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Blowup in infinite time in the simplified system of chemotaxis

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1 Introduction

The purpose of the present paper is to study the parabolic-elliptic system of chemotaxis,

$$\begin{aligned} & u_t = \nabla \cdot (\nabla u - u \nabla v) \qquad \text{in } \Omega \times (0,T) \\ & v(\cdot, t) = (G * u)(\cdot, t) \qquad \text{in } \Omega \times (0,T) \\ & \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\ & u(\cdot, 0) = u_0 \qquad \text{in } \Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, $u_0 = u_0(x)$ is a non-negative smooth function defined on $\overline{\Omega}$, and

$$(G * u)(x,t) = \int_\Omega G(x,x')u(x',t)dx',$$

with $G = G(x,x')$ standing for the Green's function of a second order linear elliptic boundary value problem. This system is proposed to describe the chemotactic aggregation of cellular slime molds [16, 24], the motion of the mean field of many self-gravitating particles [2, 34], and that of moleculars under the chemical reaction [11]. Existence of the solution globally in time, particularly in the context of the threshold of the total mass $\lambda = \|u_0\|_1$, has been studied by several authors [15, 21, 22, 4, 12], while its counter part, the blowup of the solution in finite time, is summarized as the formation of collapses with the quantized mass [33].

The asymptotic behavior of the solution globally in time, on the other hand, has not been clarified so satisfactorily, in spite of several suggestions obtained from the study of stationary solutions [27]. Its counter part is the classification of the solution blowing-up in infinite time, and [30] conjectured that this is the case only when the total mass $\lambda = \|u_0\|_1$ is so quantized as 8π or 4π times integer, according to the profile of $G(x, x')$ on the boundary. In more details, each solution, existing globally in time, will converge to a regular stationary solution if λ is disquantized, while the convergence to a singular limit of the stationary solution will occur in the other case. This paper continues the study, and shows, among other things, that if the free energy, defined below, is bounded and the total mass is disquantized, then the collapses formed in infinite time vanishes almost every moment. This suggests that the blowup in infinite time does not occur in this case; the disquantized total mass and bounded free energy.

To describe the results proven in this paper precisely, we refer to several fundamental facts on (1). See [30, 29, 32, 33] for the proof of them. First, (1) is written as

$$
\begin{aligned}
& u_t = \Delta u - f(u) \qquad \text{in } \Omega \times (0,T) \\
& \frac{\partial u}{\partial \nu} = g(u) \qquad \text{on } \partial\Omega \times (0,T) \\
& u(\cdot, 0) = u_0 \qquad \text{in } \Omega
\end{aligned}
$$

for

$$
\begin{aligned}
& f(u) = \nabla u \cdot \nabla G * u + u\Delta(G * u) \\
& g(u) = u \frac{\partial G * u}{\partial \nu}\bigg|_{\partial\Omega}
\end{aligned}
$$

and the elliptic regularity of $G(x, x')$ combined with the standard fixed point argument [17] guarantees the unique existence of the solution $u = u(x,t) \in C^{2+\theta, 1+\theta/2}(\overline{\Omega} \times [0,T])$ with $T > 0$ estimated from below by $\|u_0\|_{C^{2+\theta}(\overline{\Omega})}$, where

$0 < \theta < 1$, and henceforth the supremum of its existence time is denoted by $T_{\max} \in (0, +\infty]$. This solution is non-negative, and preserves the total mass;

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx (= \lambda). \tag{2}$$

Second, the free energy, denoted by $\mathcal{F} = \mathcal{F}(u)$, acts as a Lyapunov function, and it holds that

$$\frac{d\mathcal{F}}{dt} + \int_\Omega u \left|\nabla(\log u - v)\right|^2 dx = 0, \tag{3}$$

where

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)dx - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx'.$$

In the stationary state, in particular, we have $\log u - v =$ constant, or $u = \frac{\lambda e^v}{\int_\Omega e^v dx}$ by $\|u\|_1 = \lambda$, and therefore, it follows that

$$G * u = v \qquad \text{and} \qquad u = \frac{\lambda e^v}{\int_\Omega e^v dx}. \tag{4}$$

Henceforth, we consider the case that $G(x,x')$ is the Green's function to one of the following elliptic problems;

$$\begin{array}{lll} -\Delta v + v = u \quad \text{in } \Omega, & \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega & \\ -\Delta v = u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u dx \quad \text{in } \Omega, & \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, & \displaystyle\int_\Omega v dx = 0 \\ -\Delta v = u \quad \text{in } \Omega, & v = 0 \quad \text{on } \partial\Omega. & \end{array}$$

These problems are referred to as the (N), (JL), and (D) fields, respectively. Then, considering

$$\begin{aligned} V &= H^1(\Omega) \\ V &= \left\{ v \in H^1(\Omega) \mid \int_\Omega v dx = 0 \right\} \\ V &= H_0^1(\Omega), \end{aligned}$$

provided with the norms

$$\begin{aligned}\|v\|_V &= \left(\|\nabla v\|_2^2 + \|v\|_2^2\right)^{1/2}\\ \|v\|_V &= \|\nabla v\|_2\\ \|v\|_V &= \|\nabla v\|_2\,,\end{aligned}$$

we obtain the isomorphism

$$u \in V' \quad \mapsto \quad v = \int_\Omega G(\cdot, x')u(x')dx' \in V,$$

and also the Lagrange functional,

$$\mathcal{W}(u,v) = \int_\Omega u(\log u - 1)dx + \frac{1}{2}\|v\|_V^2 - \int_\Omega uvdx \tag{5}$$

defined for $(u,v) \in M_\lambda \times V$, where $M_\lambda = \{u \geq 0 \mid \|u\|_1 = \lambda\}$. This functional satisfies

$$\mathcal{W}(u,v)|_{v=G*u} = \mathcal{F}(u) \qquad \text{and} \qquad \mathcal{W}(u,v)|_{u=\lambda e^v / \int_\Omega e^v dx} = \mathcal{J}_\lambda(v)$$

for $(u,v) \in M_\lambda \times V$, where

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|v\|_V^2 - \lambda \log\left(\int_\Omega e^v dx\right) + \lambda \log \lambda - \lambda,$$

and both $\mathcal{F}$ and $\mathcal{J}_\lambda$ defined on M_λ and V, respectively, provide equivalent variational structures to the stationary problem (4).

More precisely, if u_∞ is a critical point of $\mathcal{F}$ defined on M_λ, then $v_\infty = G * u_\infty$ is a critical point of $\mathcal{J}_\lambda$ defined on V, and conversely, if v_∞ is a critical point of $\mathcal{J}_\lambda$ defined on V, then $u_\infty = \lambda e^{v_\infty} / \int_\Omega e^{v_\infty}$ is a crititical point of $\mathcal{F}$ defined on M_λ, and in both cases it holds that $\mathcal{F}(u_\infty) = \mathcal{J}_\lambda(v_\infty)$. Henceforth, E_λ denotes the set of stationary solutions of v, i.e.,

$$\begin{aligned}E_\lambda &= \left\{v \in V \mid v = G * u, \quad u = \lambda \frac{e^v}{\int_\Omega e^v dx}\right\}\\ &= \{v \in V \mid \delta \mathcal{J}_\lambda(v) = 0\}\,,\end{aligned}$$

where $\lambda = \|u_0\|_1$.

As we mentioned, in the case of $T_{\max} < +\infty$, there is a formation of collapses with the quantized mass [33]. More precisely, if $G(x,x')$ is associated with the (N) or (JL) field, then it holds that

$$u(x,t)dx \rightarrow \sum_{x_0 \in \mathcal{S}} m_*(x_0)\delta_{x_0}(dx) + f(x)dx \qquad \text{*-weakly in } \mathcal{M}(\overline{\Omega}) \tag{6}$$

as $t \uparrow T_{\max}$, where $\mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$ denotes the set of measures on $\overline{\Omega}$,

$$\mathcal{S} = \left\{ x_0 \in \overline{\Omega} \mid \text{there exists } (x_k, t_k) \rightarrow (x_0, T_{\max}) \text{ such that } u(x_k, t_k) \rightarrow +\infty \right\}$$

denotes the blowup set, $0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$, and

$$m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

Since the total mass is preserved (2), this implies the finiteness of the blowup points, more precisely,

$$2 \cdot \sharp(\Omega \cap \mathcal{S}) + \sharp(\partial\Omega \cap \mathcal{S}) \leq \|u_0\|_1 / (4\pi).$$

A similar fact is proven for the case of $T_{\max} = +\infty$ ([30]), that is, in the (N) or (JL) field, any $t_k \rightarrow +\infty$ admits $\{t'_k\} \subset \{t_k\}$ such that

$$u(x,t'_k)dx \rightarrow \sum_{x_0 \in \mathcal{S}'} m_*(x_0)\delta_{x_0}(dx) + f(x)dx \qquad \text{*-weakly in } \mathcal{M}(\overline{\Omega}), \tag{7}$$

where $\mathcal{S}'$ denotes the set of "exhausted" blowup points of $\{u(\cdot, t'_k)\}$:

$$\mathcal{S}' = \left\{ x_0 \in \overline{\Omega} \mid \text{there exists } x'_k \rightarrow x_0 \text{ such that } u(x'_k, t'_k) \rightarrow +\infty \right\}.$$

Our conjecture on the blowup in infinite time, therefore, is proven in the affirmative in the (N) or (JL) field, if we can deduce $f = 0$ from $\mathcal{S}' \neq \emptyset$ in (7), because the total mass of the solution is preserved as (2) and hence it follows that

$$\lambda = \sum_{x_0 \in \mathcal{S}'} m_*(x_0) + \|f\|_1$$

from (7). More precisely, if we can show $f = 0$ by $\mathcal{S}' \neq \emptyset$, then $T_{\max} = +\infty$ and

$$\lim_{k \rightarrow \infty} \|u(\cdot, t_k)\|_\infty = +\infty \qquad \text{with} \qquad t_k \rightarrow +\infty$$

is admitted only when $\lambda = \|u_0\|_1 = \sum_{x_0 \in \mathcal{S}'} m_*(x_0) \in 4\pi\mathcal{N}$.

