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Movement of Hot Spots on the Exterior Domain of a Ball

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1 Introduction

We consider the initial-boundary value problems of the heat equation in the exterior domain of a ball,

$$(1.1) \qquad \begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \Omega \end{cases}$$

and

$$(1.2) \qquad \begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \Omega, \end{cases}$$

where

$$\Omega = \mathbf{R}^N \setminus \overline{B(0,L)}, \quad L > 0, \qquad N \geq 2.$$

Here $\partial_t = \partial/\partial t$, $\partial_\nu = \partial/\partial\nu$, $\nu = \nu(x)$ is the outer unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$, and $B(0,L) = \{x \in \mathbf{R}^N : |x| < L\}$. Throughout this paper we assume that

$$\phi \in L^2(\Omega, e^{\lambda|x|^2}dx)$$

for some $\lambda > 0$. For any $t > 0$, we may denote by $H(t)$ the set of the maximum points of $u(\cdot,t)$, that is,

$$H(t) = \left\{ x \in \overline{\Omega} : u(x,t) = \max_{y\in\overline{\Omega}} u(y,t) \right\},$$

and call $H(t)$ the set of hot spots of the solution u at the time t. In this paper we study the movement of hot spots $H(t)$ of the solution u of (1.1) or (1.2) as $t \to \infty$.

Chavel and Karp [3] studied the heat equation $\partial_t u = \Delta u$ in several Riemannian manifolds, and obtained some asymptotic properties of solutions concerning the movement of hot spots of the solution. In particular, for the Euclidean space $\mathbf{R}^N$, they proved that, for any nonzero, nonnegative initial data $\phi \in L_c^\infty(\mathbf{R}^N)$, the hot spots $H(t)$ of the solution at each time $t > 0$ are contained in the closed convex hull of the support of ϕ, and $H(t)$ tends to the center of mass of φ as $t \to \infty$. Subsequently, Jimbo and Sakaguchi [11] studied the movement of hot spots of the solution of the heat equation in the half space $\mathbf{R}_+^N$ and in the exterior domain of a ball, under boundary conditions. In particular, for the Cauchy-Neumann problem (1.1) in the exterior domain $\Omega = \mathbf{R}^N \setminus \overline{B(0,L)}$ with the nonzero, nonnegative, radially symmetric initial data $\phi \in L_c^\infty(\Omega)$, they proved that the hot spots $H(t)$ satisfies

$$H(t) \subset \partial\Omega = \partial B(0,L) \tag{1.3}$$

for all sufficiently large t. Furthermore, for the Cauchy-Dirichlet problem in the exterior domain $\Omega = \mathbf{R}^3 \setminus \overline{B(0,L)}$ with the nonzero, nonnegative, radially symmetric initial data $\phi \in L_c^\infty(\Omega)$, they proved that there exist a positive constant T and a continuous function $r = r(t) \in C([T,\infty) : (L,\infty))$ such that

$$\lim_{t\to\infty} r(t)^3 t^{-1} = 2 \tag{1.4}$$

and

$$H(t) = \{x \in \mathbf{R}^N : |x| = r(t)\}, \quad t \geq T.$$

Their proofs of (1.3) and (1.4) heavily depend on the radially symmetry of the solutions and the properties of zero sets of the heat equation in $\mathbf{R}$, and it seems so difficult to apply their proofs to the solutions without the radially symmetry. (For the movement of hot spots of the solution for the Cauchy-Neumann problem in bounded domains, see [1], [2], [10], [12], and [14].)

In this paper we study the movement of hot spots $H(t)$ of the solutions of the Cauchy-Neumann problem (1.1) or the Cauchy-Dirichlet problem (1.2) in the exterior domain Ω of a ball, without the radially symmetry of the solutions. In Sections 2 and 3, we give the results on the movement of the set of hot spots $H(t)$ for the problems (1.1) and (1.2), respectively.

2 On the Cauchy-Neumann Problem (1.1)

In this section we assume

$$\phi \in L^2(\Omega, e^{\lambda|x|^2}dx), \qquad \int_\Omega \phi(x)dx > 0, \tag{2.1}$$

and give some results on the movement of the hot spots $H(t)$ for the solution of (1.1) as $t \to \infty$. We first give a sufficient condition for the hot spots $H(t)$ to exist only on the boundary $\partial\Omega$ for all sufficiently large t.

