数理解析研究所講究録 1416 巻 2005 年 105-119

L^p and L^∞ a priori estimates for some chemotaxis models and applications to the Cauchy problem

L. Corrias [a], B. Perthame [b] and H. Zaag [b]

January 20, 2005

Abstract

In this note, we generalize some a priori estimates from our earlier work concerning two models of chemotaxis. We also give a new L^∞ bound for these models which yields uniqueness for one of them.

Key words. Chemotaxis, angiogenesis, degenerate parabolic equations, global weak solutions, blow-up.

AMS subject classifications. 35B60; 35Q80; 92C17; 92C50.

1 Introduction.

Chemotaxis is the directed movement of a living system in response to a chemical stimulus. The first mathematical model for chemotaxis has been proposed by Keller and Segel [20] to describe the aggregation of the slime mold amoebae *Dictyostelium discoideum* (see also Patlak [25] for an earlier model and [19], [21] for various assessments). Since then, the mathematical description and analysis of the biological phenomenon has attracted a lot of attention. Indeed, this biochemical process has been involved in so many medical and biological applications. Moreover, from a mathematical point of view, the model has a rich structure and remains very challenging. See Horstmann [16] and [17] for a review on the subject.

This paper is concerned with a simplified version of the Keller-Segel model but however rather general, namely

$$\begin{cases} \partial_t n = \nabla \cdot [\kappa \nabla n - n\chi(c)\nabla c], \quad t>0,\ x\in\Omega, \\ \varepsilon\partial_t c = \eta\Delta c + g(n,c), \quad t>0,\ x\in\Omega, \\ n(0,x)=n_0(x), \quad c(0,x)=c_0(x), \end{cases} \tag{1}$$

where Ω is $\mathbb{R}^d$, $n(x,t)$ is the density of the population moving under chemotaxis and $c(x,t)$ is the density of the chemical substance. The function χ is the chemotactic sensitivity of the species that plays a crucial role in the chemotaxis process. Indeed, $\chi(c) > 0$ leads to positive chemotaxis (the population is attracted by the chemical), while $\chi(c) < 0$ leads to negative chemotaxis (the population is repelled by the chemical). Moreover, in a mathematical language, χ measures the balance between the diffusion effect of the Laplace term and the hyperbolic effect of the drift term and we tried to understand its impact on the dynamics. The function $g(n, c)$ contains the production and decay rate terms of the chemical and is generally taken of the form

$$g(n, c) = \beta(n, c)n - \gamma(n, c)c \ . \tag{2}$$

Finally, we take the diffusion coefficient κ of the population $n(x, t)$ constant with respect to n and c. The diffusion coefficient of the chemical η is non-negative and the parameter $\varepsilon \in \{0, 1\}$. Obviously, η and ε are not equal to zero at the same time. This means that system (1) contains the extreme case of a parabolic-elliptic system ($\varepsilon = 0$ and $\eta > 0$) and a parabolic-degenerate system ($\varepsilon = 1$ and $\eta = 0$). The goal of this paper is to summarize some ideas previously used by the authors to obtain global existence results for these two extreme cases of system (1) (see [10], [11]) and to give some hint to apply these ideas to the more general system (1) together with new results.

The parabolic-elliptic system has been extensively studied by many authors and a huge quantity of mathematical results on the existence of global in time solutions and on the blow-up of local in time solutions, have been produced. We refer to [16] and [17] for a quite complete bibliography. The parabolic-degenerate system has been much less studied than the previous one, except in one space dimension. One reason to be interested in this kind of system is that it arises in modelling the initiation of *angiogenesis*, a kind of chemotaxis process that occurs in the development of a new capillary network from a primary vascular network ([8],[9], [10], [13], [23]). Moreover, it is now clear that the asymptotic behavior of n depends strongly on the coupling effects of the dynamics of c and the chemotactic sensitivity $\chi(c)$ and that is the reason why chemotaxis systems of parabolic-degenerate type have been analyzed in [22], [24], [26], [27].

Most of the analysis presented in [11] about the parabolic-elliptic system was known before, unlike our analysis for the parabolic-degenerate system which we would like to put forward. The parabolic-degenerate system we are interested in is :

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa\Delta n - \nabla \cdot [n\chi(c)\nabla c], \quad t > 0, \ x \in \mathbb{R}^d, \\ \\ \frac{\partial}{\partial t} c = -nc^m, \quad t > 0, \ x \in \mathbb{R}^d, \\ \\ n(0, x) = n_0(x), \qquad c(0, x) = c_0(x), \qquad x \in \mathbb{R}^d \end{cases} \tag{3}$$

where $m \geq 1$ and the sensitivity function $\chi(c)$ is a given positive function on $\mathbb{R}_+$. Our main result for this system is the following:

Theorem 1 *(Existence for the angiogenesis system (3)). Assume $d \geq 2$, $m \geq 1$ and $\chi(c)$ a positive say continuous function on $[0, \infty)$. Consider some $n_0 \in L^1(\mathbb{R}^d)$ and $c_0 \in L^\infty(\mathbb{R}^d)$ such that $n_0 \geq 0$*

and $c_0 \geq 0$. *There exists a constant* $K_0(\kappa, \chi, d, \|c_0\|_{L^\infty(\mathbb{R}^d)})$ *such that if* $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0$, *then system (3) has a global (in time) weak solution* (n, c) *such that* $n \in L^\infty(\mathbb{R}^+, L^1 \cap L^{\frac{d}{2}}(\mathbb{R}^d))$, $c \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ *and for any fixed* $\max\{1; \frac{d}{2} - 1\} \leq p^* < \infty$,

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(t, K_0, p^*, \|n_0\|_{L^p(\mathbb{R}^d)}), \qquad \forall \ \max\{1; \frac{d}{2} - 1\} \leq p \leq p^*.$$

Moreover, if $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq \frac{K_0}{2}$ *and* $n_0 \in L^\infty(\mathbb{R}^d)$, *then*

$$\|n(t)\|_{L^\infty} \leq C\left(\kappa, \chi, \|n_0\|_{L^d(\mathbb{R}^d)}\right) \max\left\{\|n_0\|_{L^\infty}, \|n_0\|_{L^1}\right\}.$$

The same conclusion holds with less regular $\chi(c)$ at $c = 0$ (see [11]). Moreover, since the goal of this note is to give a general idea of the method we use to obtain the global existence of the solution of (1), we consider the system on the whole space $\mathbb{R}^d$ together with ad hoc decay condition at infinity on the densities n and c. In this way, the conservation of the initial mass is assured,

$$\int_{\mathbb{R}^d} n(t) = \int_{\mathbb{R}^d} n_0 =: M, \quad \forall t \geq 0.$$

In the case of a bounded domain Ω, the boundary conditions for n and c can be homogeneous Neumann boundary conditions or the no-flux boundary condition, so that the conservation of the initial mass is again assured and the results given here can be applied.

All the results will be dimension dependent with $d \geq 2$, since in one space dimension global existence of smooth solutions can be obtained by the classical methods of parabolic equations (see for example [13], [26]).

The paper is organized as follows. In section 2, we present a priori estimate in weighted L^p spaces for the general system (1). These estimates are valid for a large class of sub-systems, including the extreme cases mentioned before and analyzed in [11]. In sections 3 and 4, we recall results from [11] in order to show how the a priori estimates of section 2 allow to get the global existence result for the parabolic-elliptic and the parabolic-degenerate cases respectively. Moreover, for both systems, we give also the L^∞ bound not mentioned in [11] and, as a consequence, the uniqueness of the solution of the parabolic-elliptic system.

Acknowledgement. The third author presented this work during the conference "The mechanism of the spatio-temporal pattern arising in reaction diffusion system" which took place in Kyoto University in October 2004. He would like to thank T. Tsujikawa and Y. Kan-on for their kind invitation, as well as H. Matano, H. Ninomiya, D. Hilhorst and C. Oikonomides who made his stay in Japan very fruitful and pleasant.

2 Differential inequalities in weighted L^p spaces.

In this section, we give general differential inequalities for the weighted or not L^p norm of the solution n of (1). These inequalities include all the differential inequalities used in [11] and most of the known in the literature for system (1).

Let us define $u = \frac{n}{\phi(c)}$ where the function ϕ is such that

$$\text{for all } c > 0, \;\; \phi'(c) = \frac{1}{\kappa}\phi(c)\chi(c) \text{ and } \phi(0) = 1, \tag{4}$$

This means that the sensitivity function $\chi(c)$ is necessarily taken to be integrable at $c = 0$. Then, the change of variable $n \;\to\; u$ puts the equation on n in the following divergence form

$$\partial_t u = \kappa \frac{1}{\phi(c)} \nabla \cdot [\phi(c)\nabla u] - u\frac{1}{\phi(c)}\partial_t \phi(c). \tag{5}$$

The equation on u suggests naturally to consider the evolution of the quantity $\int_{\mathbb{R}^d} G(\phi)F(u)\,dx$ where G is a differentiable function on $\mathbb{R}_+$ and F is a twice differentiable function on $\mathbb{R}_+$. Therefore, using (5), we write