Taking this approach to the problem, we use the weak solution generated *during* $t'_k \to +\infty$. This fact on the generation of the weak solution is proven for the problem on the flat torus [31], and also for system (1) under the (N) or (JL) field [33].

In more details, any $t_k \to +\infty$ admits $\{t'_k\} \subset \{t_k\}$ such that

$$u(x, t'_k + t)dx \to \mu(dx, t) \qquad \text{in } C_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega})), \tag{8}$$

where $\mu = \mu(dx, t)$ is a weak solution to (1). This means

$$\int_\Omega \varphi(x)u(x, t'_k + t)dx \to \langle \varphi, \mu(dx, t)\rangle_{C(\overline{\Omega}), \mathcal{M}(\overline{\Omega})}$$

locally uniformly in $t \in (-\infty, +\infty)$ for each $\varphi \in C(\overline{\Omega})$, and if

$$\begin{aligned}
&X = \left\{ \xi \in C^2(\overline{\Omega}) \mid \frac{\partial \xi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\} \\
&\rho_\xi(x, x') = \nabla\xi(x) \cdot \nabla_x G(x, x') + \nabla\xi(x') \cdot \nabla_{x'} G(x, x') \\
&\mathcal{E}_0 = \{\rho_\eta \mid \eta \in X\} \\
&\mathcal{E} = \mathcal{E}_0 \oplus C(\overline{\Omega} \times \overline{\Omega}) \subset L^\infty(\Omega \times \Omega),
\end{aligned}$$

then there is $0 \le \nu = \nu(t)$ belonging to $L^\infty_*(-T, T; \mathcal{E}')$ for any $T > 0$ such that

$$\nu(t)|_{C(\overline{\Omega} \times \overline{\Omega})} = \mu \otimes \mu(dxdx', t) \qquad \text{a.e. } t \in (-\infty, +\infty).$$

Furthermore, the mapping

$$t \in (-\infty, +\infty) \mapsto \langle \xi, \mu(dx, t)\rangle_{C(\overline{\Omega}), \mathcal{M}(\overline{\Omega})}$$

is locally absolutely continuous and satisfies

$$\begin{aligned}
&\frac{d}{dt}\langle \xi, \mu(dx, t)\rangle_{C(\overline{\Omega}), \mathcal{M}(\overline{\Omega})} = \langle \Delta\xi, \mu(dx, t)\rangle_{C(\overline{\Omega}), \mathcal{M}(\overline{\Omega})} \\
&+\frac{1}{2}\langle \rho_\xi, \nu(t)\rangle_{\mathcal{E}, \mathcal{E}'} \qquad \text{a.e. } t \in (-\infty, +\infty)
\end{aligned}$$

for each $\xi \in X$.

From (7), the Radon-Nikodym-Lebsesgue decomposition of this $\mu(dx,t)$ has the form

$$\begin{aligned}\mu(dx,t) &= \mu_s(dx,t) + \mu_{a.c.}(dx,t) \\ &= \sum_{i=1}^{n(t)} m_*(x_i(t))\delta_{x_i(t)}(dx) + f(x,t)dx\end{aligned}$$

for each $t \in (-\infty,+\infty)$, where $\mathcal{S}_t = \{x_i(t) \mid 1 \leq i \leq n(t)\}$ denotes the set of exhausted blowup points of $\{u(\cdot, t'_k + t)\}$ as $t'_k \to +\infty$, and $0 \leq f = f(\cdot,t) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S}_t)$.

The first result proven in this paper is stated as follows.

Theorem 1 *If $G(x,x')$ is associated with the (N) or (JL) field, and*

$$\begin{aligned}&\lambda = \|u_0\|_1 \not\in 4\pi\mathcal{N} \\ &T_{\max} = +\infty \\ &\lim_{t\to+\infty} \mathcal{F}(u(\cdot,t)) > -\infty,\end{aligned}$$

then $\mu_s(dx,t) = 0$ a.e. $t \in (-\infty,+\infty)$.

Unfortunately, $t \in (-\infty,+\infty) \mapsto \mu_s(dx,t) \in \mathcal{M}(\overline{\Omega})$ is generally only $*$-weakly upper semi-continuous, and the above theorem is not sufficient to deduce $\mu_s(dx,0) = 0$, although if this is the case, then we can infer $\lambda \in 4\pi\mathcal{N}$ from

$$\begin{aligned}&T_{\max} = +\infty \\ &\limsup_{t\to+\infty} \|u(\cdot,t)\|_\infty = +\infty \\ &\lim_{t\to+\infty} \mathcal{F}(u(\cdot,t)) > -\infty.\end{aligned}$$

If the free energy is unbounded, on the contrary, the solution blows-up in finite or infinite time; more precisely [29],

$$\lim_{t\uparrow T_{\max}} \mathcal{F}(u(\cdot,t)) = -\infty \quad \Rightarrow \quad \lim_{t\uparrow T_{\max}} \int_\Omega (u\log u)(x,t)dx = +\infty. \tag{9}$$

This means a kind of concentration as the blowup time approaches, and $T_{\max} < +\infty$ may occur always in this case, namely, we suspect that $T_{\max} = +\infty$ implies $\lim_{t\uparrow+\infty} \mathcal{F}(u(\cdot,t)) > -\infty$.

The other conjecture of ours is the convergence to a singular limit of the stationary solution of the total mass quantized non-stationary solution blowing-up in infinite time. The second theorem of this paper illustrates such a profile of the solution in a specific case.

Since this theorem is concerned with the (D) field, here we mention some differences of this problem from the other cases. Actually, in the study of the (D) field, we have not been able to exclude the boundary blowup point in both cases of blowing-up in finite time and infinite time. Consequently, (6) or (7) holds with $\mathcal{M}(\overline{\Omega})$ and $\mathcal{S}$ replaced by $\mathcal{M}(\Omega) = C_0(\overline{\Omega})'$ and $\mathcal{S} \cap \Omega$, respectively, where $C_0(\overline{\Omega})$ denotes the set of continuous functions on $\overline{\Omega}$ with the value zero on $\partial\Omega$. This difficulty arises because $C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ is not dense in $C(\overline{\Omega})$. Similarly, we have (8) with $C_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega}))$ replaced by $C_*(-\infty, +\infty; \mathcal{M}(\Omega))$ when $G(x, x')$ is associated with the (D) field.

In spite of these obstructions, we can show the following theorem.

Theorem 2 *If $G(x, x')$ is associated with the (D) field, $\lambda = \|u_0\|_1 = 8\pi$, $T_{\max} = +\infty$, and $E_{8\pi} = \emptyset$, then any $t_k \to +\infty$ admits $\{t'_k\} \subset \{t_k\}$ such that*

$$\begin{aligned}
& u(x, t'_k + t)dx \rightharpoonup 8\pi\delta_{x(t)}(dx) \qquad && \textit{in } L^\infty_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega})) \\
& t \in (-\infty, +\infty) \mapsto x(t) \in \Omega && \textit{is absolutely continuous} \\
& \liminf_{t \to \pm\infty} \mathit{dist}(x(t), \partial\Omega) > 0 && \\
& \frac{dx}{dt} = 4\pi \nabla R(x(t)) \qquad (-\infty < t < +\infty), &&
\end{aligned} \tag{10}$$

where $R(x) = \left[G(x, x') + \frac{1}{2\pi}\log|x - x'|\right]_{x'=x}$ indicates the Robin function.

The first relation of (10) implies that the local L^1 norm of $u(\cdot, t + t_k)$ near $\partial\Omega$ becomes arbitrarily small locally uniformly in $t \in \mathbf{R}$. Still this is not enough to exclude the boundary blowup point, but we hope that this convergence holds actually in $C_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega}))$.