Theorem 2.1 (See Theorem 1.1 in [8].)
Let u be a solution of the Cauchy-Neumann problem (1.1) *under the condition* (2.1). *Put*

$$A_\phi^N = \int_\Omega x\phi(x)\left(1 + \frac{L^N}{N-1}|x|^{-N}\right)dx \Big/ \int_\Omega \phi(x)dx.$$

Assume

$$A_\phi^N \in B(0,L) = \mathbf{R}^N \setminus \overline{\Omega}. \tag{2.2}$$

Then there exists a positive constant T such that

$$H(t) \subset \partial\Omega = L\left\{x \in \mathbf{R}^N : |x| = L\right\} \tag{2.3}$$

for all $t \geq T$.

In particular, we see that, under the condition (2.1), the hot spots $H(t)$ of the radial solution of (1.1) exists only on the boundary of the domain Ω for all sufficiently large t.

Remark 2.1 *Let u be a solution of the Cauchy-Neumann problem* (1.1) *under the condition* (2.1). *Let $C(t)$ a center of mass of $u(t)$, that is,*

$$C(t) = \int_\Omega xu(x,t)dx \Big/ \int_\Omega u(x,t)dx.$$

Then it does not necessarily hold that $C(t) = C(0)$ for all $t > 0$. On the other hand, we put

$$A(t)^N(t) \equiv \int_\Omega xu(x,t)\left(1 + \frac{L^N}{N-1}|x|^{-N}\right)dx \Big/ \int_\Omega u(x,t)dx, \quad t > 0.$$

Then we have $A_\phi^N(t) = A_\phi^N$ for all $t > 0$, and $\lim_{t\to\infty} C(t) = A_\phi^N$.

Next we give a result on the limit of the set $H(t)$ as $t \to \infty$.

Theorem 2.2 (See Theorem 1.2 in [8].)
Let u be a solution of the Cauchy-Neumann problem (1.1) *under the condition* (2.1). *Assume* $A_\phi^N \neq 0$. *Put*

$$x_\infty = L\frac{A_\phi^N}{|A_\phi^N|} \quad \textit{if} \quad A_\phi^N \in B(0,L) \quad \textit{and} \quad x_\infty = A_\phi^N \quad \textit{if} \quad A_\phi^N \in \overline{\Omega}$$

Then

$$\lim_{t\to\infty} \sup \{|x_\infty - y| : y \in H(t)\} = 0.$$

By Theorem 2.2, we see that the hot spots $H(t)$ tends to one point x_∞ as $t \to \infty$ if $A_\phi \neq 0$, and see that (1.3) does not hold if $A_\phi \in \Omega$.

Next we will explain the outline of the proofs of Theorems 2.1 and 2.2. As in stated in [11], it is difficult to know the sign of differential of the Neumann heat kernel even for the case that Ω is the exterior of a ball, and so it seems difficult to obtain Theorems 2.1 and 2.2 by using the fundamental properties of the Neumann heat kernel. We consider the following two eigenvalue problems,

$$(E) \qquad \begin{cases} P_0\varphi \equiv \dfrac{1}{\rho}\mathrm{div}\,(\rho\nabla\varphi) = -\lambda\varphi & \text{in} \quad \mathbf{R}^N, \\ \varphi \in H^1(\mathbf{R}^N, \rho dy), \quad \rho(y) = \exp\left(\dfrac{|y|^2}{4}\right), \end{cases}$$

and

$$(2.4) \qquad -\Delta_{\mathbf{S}^{N-1}} Q = \omega Q \qquad \text{on} \quad \mathbf{S}^{N-1},$$

such that $0 = \omega_0 < \omega_1 = N-1 < \omega_2 = 2N < \omega_3 < \cdots$, where $\Delta_{\mathbf{S}^{N-1}}$ is the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$. Let l_k be the dimension of the eigenspace of the eigenvalue problem (2.4) corresponding to $\omega = \omega_k$ and $\{Q_{k,i}\}_{i=1}^{l_k}$ the eigenfunctions of (2.4) corresponding to $\omega = \omega_k$ such that $(Q_{k,i}, Q_{k,j})_{L^2(\mathbf{S}^{N-1})} = \delta_{ij}$, $i,j = 1,\ldots,l_k$. In particular we may take

$$(2.5) \qquad Q_{1,i}\left(\frac{x}{|x|}\right) = c_q\frac{x_i}{|x|}, \qquad i = 1,\ldots,N,$$

for some positive constant $c_q = c_q(N) > 0$. Furthermore we have the following lemma on the eigenfunctions of (E) (see [5] and [13]).