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} G(\phi)F(u)\,dx = &\; -\kappa\int_{\mathbb{R}^d} G(\phi)F''(u)|\nabla u|^2\,dx - \kappa\int_{\mathbb{R}^d}\left(\frac{1}{\phi}G(\phi)\right)' F'(u)\phi\nabla\phi\cdot\nabla u\,dx \\
&+ \int_{\mathbb{R}^d}[G'(\phi)F(u) - G(\phi)F'(u)\frac{u}{\phi}]\partial_t\phi\,dx \\
= &\; -\kappa\int_{\mathbb{R}^d} G(\phi)F''(u)|\nabla u|^2\,dx + \kappa\int_{\mathbb{R}^d} F(u)\Delta\psi(\phi)\,dx \\
&+ \int_{\mathbb{R}^d}[G'(\phi)F(u) - G(\phi)F'(u)\frac{u}{\phi}]\partial_t\phi\,dx
\end{aligned} \tag{6}$$

where ψ is such that $\psi'(\phi) = \phi\left(\frac{1}{\phi}G(\phi)\right)'$.

The main differential identity (6) shows how the evolution of the chemical of density c influences the evolution of the population of density n. In order to obtain useful estimates from (6), one has to handle carefully every term in (6).

For $\varepsilon = 1$ and $\eta > 0$, one has to balance the second term in the right hand side of (6) with the third term. For example, for $G(\phi) = \phi^q$ and $F(u) = u^p$ we have

$$\int_{\mathbb{R}^d} F(u)\Delta\psi(\phi)\,dx = \frac{q-1}{q}\int_{\mathbb{R}^d} u^p\Delta\phi^q\,dx = (q-1)^2\int_{\mathbb{R}^d} u^p\phi^{q-2}|\nabla\phi|^2 + (q-1)\int_{\mathbb{R}^d} u^p\phi^{q-1}\Delta\phi\,dx \tag{7}$$

and

$$\int_{\mathbb{R}^d}[G'(\phi)F(u) - G(\phi)F'(u)\frac{u}{\phi}]\partial_t\phi\,dx = (q-p)\int_{\mathbb{R}^d} u^p\phi^{q-1}[\eta(\Delta\phi - \phi''|\nabla c|^2) + \phi' g(n,c)]\,dx\,. \tag{8}$$

Then, using (7) and (8) in (6) we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\phi^q u^p\,dx = &-\kappa p(p-1)\int_{\mathbb{R}^d}\phi^q u^{p-2}|\nabla u|^2\,dx\\ &+[\kappa(q-1)+\eta(q-p)]\int_{\mathbb{R}^d}u^p\phi^{q-1}\Delta\phi\,dx\\ &+\int_{\mathbb{R}^d}u^p\phi^{q-2}[\kappa(q-1)^2\phi'^2-\eta(q-p)\phi\phi'']|\nabla c|^2\,dx\\ &+(q-p)\int_{\mathbb{R}^d}u^p\phi^{q-1}\phi' g(n,c)\,dx.\end{aligned} \tag{9}$$

In order to eliminate the second term in the right-hand side of (9), we choose p and q such that $\kappa(q-1)+\eta(q-p)=0$, i.e. $q=\frac{p+\gamma}{\gamma+1}$ with $\gamma=\frac{\kappa}{\eta}$, and (9) becomes

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\phi^{\frac{p+\gamma}{\gamma+1}}u^p\,dx = &-4\kappa\tfrac{(p-1)}{p}\int_{\mathbb{R}^d}|\nabla u^{p/2}|^2\,dx\\ &+\tfrac{1}{\gamma+1}(p-1)\int_{\mathbb{R}^d}u^p\phi^{\frac{p+\gamma}{\gamma+1}}[\chi'(c)+\frac{1}{\kappa}\frac{p+\gamma}{\gamma+1}\chi^2(c)]|\nabla c|^2\,dx\\ &-\tfrac{1}{\kappa}\tfrac{\gamma}{\gamma+1}(p-1)\int_{\mathbb{R}^d}u^p\phi^{\frac{p+\gamma}{\gamma+1}}\chi(c)\ g(n,c)\,dx.\end{aligned} \tag{10}$$

Next, let us suppose that for the initial density c_0 non-negative and bounded, the density c stays non-negative and bounded. Let us suppose also that there exists $\overline{p}\geq 1$ such that

$$\chi'(c)+\frac{1}{\kappa}\frac{p+\gamma}{\gamma+1}\chi^2(c)\leq 0 \quad \text{for all } p\in[1,\overline{p}] \quad \text{and } c\in[0,\|c\|_{L^\infty}]. \tag{11}$$

Then, for any $p\in[1,\overline{p}]$, the following differential inequality for the weighted L^p norm of u

$$\frac{d}{dt}\int_{\mathbb{R}^d}\phi^{\frac{p+\gamma}{\gamma+1}}u^p\,dx \leq -4\kappa\frac{(p-1)}{p}\int_{\mathbb{R}^d}|\nabla u^{p/2}|^2\,dx-\frac{1}{\kappa}\frac{\gamma}{\gamma+1}(p-1)\int_{\mathbb{R}^d}u^p\phi^{\frac{p+\gamma}{\gamma+1}}\chi(c)\ g(n,c)\,dx \tag{12}$$

holds true. Finally, if the range $[1,\overline{p}]$ of p is sufficiently large with respect to to the dimension d, (12) gives us the estimate of some L^p norms of n under a smallness condition on the initial density n_0 (see subsection 3.1). The dependence of g on n is obviously crucial.