We recall also that E_λ denotes the set of stationary solutions so that $v_\infty \in E_\lambda$ if and only if it is a (regular) solution to

$$-\Delta v_\infty = \lambda \frac{e^{v_\infty}}{\int_\Omega e^{v_\infty}} \quad \text{in } \Omega, \qquad v_\infty = 0 \quad \text{on } \partial\Omega.$$

The condition $E_{8\pi} = \emptyset$ has been studied in detail [6, 20, 9]. This is actually the case, if $\Omega \subset \mathbf{R}^2$ is simply connected and close to a disc. For such a domain, any solution $u = u(\cdot, t)$, existing globally in time with $\|u_0\|_1 = 8\pi$, cannot be uniformly bounded, and therefore, $\limsup_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = +\infty$ holds true. Then, thanks to the concentration lemma [25], we can show that the location of the concentration mass formed during $t'_k \to +\infty$ is subject to the ordinary differential equation given by the last relation of (10). We note that this is a conjugate form of the vortex equation derived from the Euler equation [19]:

$$\frac{dx}{dt} = 4\pi\nabla^\perp R(x(t)) \qquad (-\infty < t < +\infty).$$

The last result of this paper proves that our conjecture holds in the affirmative if the solution is radially symmetric; more precisely,

Theorem 3 *If $\Omega = \{x \in \mathbf{R}^2 \mid |x| < R\}$ is a disc, $u_0 = u_0(|x|)$ is radially symmetric, $G(x,x')$ is associated with the (N) or (JL) field, and $\lambda = \|u_0\|_1 > 8\pi$, then the blowup in infinite time does not occur in system (1), that is,*

$$\limsup_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty < +\infty$$

holds if $T_{\max} = +\infty$.

In this radially symmetric case, if $\lambda \in (0, 8\pi)$ then the solution $u = u(x,t)$ is uniformly bounded, and the stationary problem admits the unique (constant) solution, denoted by $\underline{u}_\lambda$, and furthermore, we have

$$\lim_{t\to+\infty} \|u(\cdot,t) - \underline{u}_\lambda\|_\infty = 0.$$

See [21, 27, 33] and the dicussion in the next section. On the other hand, the above theorem guarantees the generic blowup in finite time in this problem if $\lambda > 8\pi$; see [30]. Thus, behavior of the solution global in time has been almost classifed in this case, using $\lambda = \|u_0\|_1$.

This paper is composed of five sections and two appendices. We take preliminaries in the following section, and prove Theorems 1, 2, and 3 in §§3, 4, and 5, respectively. In the first appendix, we show the proof of (9) by the method of [29]. The second appendix is devoted to the proof of a conentration lemma [25] used in the proof of Theorem 2.

2 Preliminaries

In this section, we take several preliminaries and describe the relation between other works and our theorems. See [30, 29, 32, 33] for details of the result referred to in this section.

First, as is mentioned in the introduction, the stationary problem (4) has an equivalent varitational structures, $\mathcal{F}$ on M_λ and $\mathcal{J}_\lambda$ on V. These variational structures are regarded as an "unfolding" of the Lagrange functional, and in particular, it holds that

$$\mathcal{W}(u,v) \geq \max\{\mathcal{F}(u), \mathcal{J}_\lambda(v)\} \qquad \text{for } (u,v) \in M_\lambda \times V.$$

This inequality means

$$\int_\Omega \{u(\log u - 1) - uv\}\, dx + \lambda \log\left(\int_\Omega e^v dx\right) - \lambda\log\lambda + \lambda \geq 0 \tag{11}$$

for $(u,v) \in M_\lambda \times V$, and can be proved directly using Jensen's inequality [22, 4, 12]. In any case, it holds that

$$\mathcal{F}(u(\cdot,t)) \geq \mathcal{J}_\lambda(v(\cdot,t)) \qquad (t \in [0, T_{\max})) \tag{12}$$

for the solution $(u,v) = (u(\cdot,t), v(\cdot,t))$ to (1) with $\|u_0\|_1 = \lambda$, because $v = G * u$ and therefore, $\mathcal{F} = \mathcal{W}$ holds in this system.

Next, if $u = u(x,t)$ is a solution to (1), then it holds that

$$\frac{dJ}{dt} \leq C\lambda^2 + 3|\Omega| \exp(4K^2 J) \qquad (t \in [0, T_{\max}))$$

for

$$J = J(u) = \int_\Omega (u \log u + e^{-1}),$$

where C, K are positive constants determined by Ω, and therefore, in the case of

$$T_{\max} = +\infty \qquad \text{and} \qquad \liminf_{t\uparrow+\infty} \int_\Omega (u\log u)(x,t)dx < +\infty \tag{13}$$

there are $t_k \to +\infty$, $\delta > 0$, and $C > 0$ such that

$$\int_\Omega (u\log u)(x,t)dx \leq C \qquad (t \in [t_k, t_k + 2\delta),\ k = 1, 2, \dots).$$

Then, Moser's iteration scheme guarantees $\|u(\cdot,t)\|_\infty \le C$ with a constant C independent of $t \in [t_k, t_k+\delta]$ and $k = 1, 2, \ldots$, and therefore, $\omega(u_0) \neq \emptyset$ follows from the parabolic regularity, where

$$\omega(u_0) = \left\{u_\infty \mid \text{there exists } t_k \to +\infty \text{ such that } u(\cdot,t_k) \to u_\infty \text{ in } C^{2+\theta}(\overline{\Omega})\right\}$$

denotes the ω-limit set of $u = u(\cdot,t)$ obtained from the initial value u_0. This argument of iteration is also valid to the other case, i.e., we obtain

$$\lim_{t\uparrow T_{\max}} \int_\Omega (u\log u)(x,t)dx = +\infty$$

if $T_{\max} < +\infty$.

Since system (1) is provided with the Lyapunov function, the standard argument of the dynamical system [13] guarantees that any $u_\infty \in \omega(u_0)$ is a critical point of $\mathcal{F}$ defined on M_λ. In fact, first, if $u_1, u_2 \in \omega(u_0)$, then there are $t_k^1 \to +\infty$ and $t_k^2 \to +\infty$ such that $u(\cdot,t_k^1) \to u_1$ and $u(\cdot,t_k^2) \to u_2$ in $C^{2+\theta}(\overline{\Omega})$. We may assume $t_k^1 < t_k^2 < t_{k+1}^1$ for $k = 1, 2, \cdots$, and therefore, it follows that $\mathcal{F}(u(\cdot,t_{k+1}^1)) \ge \mathcal{F}(u(\cdot,t_k^2)) \ge \mathcal{F}(u(\cdot,t_k^1))$. This implies $\mathcal{F}(u_1) \ge \mathcal{F}(u_2) \ge \mathcal{F}(u_1)$ and hence $\mathcal{F}$ is constant on $\omega(u_0)$.

If $u_\infty \in \omega(u_0)$, on the other hand, the solution to (1) with the initial value u_∞, denoted by $T_t u_\infty$, exists globally in time from the local well-posedness of (1), and it holds that $T_t u_\infty \in \omega(u_0)$ for each $t \ge 0$ by the definition. This implies

$$\mathcal{F}(T_t u_\infty) = \mathcal{F}(u_\infty) \qquad (t \ge 0)$$

and therefore,

$$\left.\frac{d}{dt}\mathcal{F}(T_t u_\infty)\right|_{t=0} = 0.$$

Then, we obtain

$$u_\infty = \lambda \frac{e^{v_\infty}}{\int_\Omega e^{v_\infty} dx}$$

for $v_\infty = G * u_\infty$ by (3), and therefore, u_∞ is a stationary solution to (1).

Thus, $v_\infty = G * u_\infty$ is a critical point of $\mathcal{J}_\lambda$ defined on V for each $u_\infty \in \omega(u_0)$. It holds also that $\mathcal{J}_\lambda(v_\infty) = \mathcal{F}(u_\infty)$ from the general theory of dual variation mentioned in the introduction. From the mass quantization of the non-compact stationary solution sequence [23, 26], on the other hand, it follows that

$$j_\lambda \equiv \inf_{v \in E_\lambda} \mathcal{J}_\lambda(v) > -\infty$$

for $\lambda \notin 4\pi\mathcal{N}$ in the cases of the (N) and (JL) fields, and for $\lambda \notin 8\pi\mathcal{N}$ in the case of the (D) field. Therefore, we obtain the following fact [14, 29].

Theorem 4 *If* $\mathcal{F}(u_0) < j_\lambda$, *then* $\lim_{t\uparrow T_{\max}} \int_\Omega (u \log u)(x,t)dx = +\infty$.

Both cases $T_{\max} = +\infty$ and $T_{\max} < +\infty$ are permitted in the above theorem, but we suspect that $\mathcal{F}(u_0) < j_\lambda$ always implies $T_{\max} < +\infty$. Actually, if the assumptions of Theorem 1 hold, then we have $\omega(u_0) \neq \emptyset$ from the conclusion, and this is impossible in the case of $\mathcal{F}(u_0) < j_\lambda$. Thus, we obtain the following theorem.

Theorem 5 *If* $G(x,x')$ *is associated with the (N) or (JL) field, if*

$$\mathcal{F}(u_0) < j_\lambda$$

with $\lambda = \|u_0\|_1 \notin 4\pi\mathcal{N}$, *and if* $T_{\max} = +\infty$ *holds in the previous theorem, then* $\lim_{t\uparrow T_{\max}} \mathcal{F}(u(\cdot,t)) = -\infty$.