Lemma 2.1 *Let $k = 0, 1, 2, \ldots$. Let $\{\lambda_{k,i}\}_{i=0}^{\infty}$ be the eigenvalues of*

$$(E_k) \qquad \begin{cases} P_k\varphi \equiv P_0\varphi - \dfrac{\omega_k}{|y|^2}\varphi = -\lambda\varphi & \text{in } \mathbf{R}^N, \\ \varphi \text{ is a radial function in } \mathbf{R}^N, \\ \varphi \in L^2(\mathbf{R}^N, \rho dy), \end{cases}$$

such that $\lambda_{k,0} < \lambda_{k,1} < \lambda_{k,2} < \ldots$ and $\varphi_{k,i}$ the eigenfunction corresponding to $\lambda_{k,i}$ such that $\|\varphi_{k,i}\|_{L^2(\Omega,\rho dx)} = 1$. Then

$$\lambda_{k,i} = \frac{N+k}{2} + i, \qquad \varphi_{k,0}(y) = c_k|y|^k \exp\left(-\frac{|y|^2}{4}\right)$$

for some constants c_k. Furthermore $\{\lambda_{k,i}\}_{k,i=0}^{\infty}$ give all eigenvalue of (E), and the eigenspace of (E) corresponding to λ are spanned by the eigenfunctions $\{\varphi_{k,i}(y)Q_{k,j}(y/|y|)\}_{j=1}^{l_k}$ with $\lambda = \lambda_{k,i}$.

In order to prove Theorems 2.1 and 2.2, we may assume, without loss of generarilty, that $\phi \in L^2(\Omega, \rho dx)$. Then, by Lemma 2.1, there exist radial functions $\{\phi_{k,j}\}_{k\in\mathbf{N}\cup\{0\},j=1,\ldots,l_k}$, such that $\phi_{k,i} \in L^2(\Omega, \rho dx)$ and

$$(2.6) \qquad \phi = \sum_{k=0}^{\infty}\sum_{j=1}^{l_k} \phi_{k,j}(|x|)Q_{k,j}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\Omega, \rho dx),$$

Furthermore let $v_{k,j}$ be the radial solution of the Cauhy-Neumann problem

$$(L_k^N) \qquad \begin{cases} \partial_t v = \mathcal{L}_k v \equiv \Delta v - \dfrac{\omega_k}{|x|^2}v_k & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ v(x,0) = \phi_{k,j}(x) & \text{in} \quad \Omega. \end{cases}$$

Then the function

$$v_{k,j}(x,t)Q_{k,j}\left(\frac{x}{|x|}\right)$$

is a solution of (1.1) with the initial data $\phi_{k,j}(x)Q_{k,j}(x/|x|)$. Furthermore we see that

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{j=1}^{l_k} u_{k,j}(x,t) \quad \text{in} \quad C^2(\overline{\Omega}),$$

for all $t > 0$. Therefore we have only to study the asymptotic behavior of the radial solution of the Cauchy-Neumann problem (L_k^N) in order to study the one of the solution u of (1.1).

Let v_k be the solution of the Cauchy-Neumann problem (L_k^N) with the initial data φ, where φ is a radial function belonging to $L^2(\Omega, \rho dx)$. In order to study the asymptotic behavior of the solution v_k, we define a rescaled function w_k of the solution v_k as follows:

$$w_k(y,s) = (1+t)^{\frac{N+k}{2}} v_k(x,t), \quad y = (1+t)^{-\frac{1}{2}}x, \quad s = \log(1+t).$$

Then the function w_k satisfies

$$(P_k^N) \qquad \begin{cases} \partial_s w_k = P_k w_k + \dfrac{N+k}{2} w_k & \text{in } W, \\ \partial_\nu w_k = 0 & \text{on } \partial W, \\ w_k(y,0) = \varphi(y) & \text{in } \Omega, \end{cases}$$

where

$$\Omega(s) = e^{-s/2}\Omega, \; W = \bigcup_{0<s<\infty} (\Omega(s) \times \{s\}), \; \partial W = \bigcup_{0<s<\infty} (\partial\Omega(s) \times \{s\}).$$

We study the asymptotic behavior of the first eigenvalue and the first eigenfunction of the operator P_k, and obtain the asymptotic behavior of the solution w_k in the space L^2 with weight ρ. Furthermore, for $k = 0, 1, 2$, by using the radially symmetry of v_k, the equations (L_k^N) and (P_k^N), and the Ascoli-Arzera theorem, we study the asymptotic behavior of v_k, $\partial_r v_k$, and $\partial_r^2 v_k$ as $t \to \infty$.