Please note that (11) implies that $\chi(c)$ must be decreasing and in particular that $\chi(c)$ cannot be identically constant.

Let us also observe that, without any hypothesis on $\chi(c)$, letting $\eta\to 0^+$ so that $\gamma\to+\infty$ in (10), we obtain the differential identity below for the parabolic-degenerate case already used in [11]

$$\frac{d}{dt}\int_{\mathbb{R}^d}\phi(c)u^p\,dx=-4\kappa\frac{(p-1)}{p}\int_{\mathbb{R}^d}|\nabla u^{p/2}|^2\,dx-\frac{1}{\kappa}(p-1)\int_{\mathbb{R}^d}u^p\phi(c)\chi(c)g(n,c)\,dx. \tag{13}$$

For $\varepsilon = 0$, the third term in the right hand side of (6) has to be eliminated. This is possible for example taking $G(\phi) = \phi^p$ and $F(u) = u^p$. With this choice and after some computation, (6) becomes nothing else than the evolution equation for the L^p norm of n

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} n^p\,dx &= -\kappa p(p-1)\int_{\mathbb{R}^d}\phi^p u^{p-2}|\nabla u|^2\,dx - \kappa p(p-1)\int_{\mathbb{R}^d} n^{p-1}\nabla\phi\cdot\nabla u\,dx \\ &= -4\kappa\tfrac{(p-1)}{p}\int_{\mathbb{R}^d}|\nabla n^{p/2}|^2\,dx + (p-1)\int_{\mathbb{R}^d}\chi(c)\nabla n^p\cdot\nabla c\,dx.\end{aligned} \tag{14}$$

In the case of constant sensitivity, from (14), one obtains an estimate on the L^p norm of n. See Section 3 below.

Remark. Following [15], one can also obtain Lyapunov functions for the system (1) with $\eta > 0$. Indeed, for $G(\phi) = \phi$ and if $[F(u) - uF'(u)]\phi' = -g(n, c)$, then the function

$$H(n(t), c(t)) = \int_{\mathbb{R}^d}\phi(c)F\left(\frac{n}{\phi(c)}\right)\,dx + \frac{\eta}{2}\int_{\mathbb{R}^d}|\nabla c|^2\,dx$$

is a Lyapunov function since from (6) we have

$$\begin{aligned}\frac{d}{dt}H(n(t), c(t)) &= -\kappa\int_{\mathbb{R}^d}\phi F''(u)|\nabla u|^2\,dx - \int_{\mathbb{R}^d} g(n,c)\partial_t c\,dx - \eta\int_{\mathbb{R}^d}\Delta c\,\partial_t c\,dx \\ &= -\kappa\int_{\mathbb{R}^d}\phi F''\left(\frac{n}{\phi(c)}\right)\left|\frac{\nabla n}{\phi} - \frac{n}{\phi^2}\nabla\phi\right|^2\,dx - \varepsilon\int_{\mathbb{R}^d}(\partial_t c)^2\,dx \le 0.\end{aligned} \tag{15}$$

3 The parabolic-elliptic system.

In this paragraph, we consider the parabolic-elliptic system given by (1) when $\varepsilon = 0$, i.e.

$$\begin{cases} \frac{\partial}{\partial t}n = \kappa\Delta n - \chi\nabla\cdot[n\nabla c], \quad t > 0,\ x\in\Omega, \\ -\Delta c = n - cf(c), \quad t>0,\ x\in\Omega, \\ n(0,x) = n_0(x), \qquad x\in\Omega. \end{cases} \tag{16}$$

where f is a non-negative function, $\chi(c)$ is constant with respect to c and for simplicity in the notations, we take $\eta = 1$.

When $f(c) \equiv 0$, the validity of (16) in the framework of chemotaxis is supported by some experiments on the *Escherichia Coli* bacterium (see [6] and the references therein), even if this model does not seem to reproduce some of the observed chemotactic movement. Moreover, in this case ($f \equiv 0$), system (16) has other interesting physical interpretations. For example, it arises in astrophysics and in statistical mechanics (see [2], [3], [4] and the references therein).

Concerning the chemotaxis system (16), the situation is well established and we recall the following results.