We emphasize again what we suspect, that is, $T_{\max} = +\infty$ with

$$\lim_{t\uparrow+\infty} \mathcal{F}(u(\cdot,t)) = -\infty$$

will not occur, and therefore, $T_{\max} < +\infty$ will hold under the assumption of Theorem 5. See the descriptions below Theorem 1.

3 Proof of Theorem 1

Given $t_k \to +\infty$, we have $\{t'_k\} \subset \{t_k\}$ satisfying (8), where $\mu = \mu(dx,t)$ is a weak solution to (1). We shall write t_k for t'_k, and furthermore, given $T > 0$, we may assume $t_k + 2T < t_{k+1}$, passing to a subsequence. From the assumption $\lim_{t\uparrow+\infty} \mathcal{F}(u(\cdot,t)) > -\infty$, then we have

$$\sum_k \int_{t_k-T}^{t_k+T} dt \int_\Omega u \left|\nabla(\log u - v)\right|^2 (x,t)dx < +\infty$$

and hence it holds that

$$\lim_{k\to\infty}\int_{t_k-T}^{t_k+T} dt \int_\Omega u\left|\nabla(\log u - v)\right|^2(x,t)dx = 0.$$

We have $G(x,x') \geq -A$, and therefore, $v(x) \geq -A\lambda$, where A is a constant determined by Ω ([1]). This implies

$$u\left|\nabla(\log u - v)\right|^2 = 4e^v\left|\nabla(ue^{-v})^{1/2})\right|^2 \geq 4e^{-A\lambda}\left|\nabla(ue^{-v})^{1/2}\right|^2,$$

and therefore,

$$f_k(x,t) = (ue^{-v})^{1/2}(x,t+t_k) - \frac{1}{|\Omega|}\int_\Omega (ue^{-v})^{1/2}(x,t+t_k)dx$$

satisfies

$$\lim_{k\to\infty}\int_{-T}^{T} dt \int_\Omega \left|\nabla f_k(x,t)\right|^2 dx = 0, \qquad \int_\Omega f_k(x,t)dx = 0.$$

This means

$$f_k \to 0 \qquad \text{in } L^2(-T,T;H^1(\Omega)),$$

and passing to a subsequence (denoted by the same symbol), we obtain

$$f_k(x,t) \to 0 \qquad \text{a.e. } (x,t) \in \Omega \times (-T,T).$$

On the other hand, we have

$$\frac{1}{|\Omega|}\int_\Omega (ue^{-v})^{1/2}dx \leq \left\{\frac{1}{|\Omega|}\int_\Omega ue^{-v}dx\right\}^{1/2} \leq (|\Omega|\,\lambda e^{-A\lambda})^{-1/2}$$

and therefore, for a.e. $t \in (-T,T)$, there is $\{t_k'\} \subset \{t_k\}$ and $C_0(t) \geq 0$ such that

$$(ue^{-v})^{1/2}(x,t_k'+t) \to C_0(t) \qquad \text{a.e. } x \in \Omega,$$

i.e.,

$$(ue^{-v})(x,t_k'+t) \to C_0(t)^2 \qquad \text{a.e. } x \in \Omega. \tag{14}$$

Now, relation (8) implies

$$v(x, t_k' + t) \to \sum_{i=1}^{n(t)} m_*(x_i(t))G(x, x_i(t)) + \int_\Omega G(x, x')f(x', t)dx'$$

weakly in $W^{1,q}(\Omega)$ for $1 < q < 2$ by the L^1 elliptic estimate [5] applied to the second equation of (1). This convergence is strong in $L^p(\Omega)$ for $1 \le p < \infty$ by Rellich-Kondrachov's theorem, and hence a.e. $x \in \Omega$, passing to a subsequence. In case $n(t) \ge 1$ and $C_0(t) > 0$, this implies

$$\int_\Omega \lim_{k\to\infty} \left\{ e^{v(x,t_k'+t)} \cdot e^{-v(x,t_k'+t)} u(x, t_k' + t) \right\} dx = +\infty$$

by $m_* \ge 4\pi$, but the left-hand side is estimated above by

$$\liminf_{k\to\infty} \int_\Omega u(x, t_k' + t)dx = \lambda$$

from Fatou's lemma. This is impossible, and therefore, $\mu_s(dx, t) \ne 0$ implies $C_0(t) = 0$, i.e.,

$$(ue^{-v})(x, t_k' + t) \to 0 \qquad \text{a.e. } x \in \Omega. \tag{15}$$

On the other hand, $\mathcal{S}_t = \{x_i(t) \mid 1 \le i \le n(t)\}$ is the set of exhausted blowup points of $\{u(\cdot, t_k' + t)\}$ as $t_k' \to \infty$, and therefore, $\{v(\cdot, t_k' + t)\}$ is locally uniformly bounded in $\overline{\Omega} \setminus \mathcal{S}_t$ by the elliptic regularity. This implies

$$u(x, t_k' + t) \to 0 \qquad \text{a.e. } x \in \Omega \tag{16}$$

by (15). The parabolic regularity guarantees, on the other hand,

$$u(\cdot, t_k' + t) \to f(\cdot, t) \qquad \text{locally uniformly in } \overline{\Omega} \setminus \mathcal{B}_t$$

in (8), passing to a subsequence, and therefore, $f(x, t) = 0$ a.e. $x \in \Omega$ by (16). This implies the mass quantization, $\lambda \in 4\pi\mathcal{N}$, which contradicts the assumption. Thus, we obtain $\mu_s(dx, t) = 0$ a.e. $t \in (-T, T)$, and hence a.e. $t \in (-\infty, +\infty)$. The proof is complete.

4 Proof of Theorem 2

It is obvious that this theorem follows from the following lemma, where

$$\mathcal{K}(u) = \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx'$$

denotes -1 times the inner (potential) energy. In fact, we have only to confirm that the first condition of (17), described below, is satisfied for $u^k(\cdot,t) = u(\cdot,t_k+t)$.

Lemma 6 *If $G(x,x')$ of (1) is associated with the (D) field, $\{u_0^k\}$ is a sequence of the initial values satisfying $\|u_0^k\|_1 = 8\pi$, and*

$$\begin{aligned} K_k &= \inf_{t\in(0,T)} \mathcal{K}(u^k(\cdot,t)) \to +\infty \\ F_k &= \sup_{t\in(0,T)} \mathcal{F}(u^k(\cdot,t)) \leq F < +\infty, \end{aligned} \tag{17}$$

then we have $\{u^{k'}\} \subset \{u^k\}$ such that

$$u^{k'}(x,t)dx \to 8\pi\delta_{x(t)}(dx) \qquad \text{in } L^\infty_*(0,T;\mathcal{M}(\overline{\Omega})) \tag{18}$$

as $k'\to\infty$, where $u^k = u^k(x,t)$ denotes the solution to (1) for the initial value $u_0^k(x)$, $t\in(0,T)\mapsto x(t)\in\omega$ is locally absolutely continuous, with $\omega \Subset \Omega$ determined by F, and it holds that

$$\frac{dx}{dt} = 4\pi\nabla R(x(t)) \qquad \text{a.e. } t\in(0,T). \tag{19}$$

The show the first condition of (17) for $u^k(\cdot,t) = u(\cdot,t_k+t)$, we use

$$\lim_{t\uparrow+\infty}\int_\Omega (u\log u)(x,t)dx = +\infty. \tag{20}$$

In fact, if this is not the case, then (13) holds, and therefore, there are $t_k\to+\infty$ and $v_\infty\in E_\lambda$ such that $v(\cdot,t_k)\to v_\infty$ in $C^{2+\theta}(\overline{\Omega})$. This contradicts the assumption, $E_{8\pi}=\emptyset$, and we obtain (20).

Now, we have

$$\lim_{t\uparrow+\infty}\mathcal{K}(u(\cdot,t)) = +\infty \tag{21}$$

by $\mathcal{F}(u(\cdot,t)) \leq \mathcal{F}(u_0)$, and the proof is complete.