For the case $k = 0$, we extend the domain of w_0 to $\mathbf{R}^N$, and apply the Ascoli-Arzera theorem to w_0. Then, by using the results on the asymptotic behavior of w_0 in the space L^2 with weight ρ, we obtain a result on the asymptotic behavior of v_0 and $\partial_r v_0$, where $r = |x|$. Furthermore we obtain a result on the asymptotic behavior of $\partial_r^2 v_0$ as $t \to \infty$ by using the ones of v_0 and $\partial_r v_0$.

Proposition 2.1 *Let φ be a radial function in Ω satisfying* (2.1). *Let v_0 be a radial solution of (L_0^N) with the initial data φ. Then*

$$\lim_{t\to\infty} t^{\frac{N}{2}} v_0(x,t) = (4\pi)^{-\frac{N}{2}} \int_\Omega \varphi(x)dx$$

uniformly on any compact set in $\overline{\Omega}$. *Furthermore, for any positive constants* ϵ, *there exist positive constants* C, R, *and* T *such that*

$$\partial_r v_0(x,t) \le -Ct^{-\frac{N+1}{2}} \int_\Omega \varphi(x)dx$$

for all $x \in \Omega$ *with* $\epsilon(1+t)^{1/2} \le |x| \le R(1+t)^{1/2}$ *and all* $t \ge T$.

Proposition 2.2 *Let* φ *be a radial function in* Ω *satisfying* (2.1). *Let* v_0 *be a radial solution of* (L_0^N) *with the initial data* φ. *Then there exist positive constant* R *and* T *such that*

$$\partial_r v_0(x,t) \le -\frac{1}{4}(4\pi)^{-\frac{N}{2}} t^{-\frac{N+2}{2}}(|x|-L)\int_\Omega \varphi(x)dx$$

for all $x \in \Omega$ *with* $|x| \le L + R(1+t)^{1/2}$ *and* $t \ge T$, *where* $r = |x|$. *Furthermore, for any* $R > L$,

$$\begin{aligned}
&\partial_r v_0(x,t)\\
&= -\frac{1}{2}(4\pi)^{-\frac{N}{2}}(1+o(1))|x|(1-L^N|x|^{-N})t^{-\frac{N+2}{2}}\int_\Omega \varphi(x)dx,\\
&\partial_r^2 v_0(x,t)\\
&= -\frac{1}{2}(4\pi)^{-\frac{N}{2}}(1+o(1))(1+(N-1)L^N r^{-N})t^{-\frac{N+2}{2}}\int_\Omega \varphi(x)dx
\end{aligned}$$

as $t \to \infty$, *uniformly on* $\Omega \cap B(0,R)$.

On the other hand, for the case $k = 1$, the inequality

$$\sup_{s>1} \|\nabla_y^2 w_1(\cdot,s)\|_{C(\Omega(s))} < \infty$$

does not necessarily holds, and $w(y,s)$ tends to 0 uniformly for all y with $|y| \le Re^{-s/2}$ with any $R > L$. So it is not useful to apply the Ascoli-Arzera theorem to w_1 for the aim at studying the asymptotic behavior of w_1 and $\partial_r w_1$ in the domain $\{y \in \Omega(s) : |y| \le Re^{-s/2}\}$, as $s \to \infty$. To overcome this difficulty, we may apply the Ascoli-Arzera theorem w_1 in the any annulus $D(\epsilon, R) = \{y \in \mathbf{R}^N : \epsilon \le |y| \le R\}$ with $0 < \epsilon < R$, and obtain the asymptotic behavior of w_1 in the annulus $D(\epsilon, R)$. Furthermore, by using the equation (L_1) effectively, we study the asymptotic behavior of v_1, $\partial_r v_1$ and $\partial_r^2 v_1$ as $t \to \infty$, and obtain the following proposition.