Theorem 2 *(Existence and uniqueness for the chemotaxis system (16)) Assume $d \geq 2$ and consider some $n_0 \in L^1(\mathbb{R}^d)$ such that $n_0 \geq 0$. There exists a constant $K_0(\kappa, \chi, d)$ such that if $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0$, then system (16) has a global (in time) weak solution (n, c) such that for all $t > 0$*

$$\|n(t)\|_{L^1(\mathbb{R}^d)} = \|n_0\|_{L^1(\mathbb{R}^d)}, \qquad \|n(t)\|_{L^p(\mathbb{R}^d)} \leq \|n_0\|_{L^p(\mathbb{R}^d)}\ , \quad \max\{1; \frac{d}{2} - 1\} \leq p \leq \frac{d}{2},$$

and

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(t, K_0, \|n_0\|_{L^p(\mathbb{R}^d)}) \qquad \frac{d}{2} < p < \infty.$$

Moreover, if $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq \frac{K_0}{2}$ and if $n_0 \in L^\infty(\mathbb{R}^d)$, then the solution is unique with

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq \|n_0\|_{L^p(\mathbb{R}^d)}\ , \quad \max\{1; \frac{d}{2} - 1\} \leq p \leq d,$$

and

$$\|n(t)\|_{L^\infty} \leq C\left(\kappa, \chi, \|n_0\|_{L^d(\mathbb{R}^d)}\right) \max\{\|n_0\|_{L^1}, \|n_0\|_{L^\infty}\}.$$

The results of this theorem were known before the authors' paper [11], except for three facts :

(i) the smallness condition for the global existence is needed only in the critical norm $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$,

(ii) the L^∞ bound of n,

(iii) the uniqueness of the solution in the case of bounded initial data n_0.

We established (i) in the proof of Theorem 2 given in [11]. Therefore, we will give only some comments about the existence issue (for the whole proof, please see [11]). Let us just remark that a smallness assumption is nevertheless necessary in the above theorem. Indeed, it is known since [18] that blow-up may occur in two space dimensions for large initial data. The L^∞ bound and the uniqueness result were not mentioned in [11], hence we will give the proof later in this section.

Finally, let us remark that in the case of the whole space $\mathbb{R}^d$ under consideration and with the ad hoc decay conditions at infinity on n and c, the chemical density can be represented exactly

$$c(t, x) = \int_{\mathbb{R}^d} E_d(x - y) n(t, y) dy \tag{17}$$

when f is constant, say $f \equiv \alpha$, and where E_d is the fundamental solution of $(-\Delta + \alpha I)$. However, we prove Theorem 2 without using (17). That is the reason why our proof extends naturally to the case of a non constant f.

Remark. System (16) has the well known conserved energy (see for example [5], [14] and [15]) given by

$$\frac{d}{dt} \int \left\{ \kappa n(\ln n - 1) - \frac{\chi}{2}\left(\alpha c^2 + |\nabla c|^2\right) \right\} dx = -\int n |\nabla(\kappa \ln n - \chi c)|^2 dx \tag{18}$$

when f is constant, say $f(c) = \alpha$. Let us just mention that this energy follows from (6) when we take $G(\phi) = \phi$, $F(u) = u \ln u$ and χ constant. Indeed, in this case ϕ defined in (4) is given by $\phi(c) = \exp\left(\frac{\chi}{\kappa} c\right)$. Therefore, (6) becomes

$$\frac{d}{dt} \left\{ \int n \ln n - \frac{\chi}{\kappa} \int nc \right\} dx = -\kappa \int n |\nabla(\ln n - \frac{\chi}{\kappa} c)|^2 dx - \frac{\chi}{\kappa} \int n\ \partial_t c\ dx. \tag{19}$$

Using the fact that $n = -\Delta c + cf(c)$ (see (16)) in the second term of both the right and left hand side of (19), we obtain

$$\frac{d}{dt}\int\left\{n\ln n - \frac{\chi}{2\kappa}\left(|\nabla c|^2 + \alpha c^2\right) + \frac{\chi}{\kappa}[\psi(c) - c^2 f(c)]\right\}dx = -\kappa\int n|\nabla(\ln n - \frac{\chi}{\kappa}c)|^2 dx,$$

which is precisely (18).

