_1) + \phi_{52}(z_2)) + \phi_{53}(z_3))\Big) \\ & + g_6((\phi_{61}(z_1) + \phi_{62}(z_2)) + \phi_{63}(z_3))\Big). \end{aligned}$$

Let $\mathcal{A}_3$ (resp. $\mathcal{A}_2$) be a set of functions of three variables (resp. two variables) such as the set of all continuous functions of three variables (resp. two variables) or the set of all analytic functions of three variables (resp. two variables). Then, the superposition representation propositions in this case can be classified into the following two propositions:

Proposition I. For a certain positive integer k and for any function of $\mathcal{A}_3$, which is denoted by f, there exists a k-time nested superposition constructed from several functions of $\mathcal{A}_2$ by which f can be represented.

Proposition II. For any function of $\mathcal{A}_3$, which is denoted by f, there exists a positive integer k_f such that f can be represented by a k_f-time nested superposition constructed from several functions of $\mathcal{A}_2$.

Here, Proposition I is called a strong superposition representation proposition and Proposition II is called a weak superposition representation proposition. It is clear that Proposition II holds necessarily if Proposition I holds. By the same way as above, the superposition irrepresentation propositions can be classified into the following two propositions:

Property III. There exists a function of $\mathcal{A}_3$ which cannot be represented as any finite-time nested superposition constructed from several functions of $\mathcal{A}_2$.

Property IV. For any positive integer k, there exists a function of $\mathcal{A}_3$ which cannot be represented as any k-time nested superposition constructed from several functions of $\mathcal{A}_3$.

Here, Proposition III is called a strong superposition irrepresentation proposition and Proposition IV is called a weak superposition irrepresentation proposition. It is clear that Proposition IV holds necessarily if Proposition III holds. Since Proposition IV is the contraposition of Proposition I and Proposition III is the contraposition of Proposition II, we can prove only one of the following three cases, namely, the case that Proposition I holds, the case that both Proposition II and Proposition III hold, the case that Proposition IV hold can be proved affirmatively. In other words, if one of these three cases can be proved affirmatively, then the other two cases can be proved negatively.

For example, if we take the set of all continuous functions of three variables (resp. two variables) as an example of $\mathcal{A}_3$ (resp. $\mathcal{A}_2$), then Kolmogorov-Arnold theorem assures that only Proposition I can be proved. If we take the set of all polynomials of three variables (resp. two variables) as an example of $\mathcal{A}_3$ ($\mathcal{A}_2$), then both Proposition II and Proposition IV can be proved. If we take the set of all differentiable functions of three variables (resp. two variables) as an example of $\mathcal{A}_3$ (resp. $\mathcal{A}_2$), then Vituskin theorem [**2**] assures that only Proposition IV can be proved.

2 A Compression method of multidimensional numerical data

The 13th problem is closely related to the theory of data compresion of functions of several variables. Concretely speaking, in this theory, the approximate data attached to a differentiable function $f(\cdot,\cdot,\cdot)$ is defined as

$$\left\{\left(\frac{k}{n},\frac{l}{n},\frac{m}{n},f\left(\frac{k}{n},\frac{l}{n},\frac{m}{n}\right)\right);0\leq k,l,m\leq n\right\}.$$

By the same way as above, for any $0\leq i\leq 6$ and $1\leq j\leq 3$, the approximate data attached to a differentable function $g_i^f(\cdot)$ and the approximate data attached to a differentable function $\phi_{ij}(\cdot)$ are defined as

$$\left\{\left(\frac{k}{n},g_i^f\left(\frac{k}{n}\right)\right);0\leq k\leq n\right\},$$

$$\left\{\left(\frac{k}{n},\phi_{ij}\left(\frac{k}{n}\right)\right);0\leq k\leq n\right\},$$

respectively. Let n be a positive integer and let x be a real number which belongs to the closed interval $[0,1]$. Then, $[x]_n$ is defined as $\frac{[nx]}{n}$. Here, if it is assumed that we substitute

$\sum_{i=0}^{6} g_i^f \left(\left[\frac{1}{3} \sum_{j=1}^{3} \phi_{ij}([x_j]_n) \right]_n \right)$ for $f([x_1]_n, [x_2]_n, [x_3]_n)$, then it follows that the cardinal number of the approximate data attached to f can be reduced from $(n+1)^3$ elements to $21(n+1)$ elements. Therefore, this result shows that the compression of the approximate data attached to f can be realized if we can choose g_i^f and ϕ_{ij} as being differentiable, because it can be easily proved that the following equality holds:

$$\lim_{n\to\infty} \left| \sum_{i=0}^{6} g_i^f \left(\left[\frac{1}{3} \sum_{j=1}^{3} \phi_{ij}([x_j]_n) \right]_n \right) - f([x_1]_n, [x_2]_n, [x_3]_n) \right| = 0.$$

Actually, It follows from Vituskin theorem that it is impossible to realize such data compression as stated above. Here, we can prove the following:

Proposition. Not all differentiable functions of three variables can be represented as Kolmogorov-Arnold superpositions constructed from only differentiable functions of one variable.

Proof. The fact that Vituskin theorem is a generalization of the above proposition concludes the proof of this proposition. □

References

1. A. N. Kolmogorov, On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition, Dokl., 114(1957), 679-681.

2. A. G. Vitushkin, Some properties of linear superpositions of smooth functions, Dokl., 156(1964), 1003-1006.