Proposition 2.3 *Let φ be a radial function in Ω satisfying* (2.1). *Let v_1 be a radial solution of (L_1^N) with the initial data φ. Put*

$$U_L^N(r) = c_1 r\left(1 + \frac{L^N}{N-1} r^{-N}\right), \quad a_\varphi^N = \int_\Omega \varphi(x) U_L^N(|x|)dx.$$

Then there exists a positive constant C such that

$$\|\nabla v_1(x,t)\|_{L^\infty(\Omega)} \le C_1(|a_\varphi^N| + o(1))t^{-\frac{N+2}{2}}$$

for sufficiently large t. Furthermore, for any $R > L$,

$$\begin{aligned} v_1(x,t) &= (a_\varphi^N + o(1))U_L^N t^{-\frac{N+2}{2}}, \\ \partial_r v_1(x,t) &= c_1(a_\varphi^N + o(1))\left(1 - L^N r^{-N}\right) t^{-\frac{N+2}{2}}, \\ \partial_r^2 v_1(x,t) &= c_1(a_\varphi^N + o(1))NL^N r^{-(N+1)} t^{-\frac{N+2}{2}}, \end{aligned}$$

as $t \to \infty$, uniformly on $\Omega \cap B(0,R)$.

Similarly we study the asymptotic behavior of v_2, $\partial_r v_2$ and $\partial_r^2 v_2$ as $t \to \infty$, and obtain the following proposition.

Proposition 2.4 *Let φ be a radial function in Ω satisfying* (2.1). *Let v_2 be a radial solution of (L_2^N) with the initial data φ. Then there exists a positive constant C_1 such that*

$$\begin{aligned} \|v_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_1 t^{-\frac{N+2}{2}}, \\ \|\partial_r v_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_1 t^{-\frac{N+3}{2}}, \end{aligned}$$

for sufficiently large t. Furthermore, for any $R > L$, there exists a constant C_2 such that

$$|\partial_r^2 v_2(x,t)| \le C_2 t^{-\frac{N+3}{2}}$$

for all $x \in \Omega$ with $|x| \le R$ and all sufficiently large t.

By Propositions 2.1–2.4, we may obtain the asymptotic behavior of the solutions $u_{k,j}$, $k = 0,1,2$, $j = 1,\dots,l_k$. Finally, by (2.6), we put

$$\phi_3 = \phi - \sum_{k=0}^{2}\sum_{j=1}^{l_k} \phi_{k,j}(|x|)Q_{k,j}\left(\frac{x}{|x|}\right), \tag{2.7}$$

and study the solution of (1.1) with the initial data ϕ_3. Then we have

Proposition 2.5 *Assume* (2.1). *Let* ϕ_3 *be a function defined by* (2.6) *and* (2.7). *Let* u_3 *be a function of* (1.1) *with the initial data* ϕ_3. *Then there exists a constant* C *such that*

$$\|\nabla_x^k u_3(\cdot,t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{N+3}{2}}, \quad k = 0,1,2,$$

for all sufficiently large t.

By Propositions 2.1–2.5, we obtain the asymptotic behavior of u, $\nabla_x u$, and $\nabla_x^2 u$ as $t \to \infty$, and may obtain Theorems 2.1 and 2.2.

3 On the Cauchy-Neumann Problem (1.2)

In this section we assume that

$$\phi \in L^2(\Omega, \rho dx), \quad m_\phi > 0, \tag{3.1}$$

where $\rho(x) = \exp(|x|^2/4)$ and

$$m_\phi = \begin{cases} \displaystyle\int_\Omega \phi(x)\left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right)dx & \text{if } N \ge 3, \\ \displaystyle\int_\Omega \phi(x)\log\frac{|x|}{L}dx & \text{if } N \ge 2. \end{cases}$$

We first give the following results on the asymptotic behavior of the solution u of (1.2), which imply that the hot spots $H(t)$ run away from the boundary $\partial\Omega$ as $t \to \infty$.

Theorem 3.1 (See Theorem 1.1 in [9].)
Let u *be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1) *and* $N \ge 3$. *Then*

$$\lim_{t\to\infty}\int_\Omega u(x,t)dx = m_\phi > 0 \tag{3.2}$$

and

$$\lim_{t\to\infty} t^{\frac{N}{2}}u(x,t) = (4\pi)^{-\frac{N}{2}}m_\phi\left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right) \tag{3.3}$$

uniformly for all x *on any compact set in* $\overline{\Omega}$.