Lemma 6 is obtained from its discrete version, the concentration lemma [25] described below. Traditionally, such a kind of lemma is stated in terms of the convergence of the probability measure [7], and we shall adopt this formulation, putting

$$P(\Omega) = \left\{\rho \in L^1(\Omega) \mid \rho \geq 0, \quad \int_\Omega \rho(x)dx = 1\right\}$$
$$\mathcal{I}(\rho) = \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')\rho(x)\rho(x')dxdx' - \frac{1}{8\pi}\int_\Omega (\rho\log\rho)(x)dx.$$

First, the dual form of the Trudinger-Moser inequality [7, 33] assures

$$\sup\{\mathcal{I}(\rho) \mid \rho \in P(\Omega)\} < +\infty, \tag{22}$$

and therefore, the value

$$I_\Omega(x) = \sup\left\{\limsup_{k\to+\infty}\mathcal{I}(\rho_k) \mid \{\rho_k\} \subset P(\Omega),\right.$$
$$\left.\rho_k(x)dx \to \delta_x(dx) \qquad *\text{-weakly in } \mathcal{M}(\overline{\Omega})\right\} < +\infty \tag{23}$$

is well-defined for each $x \in \overline{\Omega}$. Next, we have

$$I_\Omega(x) = I_{B_1(0)}(0) + \frac{1}{2}R(x) \tag{24}$$

for $B_1(0) = \{x \in \mathbf{R}^2 \mid |x| < 1\}$ (Theorem 3.1 of [7]), and therefore, it holds that

$$\Omega_{I_\infty} \equiv \{x \in \Omega \mid I_\Omega(x) \geq I_\infty\} \Subset \Omega \tag{25}$$

for each $I_\infty \in \mathbf{R}$, i.e., there is an open set O such that

$$\Omega_{I_\infty} \subset O \subset \overline{O} \subset \Omega. \tag{26}$$

Given $u \geq 0$ with $\|u\|_1 = \lambda$, we have $f = u/\lambda \in P(\Omega)$. Then, it holds that

$$\begin{aligned}
&\mathcal{I}(f) \\
&= -\frac{1}{8\pi\lambda}\left\{\int_\Omega u(\log u - 1)dx - \frac{4\pi}{\lambda}\iint_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx'\right\} \\
&-\frac{1}{8\pi\lambda}\left\{1 - \log\lambda\right\},
\end{aligned}$$

and therefore, we have

$$\mathcal{I}(f) = -\frac{1}{64\pi^2}\mathcal{F}(u) + \text{constant}$$

in the case of $\lambda = 8\pi$. Thus, Lemma 6 is reduced to the following lemma.

Lemma 7 *If $u^k = 8\pi\rho_k(x,t)$ $(k = 1, 2, \cdots)$ is a solution seqeuence to (1) with $\rho_k \in L^1(0,T; P(\Omega))$ satisfying*

$$\inf_{t\in(0,T)} \mathcal{K}(\rho_k(\cdot,t)) \to +\infty$$
$$\inf_{t\in(0,T)} \mathcal{I}(\rho_k(\cdot,t)) \to I_\infty > -\infty,$$

then there is a subsequence $\{\rho_{k'}\} \subset \{\rho_k\}$ such that

$$\rho_{k'}(x,t)dx \to \delta_{x(t)}(dx) \qquad \text{in } L^\infty_*(0,T; \mathcal{M}(\overline{\Omega})), \tag{27}$$

where $t \in (0,T) \mapsto x_\infty(t) \in \Omega_{I_\infty} (\Subset \Omega)$ is locally absolutely continuous and satisfies

$$\frac{dx_\infty}{dt} = 4\pi\nabla R(x_\infty(t)) \qquad \text{a.e. } t \in (0,T). \tag{28}$$

To show the above lemma, we use its discrete version (concentration lemma), of which proof is given in the second appendix.

Lemma 8 *If $\{\rho_k\} \subset P(\Omega)$ satisfies*

$$\lim_{k\to\infty} \mathcal{K}(\rho_k) = +\infty$$
$$\lim_{k\to\infty} \mathcal{I}(\rho_k) = I_\infty > -\infty$$
$$\lim_{k\to\infty} \int_\Omega x\rho_k(x)dx = x_\infty$$

for some $x_\infty \in \mathbf{R}^2$, then we have $x_\infty \in \Omega_{I_\infty}$ and

$$\rho_k(x)dx \to \delta_{x_\infty}(dx) \qquad *\text{-weakly in } \mathcal{M}(\overline{\Omega}).$$

Now, we give the following.

Proof of Lemma 7: We define Ω_{I_∞} by (25) for

$$I_\infty = \lim_{k\to\infty} \inf_{t\in(0,T)} \mathcal{I}(\rho_k(\cdot,t)) > -\infty$$

and take the open set O and $\xi \in C_0^\infty(\Omega)$ satisfying (26) and $\xi|_O = 1$, respectively.

From the assumption, $u^k(x,t) = 8\pi\rho_k(x,t)$ is a solution to (1). We take an arbitrary $\eta \in C_0^\infty(0,T)$, and multiply the first equation of (1) by $\eta\xi x_i$ for $i = 1,2$, where $x = (x_1, x_2)$. Then, using the second equation of (1), we obtain

$$\begin{aligned}
&-\iint_{\Omega\times(0,T)} \eta'(t)x_i\xi(x)\rho_k(x,t)dxdt \\
&= \int_0^T \eta(t)\left(\int_\Omega \Delta(x_i\xi(x))\rho_k(x,t)dx\right)dt \\
&+4\pi\int_0^T \eta(t)\left(\iint_{\Omega\times\Omega} \Xi_i(x,x')\rho_k(x,t)\rho_k(x',t)dxdx'\right)dt \qquad (29)
\end{aligned}$$

by $G(x,x') = G(x',x)$, where

$$\Xi_i(x,x') = \nabla(x_i\xi(x))\cdot\nabla_x G(x,x') + \nabla(x_i'\xi(x'))\cdot\nabla_{x'}G(x,x').$$

Here, we have

$$G(x,x') = \frac{1}{2\pi}\log\frac{1}{|x-x'|} + K(x,x')$$

with $K \in C^{\theta,2+\theta}(\overline{\Omega}\times\Omega)\cap C^{2+\theta,\theta}(\Omega\times\overline{\Omega})$ for $0<\theta<1$, and therefore, it holds that

$$\begin{aligned}
\Xi_i(x,x') &= -\frac{(x-x')\cdot(x_i\nabla\xi(x) - x_i'\nabla\xi(x'))}{2\pi|x-x'|^2} - \frac{(x_i-x_i')(\xi(x)-\xi(x'))}{2\pi|x-x'|^2} \\
&+\nabla_x K(x,x')\cdot\nabla(x_i\xi(x)) + \nabla_{x'}K(x,x')\cdot\nabla(x_i'\xi(x')). \qquad (30)
\end{aligned}$$

We have also

$$\|\Xi_i\|_\infty \le C\,\|\xi\|_{C^2(\overline{\Omega})},$$

and therefore, $\{\int_\Omega x_i\xi(x)\rho_k(x,\cdot)dx\}$ is uniformly bounded and locally equi-continuous in $(0,T)$. Consequently, there is $\{\rho_{k'}\}\subset\{\rho_k\}$ that admits the continuous

$$t\in(0,T)\mapsto x_\infty(t)=\lim_{k'\to\infty}\int_\Omega x\xi(x)\rho_{k'}(x,t)dx\in\mathbf{R}^2,$$

and then, we have $x_\infty(t)\in\Omega_{I_\infty}$ and

$$\rho_{k'}(x,t)dx\to\delta_{x_\infty(t)}(dx)\qquad *\text{-weakly in } \mathcal{M}(\overline{\Omega}).$$

for each $t\in(0,T)$ by Lemma 8. This means (27).

We have also

$$\lim_{k'\to\infty}\int_\Omega[\Delta(x\xi(x))]\rho_{k'}(x,t)dx=0\qquad(t\in(0,T))$$

and

$$\lim_{k'\to\infty}\iint_{\Omega\times\Omega}\Xi_i(x,x')\rho_{k'}(x,t)\rho_{k'}(x',t)dxdx'=\frac{\partial R}{\partial x_i}(x_\infty(t))$$

by $\xi|_O=1$, and therefore, it follow that

$$-\int_0^T\eta'(t)x_\infty^i(t)dt=4\pi\int_0^T\eta(t)\frac{\partial R}{\partial x_i}(x_\infty(t))dt$$

from (29). Thus, $t\in(0,T)\mapsto x_\infty(t)\in\mathbf{R}^2$ is locally absolutely continuous, and satisfies (28). The proof is complete.

5 Proof of Theorem 3

We shall descibe the case that $G(x,x')$ is associated with the (JL) field, because the proof is similar to the other case of the (N) field. This system is defined by

$$\begin{aligned}&u_t=\nabla\cdot(\nabla u-u\nabla v)\qquad\text{in }\Omega\times(0,\infty),\\&0=\Delta v-\frac{\lambda}{|\Omega|}+u\qquad\text{in }\Omega\times(0,\infty)\\&\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0\qquad\text{on }\partial\Omega\times(0,\infty)\end{aligned}$$

with $u = u(r,t)$, $r = |x|$, and $\lambda = \|u_0\|_1$, and therefore, it holds that

$$\frac{\partial}{\partial t}\int_{|x|<r} u(x,t)dx = 2\pi\left(\frac{\partial u}{\partial r} - u\frac{\partial v}{\partial r}\right) r \tag{31}$$

$$-2\pi r\frac{\partial v}{\partial r} = \int_{|x|<r}\left(u(x,t) - \frac{\lambda}{|\Omega|}\right) dx. \tag{32}$$