3.1 The existence issue in Theorem 2.

First of all, let us observe that by the maximum principle $n > 0$ and $c \geq 0$ if $n_0 \geq 0$, while with the decay at infinity condition on n and c, the initial mass $M = \int_{\mathbb{R}^d} n_0(x)\, dx$ is conserved. Next, for constant χ the identity (14) becomes

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^p + 4\kappa\frac{p-1}{p}\int_{\mathbb{R}^d}|\nabla n^{p/2}|^2 \leq \chi(p-1)\int_{\mathbb{R}^d} n^{p+1}\,. \tag{20}$$

Using the Gagliardo-Nirenberg-Sobolev inequality (see [7], [12]) in space dimension $d > 2$, we get for any $p \geq \frac{d}{2} - 1$ (so that : $\frac{d}{2} \leq p+1 \leq \frac{dp}{d-2}$)

$$\int_{\mathbb{R}^d} n^{p+1} \leq \|n\|^p_{L^{\frac{pd}{d-2}}(\mathbb{R}^d)}\,\|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} = \|n^{p/2}\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}\,\|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq C(d)\|\nabla n^{p/2}\|^2_{L^2(\mathbb{R}^d)}\,\|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}\,. \tag{21}$$

In dimension $d = 2$, following [18] (see also [7]), we have in a similar way for any $p > 0$

$$\int_{\mathbb{R}^2} n^{p+1} \leq C\left(\frac{p+1}{p}\right)^2 \|\nabla n^{p/2}\|^2_{L^2(\mathbb{R}^2)}\,\|n\|_{L^1(\mathbb{R}^2)}\,. \tag{22}$$

Inequalities (21) and (22), together with (20), lead to the main differential inequality in any dimension $d \geq 2$

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^p \leq (p-1)\|\nabla n^{p/2}\|^2_{L^2(\mathbb{R}^d)}\left[\chi\tilde{C}(d)\|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p}\right], \qquad \forall\ \max\{1; \frac{d}{2} - 1\} \leq p < \infty\,. \tag{23}$$

In dimension $d = 2$, (23) means that if the initial mass M is sufficiently small, in terms of $p \geq 1$, then the $\|n(t)\|_{L^p(\mathbb{R}^2)}$ norm (for the same p) decreases for all times $t \geq 0$.

In dimension $d > 2$ and for $p = \frac{d}{2}$, (23) gives us that whenever we have initially

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{8\kappa}{d} \leq 0\,, \tag{24}$$

the $\|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ norm decreases for all times $t \geq 0$. As a consequence, whenever (24) holds true, all the $\|n(t)\|_{L^p(\mathbb{R}^d)}$ norms, with $\max\{1; \frac{d}{2} - 1\} \leq p \leq \frac{d}{2}$, decrease for all times $t \geq 0$. In the same way, if the initial density n_0 satisfies

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p^*} \leq 0\,, \tag{25}$$

for $\frac{d}{2} < p^* < \infty$, then (24) follows again and all the $\|n(t)\|_{L^p(\mathbb{R}^d)}$ norms with $\max\{1; \frac{d}{2} - 1\} \le p \le p^*$ decrease for all times $t \ge 0$.

In conclusion, a smallness condition depending on $\frac{1}{p}$ with $\max\{1; \frac{d}{2} - 1\} \le p < \infty$ on the initial density n_0 is sufficient to get a decreasing L^p norm of n, in any dimension $d \ge 2$. However, if one writes (20) for $(n - K)_+$ with $K > 0$ large enough and uses the same techniques, then we get rid of the dependence of the smallness condition on p (see [11] for details). However, the L^p bound we get this way depends also on t.

Finally, using a regularizing method, we derive from the a priori estimates the existence of solutions for system (16) and we get the first part of Theorem 2.

3.2 The L^∞ bound in Theorem 2.

Let us prove now that it is sufficient to control the L^d norm of n in order to obtain the L^∞ bound of n, whenever $n_0 \in L^\infty$. Indeed, using straightforward interpolation and the Gagliardo-Nirenberg-Sobolev inequality, we have in dimension $d = 2$

$$\int_{\mathbb{R}^d} n^{p+1} \le \|n\|_{L^2} \|n^{\frac{p}{2}}\|_{L^4}^2 \le C \|n\|_{L^2} \|n^{\frac{p}{2}}\|_{L^2} \|n^{\frac{p}{2}}\|_{H^1} , \tag{26}$$

while in dimension $d > 2$ it holds true that

$$\int_{\mathbb{R}^d} n^{p+1} \le \|n^{\frac{p}{2}}\|_{L^{\frac{2d}{d-2}}} \|n^{\frac{p}{2}+1}\|_{L^{\frac{2d}{d+2}}} \le C \|\nabla n^{\frac{p}{2}}\|_{L^2} \|n\|_{L^p}^{\frac{p}{2}} \|n\|_{L^d} . \tag{27}$$

Then, putting (26) and (27) in (20) and using the Young's inequality in an appropriate way, we obtain in any dimension $d \ge 2$

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^p + 4\kappa\frac{p-1}{p}\int_{\mathbb{R}^d} |\nabla n^{p/2}|^2 \le 2\kappa\frac{p-1}{p}\int_{\mathbb{R}^d} |\nabla n^{p/2}|^2 + p(p-1)C(\chi, \kappa, \|n\|_{L^d}^2)\|n\|_{L^p}^p \tag{28}$$