Theorem 3.2 (See Theorem 1.2 in [9].)
Let u be a solution of the Cauchy-Dirichlet problem (1.2) *under the condition* (3.1) *and $N = 2$. Then there exists a constant C such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le C(\log t)^{-1}\|\phi\|_{L^2(\Omega,\rho dx)} \tag{3.4}$$

for all $t \ge 1$. Furthermore

$$\lim_{t\to\infty}(\log t)\int_\Omega u(x,t)dx = 2m_\phi \tag{3.5}$$

and

$$\lim_{t\to\infty} t(\log t)^2 u(x,t) = \frac{1}{\pi}m_\phi \log\frac{|x|}{L} \tag{3.6}$$

uniformly for all x on any compact set in $\overline{\Omega}$.

Remark 3.1 *Collet, Martínes, and Martín* [4] *used the probability method to prove the asymptotic behavior of the Dirichlet heat kernel $G = G(x,y,t)$ on the exterior domain of a compact set as $t \to \infty$. In particular, for the exterior domain $\mathbf{R}^N \setminus \overline{B(0,L)}$, they obtained that*

$$\lim_{t\to\infty} t^{\frac{N}{2}} G(x,y,t) = (4\pi)^{-\frac{N}{2}}\left(1-\frac{L^{N-2}}{|x|^{N-2}}\right)\left(1-\frac{L^{N-2}}{|y|^{N-2}}\right) \quad \text{if } N \ge 3, \tag{3.7}$$

$$\lim_{t\to\infty} t(\log t)^2 G(x,y,t) = \frac{1}{\pi}\log\frac{|x|}{L}\log\frac{|y|}{L} \qquad \text{if } N = 2, \tag{3.8}$$

for all $x, y \in \Omega$ (see also [6]*). By* (3.3) *and* (3.6)*, we may obtain* (3.7) *and* (3.8)*, and the proof of this paper is complete different from the one of* [4]*. Furthermore we remark that Herraiz* [7] *applied the comparison method to the Cauchy-Dirichlet problem* (1.2) *in the exterior domain of a compact set, and obtained the similar results to Theorems* 3.1 *and* 3.2 *for nonnegative initial data ϕ.*

Next we give a result on the rate for the hot spots $H(t)$ to run away from the boundary Ω as $t \to \infty$.

Theorem 3.3 (See Theorem 1.3 in [9].)
Let u be a solution of the Cauchy-Dirichlet problem (1.2) *under the condition* (3.1)*. Put*

$$\zeta(t) = 2(N-2)L^{N-2}t \quad \text{if } N \ge 3, \qquad \zeta(t) = 2t(\log t)^{-1} \quad \text{if } N = 2.$$

Then

(3.9)
$$\lim_{t\to\infty} \sup_{x\in H(t)} \left| \zeta(t)^{-1}|x|^N - 1 \right| = 0.$$

Furthermore there exists a positive constant T such that, if $x \in H(t)$ and $t \geq T$, then

(3.10)
$$H(t) \cap l_x = \{x\},$$

where $l_x = \{kx/|x| \ : \ k \geq 0\}$.

Next we give a sufficient condition for the hot spots $H(t)$ to consist of one point $x(t)$ after a finite time. Furthermore we give the limit of $x(t)/|x(t)|$ as $t \to \infty$.

Theorem 3.4 (See Theorem 1.4 in [9].)
Let u be a solution of the Cauchy-Dirichlet problem (1.2) *under the condition* (3.1). *Assume that*

$$A_\phi^D \equiv \int_\Omega x\phi(x) \left(1 - \frac{L^N}{|x|^N}\right) dx \neq 0.$$

Then there exist a positive constant T and a smooth curve $x = x(t) \in C^\infty([T,\infty) : \Omega)$ such that $H(t) = \{x(t)\}$ for all $t \geq T$ and

(3.11)
$$\lim_{t\to\infty} \frac{x(t)}{|x(t)|} = \frac{A_\phi^D}{|A_\phi^D|}.$$

Therefore, by Theorems 3.3 and 3.4, we see that, under the assumptions (3.1) and $A_\phi^D \neq 0$, the set of hot spots $H(t)$ consists of one points $x(t)$ after a finite time, and

$$\lim_{t\to\infty} \zeta(t)^{-1/N}|x(t)| = 1, \qquad \lim_{t\to\infty} x(t)/|x(t)| = A_\phi^D/|A_\phi^D|.$$