Still we have (7) with $\sharp\mathcal{S} < +\infty$, and therefore, if $T_{\max} = +\infty$, $u = u(|x|,t)$, and $\lim_{k\to\infty}\|u(\cdot,t_k)\|_\infty = +\infty$ with some $t_k \to +\infty$, then there is $\{t'_k\} \subset \{t_k\}$ such that

$$u(x,t'_k)dx \to 8\pi\delta_0(dx) + f(x) \qquad \ast\text{-weakly in } \mathcal{M}(\overline{\Omega}) \tag{33}$$

with $0 \leq f = f(|x|) \in L^1(\Omega) \cap C(\overline{\Omega}\setminus\{0\})$. Moreover, by the parabolic and elliptic regularity [21] we obtain the following inequalities, where C is a constant determined by $\varepsilon \in (0,R)$, $\lambda = \|u_0\|_1$, and $\|u_0\|_\infty$, and $\Omega_\varepsilon = \{x \in \mathbf{R}^2 \mid \varepsilon < |x| < R\}$:

$$\sup_{t\geq 0}\|u(\cdot,t)\|_{L^\infty(\Omega_\varepsilon)} \leq C$$
$$\sup_{t\geq 0}\|\nabla u(\cdot,t)\|_{L^2(\Omega_\varepsilon)} \leq C$$
$$\sup_{t\geq 0}\int_t^{t+1}\|u_t(\cdot,s)\|^2_{L^2(\Omega_\varepsilon)}\,ds \leq C. \tag{34}$$

We define

$$z(r,t) \equiv \frac{1}{2\pi}\int_{|x|<r}\left(u(x,t) - \frac{\lambda}{|\Omega|}\right) dx \qquad (0<r<R, t>0),$$

satisfying

$$z_r = r\left(u - \frac{\lambda}{|\Omega|}\right), \qquad z_{rr} = ru_r + u - \frac{\lambda}{|\Omega|},$$

and then it follows that

$$\mathcal{L}(z) \equiv z_t - z_{rr} + \frac{1}{r}z_r - \frac{1}{r}zz_r - \frac{\lambda}{|\Omega|}z = 0 \qquad \text{in } (0,R)\times(0,\infty)$$
$$z(0,t) = z(R,t) = 0 \qquad \text{in } (0,\infty). \tag{35}$$

Lemma 9 *We have $W(R,t) < -R^2$ for $t > 0$, where*

$$W(r,t) = \int_0^r z(s,t)sds.$$

Proof: From the first equation of (1), we have

$$\begin{aligned}\int_\Omega |x|^2 u_t dx &= -\int_\Omega 2x \cdot (\nabla u - u\nabla v)dx \\ &= -2\int_{\partial\Omega} [(x\cdot\nu)u]d\sigma + 4\lambda + 4\pi \int_0^R r^2(uv_r)dr,\end{aligned}$$

while (32) implies

$$-rv_r(r,t) = \int_0^r s\left(u(s,t) - \frac{\lambda}{|\Omega|}\right)ds \qquad (0 < r < R).$$

Thus, we obtain

$$\begin{aligned}&\int_0^R (uv_r)(r,t)r^2dr = -\int_0^R ru(r,t)\left\{\int_0^r su(s,t)ds - \frac{\lambda r^2}{2|\Omega|}\right\}dr \\ &= -\frac{1}{2}\left\{\int_0^R ru(r,t)dr\right\}^2 + \frac{\lambda}{2|\Omega|}\int_0^R r^3u(r,t)dr \\ &= -\frac{\lambda^2}{8\pi^2} + \frac{\lambda}{4\pi|\Omega|}\int_\Omega |x|^2 udx,\end{aligned}$$

and therefore,

$$\frac{dm}{dt} = 4\lambda - \frac{\lambda^2}{2\pi} + \frac{\lambda}{|\Omega|}m - 4\pi R^2 u(R,t) \tag{36}$$

for

$$m(t) = \int_\Omega |x|^2 u(x,t)dx.$$

We shall show

$$4\lambda - \frac{\lambda^2}{2\pi} + \frac{\lambda}{|\Omega|}m(t) > 0 \qquad (t > 0). \tag{37}$$

In fact, if this is not the case, we have $t_1 > 0$ such that

$$4\lambda - \frac{\lambda^2}{2\pi} + \frac{\lambda}{|\Omega|} m(t_1) \leq 0.$$

Then, the standard continuity argument applied to (36) guarantees $m_t < 0$ in (t_1, ∞), and also $m(t_2) < 0$ for some $t_2 > t_1$. This is a contradiction, and hence we obtain (37).

We have, on the other hand,

$$\begin{aligned} m(t) &= 2\pi \int_0^R r^3 u(r,t) dr = 2\pi \int_0^R r^2 \left[\frac{d}{dr} \int_0^r u(\rho,t)\rho d\rho \right] dr \\ &= 2\pi \left\{ \left[r^2 \int_0^r u(\rho,t)\rho d\rho \right]_{r=0}^{r=R} - \int_0^R 2r \left[\int_0^r u(\rho,t)\rho d\rho \right] dr \right\} \\ &= \frac{\lambda R^2}{2} - 4\pi W(R,t), \end{aligned}$$

and therefore, $W(R,t) < R^2$ for $t > 0$ by (37). The proof is complete.

Henceforth, we write $t'_k = t_k$ in (33) for simplicity. Then, $z^k(r,t) = z(r, t + t_k)$ $(k = 1, 2, \cdots)$ is uniformly bounded and locally equi-continuous in $(0, R) \times (-\infty, +\infty)$ by the second inequality of (34), and therefore, there is a subsequence, denoted by the same symbol, converging locally uniformly in $(0, R) \times (-\infty, \infty)$. From the parabolic regularity, this limit function, denoted by $z^\infty = z^\infty(r,t)$ belongs to $C^{2,1}((0, R) \times (-\infty, \infty))$ and satisfies

$$\mathcal{L}(z^\infty) = 0 \qquad \text{in } (0, R) \times (-\infty, \infty). \tag{38}$$

We have, furthermore, $z^\infty \in C([0, R] \times (-\infty, \infty))$ and

$$\begin{aligned} &z^\infty(0,t) = 4, \quad z^\infty(R,t) = 0 && \text{for } t \in (-\infty, \infty) \\ &z^\infty(r,t) \geq 4 - \frac{\lambda r^2}{2\pi R^2} && \text{for } (r,t) \in (0, R) \times (-\infty, \infty) \end{aligned} \tag{39}$$

by (33) and (35).

Proof of Theorem 3: Using $\lambda > 8\pi$, we take $\varepsilon \in (0, \min(\lambda - 8\pi, \pi))$ and then define

$$z_*(r,t) = 4 - \frac{\lambda r^2}{2\pi R^2} + \frac{\varepsilon}{2\pi R^2}(r - \ell(t))_+^2,$$

where

$$\ell(t) = R\exp\left(-\left(1-\frac{\varepsilon}{\pi}\right)\frac{t}{R^2}\right) \in (0, R).$$

We have

$$\mathcal{L}(z_*) = \left[-\frac{\varepsilon}{\pi R^2}(r-\ell(t))_+\ell'(t)\right] - \left[-\frac{\lambda}{\pi R^2} + \frac{\varepsilon}{\pi R^2}\chi_{[\ell(t),R]}(r)\right]$$
$$+\frac{1}{r}\left[-\frac{\lambda r}{\pi R^2} + \frac{\varepsilon}{\pi R^2}(r-\ell(t))_+\right] - \frac{1}{r}\left[-\frac{\lambda r}{\pi R^2} + \frac{\varepsilon}{\pi R^2}(r-\ell(t))_+\right]z_* - \frac{\lambda z_*}{|\Omega|}$$

for

$$\chi_E(r) = \begin{cases} 1 & \text{if} \quad r \in E \\ 0 & \text{if} \quad r \notin E, \end{cases}$$

and therefore, if $r \in (0, \ell(t))$, we have

$$\mathcal{L}(z_*) = \frac{\lambda}{\pi R^2} - \frac{\lambda}{\pi R^2} + \frac{\lambda}{\pi R^2}z_* - \frac{\lambda}{|\Omega|}z_* = 0.$$

In the other case of $r \in (\ell(t), R)$, we have

$$\mathcal{L}(z_*) = -\frac{\varepsilon}{\pi R^2}(r-\ell(t))\ell'(t) - \frac{\varepsilon}{\pi R^2} + \frac{\varepsilon}{\pi R^2 r}(r-\ell(t))$$
$$-\frac{\varepsilon}{\pi R^2 r}(r-\ell(t))\left[4 - \frac{\lambda r^2}{2\pi R^2} + \frac{\varepsilon}{2\pi R^2}(r-\ell(t))^2\right]$$
$$= -\frac{\varepsilon}{\pi R^2}(r-\ell(t))\left[\ell'(t) + \frac{\ell(t)}{r(r-\ell(t))}\right.$$
$$\left.+\frac{1}{r}\left(4 - \frac{\lambda r^2}{2\pi R^2}\right) + \frac{\varepsilon}{2\pi R^2 r}\left(r^2 - 2r\ell(t) + \ell^2(t)\right)\right]$$
$$\leq -\frac{\varepsilon}{\pi R^2}(r-\ell(t))\left[\ell'(t) + \frac{\ell(t)}{r(r-\ell(t))} - \frac{\varepsilon\ell(t)}{\pi R^2} + \frac{1}{r}\left(4 - \frac{\lambda-\varepsilon}{2\pi R^2}r^2\right)\right]$$
$$\leq -\frac{\varepsilon}{\pi R^2}(r-\ell(t))\left[\ell'(t) + \frac{\ell(t)}{R^2}\left(1-\frac{\varepsilon}{\pi}\right) + \frac{1}{r}\left(4 - \frac{\lambda-\varepsilon}{2\pi R^2}r^2\right)\right]$$
$$\leq -\frac{\varepsilon}{\pi R^2}(r-\ell(t))\left[\ell'(t) + \left(1-\frac{\varepsilon}{\pi}\right)\frac{\ell(t)}{R^2}\right] = 0.$$