Finally, taking $n_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ and the smallness condition (25) on $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ so that $\|n(t)\|_{L^d} \le \|n_0\|_{L^d}$, we have the main differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^p + 2\kappa\frac{p-1}{p}\int_{\mathbb{R}^d} |\nabla n^{p/2}|^2 \le Cp(p-1)\int_{\mathbb{R}^d} n^p \tag{29}$$

giving the following L^∞ bound of n

$$\sup_{t \ge 0} \|n(t)\|_{L^\infty} \le C \max\{\|n_0\|_{L^1}; \|n_0\|_{L^\infty}\} \tag{30}$$

by the iterative technique of Alikakos [1], where the constant C in the estimate (30) depends on $\kappa, \chi, \|n_0\|_{L^d(\mathbb{R}^d)}$.

For the sake of completeness, we present the argument of [1]. The constants may change from line to line. Let define $p_k = 2^k$, for $k \in \mathbb{N}$, and $v = n^{p_{k-1}}$. Then, inequality (29) becomes

$$\frac{d}{dt}\int_{\mathbb{R}^d} v^2 + 2\kappa\frac{p_k - 1}{p_k}\int_{\mathbb{R}^d} |\nabla v|^2 \le C_1 p_k^2 \int_{\mathbb{R}^d} v^2. \tag{31}$$

Next, by the Gagliardo-Nirenberg-Sobolev inequality and Young's inequality, we have (without making any difference between $d = 2$ and $d > 2$)

$$\|v\|_{L^2}^2 \le C_2 \|v\|_{L^1}^{2\alpha} \|v\|_{H^1}^{2(\alpha-1)} \le C_2\Big(\frac{1}{\varepsilon^{\frac{1}{\alpha}-1}}\|v\|_{L^1}^2 + \varepsilon\|v\|_{H^1}^2\Big) \qquad \alpha \in (0,1) \tag{32}$$

and for ε sufficiently small

$$\|v\|_{L^2}^2 \le C_2\Big(\frac{1}{\varepsilon^{\frac{1}{\alpha}-1}}\|v\|_{L^1}^2 + \varepsilon\|\nabla v\|_{L^2}^2\Big) . \tag{33}$$

Multiplying (33) by $(C_1 p_k^2 + \varepsilon)$ and taking ε small enough so that

$$C_2(C_1 p_k^2 + \varepsilon)\varepsilon \le 2\kappa \frac{p_k - 1}{p_k}, \tag{34}$$

we obtain from (31)

$$\frac{d}{dt}\int_{\mathbb{R}^d} v^2 \le -\varepsilon \int_{\mathbb{R}^d} v^2 + C_2 \frac{1}{\varepsilon^{\frac{1}{\alpha}-1}}(C_1 p_k^2 + \varepsilon)\Big(\int_{\mathbb{R}^d} v\Big)^2 . \tag{35}$$

i.e. using again the function n,

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^{p_k} \le -\varepsilon \int_{\mathbb{R}^d} n^{p_k} + C_2 \frac{1}{\varepsilon^{\frac{1}{\alpha}-1}}(C_1 p_k^2 + \varepsilon)\Big(\sup_{t\ge 0}\int_{\mathbb{R}^d} n^{p_{k-1}}\Big)^2 \tag{36}$$

and by Gronwall lemma

$$\int_{\mathbb{R}^d} n^{p_k} \le \max\{\int_{\mathbb{R}^d} n_0^{p_k}; C_2 \varepsilon^{-\frac{1}{\alpha}}(C_1 p_k^2 + \varepsilon)\Big(\sup_{t\ge 0}\int_{\mathbb{R}^d} n^{p_{k-1}}\Big)^2\}, \quad \forall\, t \ge 0 . \tag{37}$$

The estimate of $\|n\|_{L^{p_k}}$ follows by (37) using an iterative argument and a careful estimate of the constants in (37). Indeed, (34) gives us that $\varepsilon = \varepsilon_k$ is of order $\frac{1}{p_k^2}$. Then, let us define

$$\delta_k \;=\; C_2 \varepsilon_k^{-\frac{1}{\alpha}}(C_1 p_k^2 \;+\; \varepsilon_k) \text{ and } L = \max\{\|n_0\|_{L^1}; \|n_0\|_{L^\infty}\}.$$

Inequality (37) now reads as

$$\int_{\mathbb{R}^d} n^{p_k} \le \max\{L^{p_k}; \delta_k\Big(\sup_{t\ge 0}\int_{\mathbb{R}^d} n^{p_{k-1}}\Big)^2\}, \quad \forall\, t \ge 0 . \tag{38}$$

Since $\delta_k \ge 1$ for any $k \in \mathbb{N}$, from (38) we obtain recursively

$$\int_{\mathbb{R}^d} n^{p_k} \le \delta_k\, \delta_{k-1}^{p_1}\, \delta_{k-2}^{p_2} \cdots \delta_1^{p_{k-1}} L^{p_k} . \tag{39}$$

Finally, let C be a constant such that $\delta_k \le C p_k^{2(\frac{1}{\alpha}+1)}$. We have

$$\|n\|_{L^{p_k}} \le C^{(1-\frac{1}{p_k})}\; 2^{2(\frac{1}{\alpha}+1)\sum_{i=1}^{k}\frac{i}{2^i}}\; L \tag{40}$$

and estimate (30) follows by taking $k \to +\infty$.