Next we explain the outline of the proofs of Theorems 3.1–3.3. In the similar way to the Cauchy-Neumann problem (1.1), we have only to study the asymptotic behavior of the radial solutions v_k of the Cauchy-Dirichlet problem

$$(L_k^D) \qquad \begin{cases} \partial_t v = \mathcal{L}_k v \equiv \Delta v - \dfrac{\omega_k}{|x|^2} v_k & \text{in} \quad \Omega \times (0,\infty), \\ v = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ v(x,0) = \varphi(x) & \text{in} \quad \Omega, \end{cases}$$

where φ is a radial function belonging to $L^2(\Omega, \rho dx)$ and $k = 0, 1, 2 \ldots$. Furthermore, by the same argument with in the Cauchy-Neumann problem (1.1), we introduce a rescaled function w_k of v_k, and study the asymptotic behavior of the rescaled functions w_k as $s \to \infty$. For the case $N \geq 3$, we study the asymptotic behavior of $w_0 = w_0(y, s)$ in the space L^2 with weight ρ, and obtain the one of $v_0 = v_0(x, t)$ for all $x \in \Omega$ with $|x| \sim t^{1/2}$ as $t \to \infty$. Furthermore, by using the radially symmetry of v_0 and (L_0), we obtain the asymptotic behavior of v_0, $\partial_r v_0$, $\partial_r^2 v_0$, and $\partial_t v_0$ for all $x \in \Omega$ with $|x| = O(t^{1/2})$ as $t \to \infty$,

Proposition 3.1 *Let φ be a radial function in Ω satisfying* (2.1). *Let v_0 be a radial solution of (L_0^D) with the initial data φ and $N \geq 3$. Put*

$$U_L^{D,0}(r) = c_0\left(1 - \frac{L^{N-2}}{r^{N-2}}\right), \quad a_\varphi^{D,0} = \int_\Omega \varphi(x) U_L^{D,0}(|x|)dx.$$

Then there hold that

$$\begin{aligned}
v_0^*(r,t) &= t^{-\frac{N}{2}}(a_\varphi^{D,0} + o(1))U_L^0(r) + \frac{N}{2}t^{-\frac{N+2}{2}}(a_0 + o(1))O(r^2) \\
&\quad + O(t^{-\frac{N+4}{2}})O(r^4), \\
(\partial_r v_0^*)(r,t) &= t^{-\frac{N}{2}}(a_\varphi^{D,0} + o(1))\partial_r U_L^0(r) \\
&\quad - \frac{Nc_0}{4} r t^{-\frac{N+2}{2}}(a_\varphi^{D,0} + o(1))(1 + O(r^{-1})) + O(t^{-\frac{N+4}{2}})O(r^3), \\
(\partial_r^2 v_0^*)(r,t) &= t^{-\frac{N}{2}}(a_\varphi^{D,0} + o(1))\partial_r^2 U_L^0(r) - U_L^0(r)\frac{N}{2}t^{-\frac{N+2}{2}}(a_\varphi^{D,0} + o(1)) \\
&\quad + O(t^{-\frac{N+4}{2}})O(r^2), \\
(\partial_t v_0^*)(r,t) &= -\frac{N}{2}t^{-\frac{N+2}{2}}(a_\varphi^{D,0} + o(1))U_L^0(r) + O(t^{-\frac{N+4}{2}})O(r^2)
\end{aligned}$$

for all $r \geq L$ and $t \geq 1$.

For the case $N = 2$, the behavior of v_0 is different from the one for the case $N \geq 3$. By the similar way to in the case $N \geq 3$, we first obtain $\max_{x \in \partial\Omega} |\partial_r v_0(x,t)| = O(t^{-1}(\log t)^{-1})$ as $t \to \infty$. This gives that $\|v_0(\cdot, t)\|_{L^1(\Omega)} = O((\log t)^{-1})$ as $t \to \infty$. By using the similar argument to in the case $N \geq 3$ again, we have $\max_{x \in \partial\Omega} |\partial_r v_0(x,t)| = O(t^{-1}(\log t)^{-2})$ as $t \to \infty$, and obtain the following proposition.