Thus, we obtain

$$\mathcal{L}(z_*) \leq 0 \qquad \text{in } (0, R) \times (-\infty, \infty).$$

We have also

$$\begin{aligned}
z_*(R,t) &= 4 - \frac{\lambda}{2\pi} + \frac{\varepsilon}{2\pi R^2}(R-\ell(t))_+^2 \le 4 - \frac{\lambda-\varepsilon}{2\pi} < 0 \\
z_*(0,t) &= 4 \\
z^\infty(r,t-T) &\ge 4 - \frac{\lambda r^2}{2\pi R^2} = z_*(r,0)
\end{aligned}$$

by (39), and therefore,

$$z^\infty(r,t) \ge z_*(r,T+t) \qquad \text{for } (r,t) \in (0,R)\times(0,\infty)$$

for any $T>0$ from the comparison theorem. By making $T\to\infty$, we obtain

$$z^\infty(r,t) \ge 4 - \frac{\lambda-\varepsilon}{2\pi R^2}r^2 \qquad \text{in } (0,R)\times(0,\infty).$$

Since $\varepsilon \in (0, \min(0, \lambda - 8\pi))$ is arbitrary, this implies

$$z^\infty(r,t) \ge 4 - \frac{\lambda_1 r^2}{R^2} \qquad \text{in } (0,R)\times(0,\infty), \tag{40}$$

where $\lambda_1 = \min(\lambda - 8\pi, \pi)$. If $\lambda \in (8\pi, 9\pi]$, then (40) reads;

$$z^\infty(r,t) \ge 4 - \frac{4r^2}{R^2} \qquad \text{in } (0,R)\times(0,\infty). \tag{41}$$

In the other case of $\lambda > 9\pi$, we define z_*, replacing λ by λ_1. Then, from the same argument it follows that

$$z^\infty(r,t) \ge 4 - \frac{\lambda_2 r^2}{2\pi R^2} \qquad \text{in } (0,R)\times(0,\infty),$$

where $\lambda_2 = \min(\lambda_1 - 8\pi, \pi) = \min(\lambda - 9\pi, \pi)$. Repeating this, we obtain (41) if $\lambda > 8\pi$.

Lemma 9, on the other hand, guarantees

$$W^\infty(R,t) \equiv \int_0^R z^\infty(r,t)rdr \le R^2,$$

while (41) implies

$$W^\infty(R,t) \ge \int_0^R \left(4 - \frac{4r^2}{R^2}\right) rdr = R^2.$$

This means

$$W^{\infty}(,Rt) = R^2 \qquad \text{in } (0,\infty),$$

or equivalently,

$$z^{\infty}(r,t) = 4 - \frac{4r^2}{R^2} \qquad \text{in } (0,R)\times(0,\infty).$$

However, this is impossible by

$$\mathcal{L}(z^{\infty}) = \frac{\lambda - 8\pi}{|\Omega|} z^{\infty} \neq 0.$$

The proof is complete.

A Proof of (9)

We use the Lagrange functional defined by (5):

$$\mathcal{W}(u,v) = \int_{\Omega} u(\log u - 1)dx + \frac{1}{2}\|v\|_V^2 - \int_{\Omega} uvdx,$$

which satisfies

$$\mathcal{W}(u(\cdot,t), v(\cdot,t)) = \mathcal{F}(u(\cdot,t)) \leq \mathcal{F}(u_0) \tag{42}$$

for the solution $(u,v) = (u(\cdot,t), v(\cdot,t))$ to (1). Seince $u \log u + e^{-1} \geq 0$, we have

$$\mathcal{W}(u,v) \geq -|\Omega|\, e^{-1} - \lambda - \int_{\Omega} uvdx$$

and therefore,

$$\lim_{t\uparrow T_{\max}} \int_{\Omega} (uv)(x,t)dx = \mathcal{K}(u(\cdot,t)) = +\infty \tag{43}$$

from the assumption $\lim_{t\uparrow T_{\max}} \mathcal{F}(u(\cdot,t)) = -\infty$. On the other hand, we can apply the L^1 elliptic estimate [5] to the second equation of (1) by $\|u(\cdot,t)\|_1 = \lambda$, and this implies

$$\sup_{t\in[0,T_{\max})} \|v(\cdot,t)\|_{W^{1,q}} < +\infty \tag{44}$$

for each $q \in [1,2)$.

By Chang-Yang's inequality [10], we have a constant K determined by Ω such that

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^w dx\right) \leq \frac{1}{8\pi}\|\nabla w\|_2^2 + \frac{1}{|\Omega|}\int_\Omega w dx + K$$

for any $w \in H^1(\Omega)$. For each $b > 0$, therefore, we have

$$\log\left(\int_\Omega e^{bv}dx\right) \leq \frac{b^2}{8\pi}\|\nabla v\|_2^2 + \frac{b}{|\Omega|}\|v\|_1 + K + \log|\Omega| = \frac{b^2}{4\pi}W(u,v)$$
$$-\frac{b^2}{4\pi}\int_\Omega u(\log u - 1)dx + \frac{b^2}{4\pi}\int_\Omega uvdx + \frac{b}{|\Omega|}\|v\|_1 + K + \log|\Omega|,$$

while

$$b\int_\Omega uvdx \leq \int_\Omega u(\log u - 1)dx + \lambda\log\left(\int_\Omega e^{bv}dx\right) - \lambda\log\lambda + \lambda$$

follows from (11). Using these inequalities, we obtain

$$b\left(1-\frac{b\lambda}{4\pi}\right)\int_\Omega uvdx \leq \left(1-\frac{b^2\lambda}{4\pi}\right)\int_\Omega u(\log u - 1)dx$$
$$+\lambda\left\{\frac{b^2}{4\pi}\mathcal{W}(u,v) + \frac{b}{|\Omega|}\|v\|_1 + K + \log|\Omega|\right\} - \lambda\log\lambda + \lambda.$$

Then, taking $0 < b < \min\left\{\frac{4\pi}{\lambda}, \left(\frac{4\pi}{\lambda}\right)^{1/2}\right\}$ and $(u,v) = (u(\cdot,t), v(\cdot,t))$, we have

$$\lim_{t\uparrow T_{\max}}\int_\Omega u(\log u - 1)(x,t)dx = +\infty$$

by (43), (44), and (42). The proof is complete.

B Proof of Lemma 8

We use several terminologies of the statistical mechanics. First, we have $G(x,x') > 0$ because it is associated with the (D) field, and therefore, (minus) potential energy is positive for $\rho \in P(\Omega)$:

$$\mathcal{K}(\rho) = \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')\rho(x)\rho(x')dxdx' > 0.$$

By Proposition 2.1 of [7], we define the entropy functional

$$\mathcal{E}(\rho) = -\int_\Omega \rho(\log \rho - 1)dx,$$

and obtain

$$E(s) \equiv \sup_{\rho \in P(\Omega), \mathcal{K}(\rho)=s} \mathcal{E}(\rho) < +\infty$$

for each $s > 0$, and furthermore, this value is attained by some element, denoted by $\rho_s \in P(\Omega)$, satisfying $\mathcal{K}(\rho_s) = s$. Then, Theorem 6.1 of [7] reads;

Theorem 10 *Given $s_k \to +\infty$, we have $\{s'_k\} \subset \{s_k\}$ such that*

$$\rho_{s'_k}(x)dx \to \delta_{x_\infty}(dx) \qquad \text{*-weakly in } \mathcal{M}(\overline{\Omega})$$

with $x_\infty \in \Omega$ satisfying $R(x_\infty) = \sup_{x\in\Omega} R(x)$.

Our Lemma 8 is an extension, and follows from a similar argument. In fact, it is easy to see that this lemma is equivalent to the following theorem, where

$$\mathcal{E}^\Delta(\rho) = E(\mathcal{K}(\rho)) - \mathcal{E}(\rho).$$

Theorem 11 *If $\{\rho_k\} \subset P(\Omega)$ satisfies*

$$\lim_{k\to\infty} \mathcal{K}(\rho_k) = +\infty$$
$$\lim_{k\to\infty} \mathcal{E}^\Delta(\rho_k) = E^\Delta_\infty < +\infty, \tag{45}$$

then we have $\{\rho'_k\} \subset \{\rho_k\}$ such that

$$\rho'_k(x)dx \to \delta_{x_\infty}(dx) \qquad \text{*-weakly in } \mathcal{M}(\overline{\Omega}) \tag{46}$$

with $x_\infty \in \Omega$ satisfying

$$R(x_\infty) \geq \sup_{x\in\Omega} R(x) - \frac{1}{4\pi}E^\Delta_\infty. \tag{47}$$

To prove this theorem, we use a fact obtained in the proof of Lemma 6.2 of [7], which is regarded as an improved dual Trudinger-Moser inequality. In fact, usual dual Trudinger-Moser inequality (22) is represented as

$$\sup_{\rho\in P(\Omega)} \mathcal{I}_{8\pi}(\rho) < +\infty,$$

where

$$\begin{aligned}\mathcal{I}_\beta(\rho) &= \mathcal{K}(\rho) + \frac{1}{\beta}\mathcal{E}(\rho) \\ &= \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')\rho(x)\rho(x')dxdx' - \frac{1}{\beta}\int_\Omega \rho(\log\rho - 1)dx.\end{aligned}$$

Lemma 12 *Each $d > 0$ admits $C = C(d)$ such that if $m > 0$, then we have $\beta = \beta(m) > 8\pi$ such that $\mathcal{I}_\beta(\rho) \leq C$ for any $\rho \in P(\Omega)$ satisfying*

$$\int_{A_1} \rho dx, \ \int_{A_2} \rho dx \geq m,$$

where $A_1, A_2 \subset \Omega$ are measurable sets with $dist(A_1, A_2) \geq d$.

Now, we give the following.

Proof of Theorem 11: First, we show

$$\lim_{k\to\infty} \{1 - Q_k(r)\} = 0 \tag{48}$$

for each $0 < r \ll 1$, where

$$Q_k(r) = \sup_{y\in\overline{\Omega}} \int_{\Omega\cap B(y,r)} \rho_k(x)dx$$

denotes the concentration function of $\rho_k = \rho_k(x)$.

In fact, defining $x_k \in \overline{\Omega}$ by

$$\int_{\Omega\cap B(x_k, r/2)} \rho_k(x)dx = Q_k(r/2),$$

we have

$$1 - Q_k(r) \leq 1 - \int_{\Omega\cap B(x_k,r)} \rho_k(x)dx = \int_{\Omega\setminus B(x_k,r)} \rho_k(x)dx,$$

and therefore,

$$\min\{Q_k(r/2), 1-Q_k(r)\} \leq \min\left\{\int_{\Omega\cap B(x_k,r/2)} \rho_k(x)dx, \int_{\Omega\setminus B(x_k,r)} \rho_k(x)dx\right\}.$$

If we apply Lemma 12 for $d = r/2$, then we have $C = C(d)$ and $\beta = \beta(m) > 8\pi$ for each $m > 0$ such that

$$m \leq \min\{Q_k(r/2), 1-Q_k(r)\} \quad \Rightarrow \quad \mathcal{I}_\beta(\rho_k) \leq C.$$

Proposition 6.1 of [7], on the other hand, guarantees

$$-8\pi s - C_1 \leq E(s)\left(= \sup_{\rho\in P(\Omega), \mathcal{K}(\rho)=s} \mathcal{E}(\rho)\right) \qquad (s \gg 1) \tag{49}$$

with a constant C_1, and hence it follows that

$$\mathcal{I}_\beta(\rho_k) = \mathcal{K}(\rho_k) + \frac{1}{\beta}\mathcal{E}(\rho_k) = \mathcal{K}(\rho_k) - \frac{1}{\beta}\mathcal{E}^\Delta(\rho_k) + \frac{1}{\beta}E(\mathcal{K}(\rho_k))$$
$$\geq \left(1 - \frac{8\pi}{\beta}\right)\mathcal{K}(\rho_k) - \frac{C_1}{\beta} - \frac{1}{\beta}\mathcal{E}^\Delta(\rho_k) \to +\infty$$

as $k \to \infty$. From these relations, we obtain

$$\lim_{k\to\infty} \min\{Q_k(r/2), 1-Q_k(r)\} = 0.$$

Here, we have $Q_k(r) \geq cr^2$ for $k = 1, 2, \cdots$ and $0 < r \ll 1$ by the standard convering argument, and therefore, (48) follows.

Next, we show that (46) holds with some $x_\infty \in \overline{\Omega}$, passing to a subsequence. In fact, since $\Omega \subset \mathbf{R}^2$ is bounded, we have

$$\overline{x_k} \equiv \int_\Omega x\rho_k(x)dx \to x_\infty \in \mathbf{R}^2,$$

passing to a subsequence. Then, for each $0 < r \ll 1$, we have $1 - Q_k(r/2) \leq r$ if k is large by (48). In this case, it holds that

$$|\overline{x_k} - x_k| = \left|\int_\Omega (x - x_k)\rho_k(x)dx\right| \leq \int_{\Omega\cap B(x_k,r)} |x - x_k|\,\rho_k(x)dx$$
$$+ \int_{\Omega\setminus B(x_k,r)} |x - x_k|\,\rho_k(x)dx \leq r + \text{diam}\,\Omega \cdot \int_{\Omega\setminus B(x_k,r/2)} \rho_k(x)dx$$
$$= r + \text{diam}\,\Omega \cdot (1 - Q_k(r/2)) \leq (1 + \text{diam}\,\Omega)\,r,$$

and therefore,

$$\lim_{k\to\infty} |\overline{x_k} - x_k| = 0.$$

In particular, it holds that $x_\infty \in \overline{\Omega}$. Similarly, we have

$$\left|\zeta(x_k) - \int_\Omega \zeta(x)\rho_k(x)dx\right| \leq \int_{\Omega\cap B(x_k,r)} |\zeta(x_k) - \zeta(x)|\,\rho_k(x)dx$$
$$+ \int_{\Omega\setminus B(x_k,r)} |\zeta(x_k) - \zeta(x)|\,\rho_k(x)dx = o(1)$$

for each $\zeta = \zeta(x) \in C(\overline{\Omega})$, and therefore,

$$\lim_{k\to\infty} \left|\zeta(x_k) - \int_\Omega \zeta(x)\rho_k(x)dx\right| = 0.$$

Thus, using

$$\left|\zeta(x_\infty) - \int_\Omega \zeta(x)\rho_k(x)dx\right| \leq |\zeta(x_\infty) - \zeta(\overline{x_k})|$$
$$+ |\zeta(\overline{x_k}) - \zeta(x_k)| + \left|\zeta(x_k) - \int_\Omega \zeta(x)\rho_k(x)dx\right|,$$

we have

$$\lim_{k\to\infty} \left|\zeta(x_\infty) - \int_\Omega \zeta(x)\rho_k(x)dx\right| = 0,$$

which means (46).

We show (47) and complete the proof. In fact, we have

$$\mathcal{I}(\rho_k) = \mathcal{I}_{8\pi}(\rho_k) = \mathcal{K}(\rho_k) + \frac{1}{8\pi}\mathcal{E}(\rho_k)$$
$$= \mathcal{K}(\rho_k) - \frac{1}{8\pi}\mathcal{E}^\Delta(\rho_k) + \frac{1}{8\pi}E(\mathcal{K}(\rho_k)) \geq -\frac{1}{8\pi}\mathcal{E}^\Delta(\rho_k) - \frac{C_1}{8\pi}$$

by (49), and therefore,

$$\liminf_{k\to\infty} \mathcal{I}(\rho_k) > -\infty \tag{50}$$

from the assumption. We have

$$G(x,x') = \frac{1}{2\pi}\log|x-x'|^{-1} + K(x,x')$$

with $R(x) = K(x,x) \to -\infty$ as $x \to \partial\Omega$, and also

$$\begin{aligned}\mathcal{I}(\rho_k) &= \frac{1}{4\pi}\iint_{\Omega\times\Omega} \log|x-x'|^{-1}\rho_k(x)\rho_k(x')dxdx' \\ &-\frac{1}{8\pi}\int_\Omega(\rho_k\log\rho_k)(x)dx + \frac{1}{2}\iint_{\Omega\times\Omega} K(x,x')\rho_k(x)\rho_k(x')dxdx' \\ &\le C + \frac{1}{2}\iint_{\Omega\times\Omega} K(x,x')\rho_k(x)\rho_k(x')dxdx'\end{aligned}$$

by the logarithmic Hardy-Littlewood-Sobolev inequality [8, 3], and therefore, (46) with $x_\infty \in \overline{\Omega}$ implies $x_\infty \in \Omega$ in the case of (50).

Equality (24), on the other hand, implies a sharp form of (49):

$$\lim_{s\uparrow+\infty}\left\{s + \frac{1}{8\pi}E(s)\right\} = \sup_{x\in\Omega} I_\Omega(x), \tag{51}$$

for $I_\Omega(x)$ defined by (23). (See the proof of Theorem 3.1 of [7].) Then, we obtain

$$\begin{aligned}I_\Omega(x_\infty) &\ge \limsup_{k\to\infty}\mathcal{I}(\rho_k) = \limsup_{k\to\infty}\left\{\mathcal{K}(\rho_k) + \frac{1}{8\pi}\mathcal{E}(\mathcal{K}(\rho_k)) - \frac{1}{8\pi}\mathcal{E}^\Delta(\rho_k)\right\} \\ &= \limsup\left\{\mathcal{K}(\rho_k) + \frac{1}{8\pi}\mathcal{E}(\mathcal{K}(\rho_k))\right\} - \frac{1}{8\pi}E_\infty^\Delta = \sup_{x\in\Omega} I_\Omega(x) - \frac{1}{8\pi}E_\infty^\Delta\end{aligned}$$

by (45). This means (47) and the proof is complete.

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