Proposition 3.2 *Let φ be a radial function in Ω satisfying* (2.1). *Let v_0 be a radial solution of (L_0^D) with the initial data φ and $N = 2$. Put*

$$\tilde{a}_\varphi^{D,0} = 4c_0^2 \int_\Omega \varphi(x) \log \frac{|x|}{L} dx.$$

Then there exists a function $\zeta_1 = \zeta_1(t)$ and $\zeta_2(t)$ with

$$\lim_{t\to\infty} t(\log t)^2 \zeta_1(t) = \tilde{a}_\varphi^{D,0}, \qquad \lim_{t\to\infty} t^2 (\log t)^2 \zeta_2(t) = \tilde{a}_\varphi^{D,0},$$

such that

$$\begin{aligned} v_0(r,t) &= \zeta_1(t) \log \frac{r}{L} + O(r^2 \log r)\zeta_1(t) + O(r^4) O(t^{-3}(\log t)^{-1}), \\ (\partial_r v_0)(r,t) &= \frac{\zeta_1(t)}{r} - \zeta_1(t) r \log r (1 + o(1)) + O(r^3) O(t^{-3}(\log t)^{-1}), \\ (\partial_r^2 v_0)(r,t) &= -\frac{\zeta_1(t)}{r^2} - U_L^0(r)\zeta_1(t) + O(r^2) O(t^{-3}(\log t)^{-1}), \\ (\partial_t v_0)(r,t) &= -\left(\log \frac{r}{L}\right) \zeta_2(t) + O(r^2) O(t^{-3}(\log t)^{-1}) \end{aligned}$$

for all $r \geq L$ and $t \geq 2$.

Furthermore, by the similar argument to the problem (1.1), we obtain the asymptotic behavior of the solutions v_1 and v_2.

Proposition 3.3 *Let φ be a radial function in Ω satisfying* (2.1). *Let v_1 be a radial solution of (L_1^D) with the initial data φ and $N \geq 2$. Put*

$$U_L^{D,1}(r) = c_1 r \left(1 - \frac{L^N}{r^N}\right), \quad a_\varphi^{D,1} = \int_\Omega \varphi(x) U_L^{D,1}(|x|) dx.$$

Then there hold that

$$\begin{aligned} v_1^*(r,t) &= t^{-\frac{N+2}{2}} (a_\varphi^{D,1} + o(1)) U_L^1(r) + O(r^2) O(t^{-\frac{N+3}{2}}), \\ \partial_r v_1^*(r,t) &= t^{-\frac{N+2}{2}} (a_\varphi^{D,1} + o(1)) \partial_r U_L^1(r) + O(r) O(t^{-\frac{N+3}{2}}), \\ \partial_r^2 v_1^*(r,t) &= t^{-\frac{N+2}{2}} (a_\varphi^{D,1} + o(1)) \partial_r^2 U_L^1(r) + O(t^{-\frac{N+3}{2}}) \end{aligned}$$

for all $r \geq L$ and $t > 1$.

Proposition 3.4 *Let φ be a radial function in Ω satisfying* (2.1). *Let v_2 be a radial solution of (L_2^D) with the initial data φ and $N \geq 2$. Then there hold that*

$$\begin{aligned} v_2^*(r,t) &= O(t^{-\frac{N+4}{2}}\log t)U_L^{D,2}(r) + O(t^{-\frac{N+4}{2}})O(r^2\log r), \\ \partial_r v_2^*(r,t) &= O(t^{-\frac{N+4}{2}}\log t)\partial_r U_L^{D,2}(r) + O(t^{-\frac{N+4}{2}})r\log\frac{r}{L}, \\ \partial_r^2 v_2^*(r,t) &= O(t^{-\frac{N+4}{2}}\log t)\partial_r^2 U_L^{D,2}(r) + O(t^{-\frac{N+4}{2}})\log\frac{r}{L} \end{aligned}$$

for all $r \geq L$ and $t > 1$, where

$$U_L^{D,2}(r) = c_2 r^2\left(1 - \frac{L^{N+2}}{r^{N+2}}\right).$$

Therefore, by the similar argument to the problem (1.1) and Propositions 3.1–3.4, we may prove Theorems 3.1–3.3. In order to prove Theorem 3.4, we study the asymptotic behavior of $x/|x|$ for all $x \in H(t)$ and all sufficiently large t, by using the asymptotic behavior of v_0 and v_1. Furthermore we compare the hot spots $H(t)$ with the radial solution of (1.2) with the initial data $\varphi \in L^2(\Omega, \rho dx)$ with $m_\varphi = m_\phi$. Then we may prove that, if t is sufficiently large, then the matrix $\{-\partial_{x_i}\partial_{x_j}u(x,t)\}_{i,j=1}^N$ is positive definite for all points near the hot spots $H(t)$, and complete the proof Theorem 3.4.

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