3.3 The uniqueness issue in Theorem 2.

In the following, we give the proof of the uniqueness of the global solution in the case of bounded data n_0. This proof follows an idea of [14] and was not mentioned in [11].

Let (n_1, c_1) and (n_2, c_2) be two solutions of (16) and let us consider the function

$$f(n_1, n_2) = n_1 \ln n_1 + n_2 \ln n_2 - (n_1 + n_2) \ln(\frac{n_1 + n_2}{2}),$$

which satisfies

$$f(n_1, n_2) \geq \frac{1}{4}(\sqrt{n_1} - \sqrt{n_2})^2.$$

Then, using the second equation in (16) to estimate the L^2 norm of $\nabla(c_1 - c_2)$, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} f(n_1,n_2) &= -\kappa \int_{\mathbb{R}^d} \frac{n_1 n_2}{n_1+n_2} |\nabla \ln(\frac{n_1}{n_2})|^2 + \int_{\mathbb{R}^d} \frac{n_1 n_2}{n_1+n_2} \nabla \ln(\frac{n_1}{n_2}) \cdot [\nabla c_1 - \nabla c_2] \\ &\leq -\tfrac{\kappa}{2} \int_{\mathbb{R}^d} \frac{n_1 n_2}{n_1+n_2} |\nabla \ln(\frac{n_1}{n_2})|^2 + \frac{1}{8\kappa} \|n_1 + n_2\|_{L^\infty} \int_{\mathbb{R}^d} |\nabla(c_1 - c_2)|^2 \\ &\leq \tfrac{\chi^2}{16\alpha\kappa} \|n_1 + n_2\|_{L^\infty} \int_{\mathbb{R}^d} (n_1 - n_2)^2 \\ &\leq \tfrac{\chi^2}{8\alpha\kappa} \|n_1 + n_2\|_{L^\infty} \|\sqrt{n_1} - \sqrt{n_2}\|_{L^2}^2, \end{aligned} \tag{41}$$

and the Gronwall lemma gives us that $f(n_1(t), n_2(t)) = 0$ for any $t > 0$ whenever n_1 and n_2 have the same initial data.

4 The parabolic-degenerate system.

In this section, we deal with the parabolic-degenerate system

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n \chi(c) \nabla c], \quad t > 0,\ x \in \Omega, \\ \frac{\partial}{\partial t} c = -n c^m, \quad t > 0,\ x \in \Omega, \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), \qquad x \in \Omega. \end{cases} \tag{42}$$

with $\Omega = \mathbb{R}^d$, $m \geq 1$ and the sensitivity function $\chi(c)$ a given positive function on $\mathbb{R}_+$, generally chosen as a decreasing function since sensitivity is lower for higher concentrations of the chemical because of saturation effects.

It is interesting to analyze systems of type (42), where the equation on c is a simple ODE, in order to understand the mechanism of the competition between the chemotactic effects due to the sensitivity function $\chi(c)$ and the dynamics of c. In this spirit, system (42) with $\chi(c)$ a positive continuous function on $[0, \infty)$ has been analyzed in [10] and [11]. In that paper, we prove Theorem 1, except for the L^∞ bound which is new. Let us note that the analysis is similar to what we did in Theorem 2 for the chemotaxis system (16) and founded on the following equivalent of (20) :

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}\right)^p \phi(c) + 4\kappa\frac{p-1}{p}\int_{\mathbb{R}^d}\phi(c)\left|\nabla\left(\frac{n}{\phi(c)}\right)^{p/2}\right|^2 = \frac{1}{\kappa}(p-1)\int_{\mathbb{R}^d}\phi^2(c)\chi(c)c^m\left(\frac{n}{\phi(c)}\right)^{p+1}. \quad (43)$$

This is exactly the identity (13) obtained in section 2. In particular, we get the L^∞ bound in Theorem 1 using (43) and the iterative argument of Alikakos [1] (see subsection 3.2). However, since the Alikakos [1] estimates handle norms with respect to the Lebesgue measure, one should use here the following norm-equivalence : for all $p \geq 1$, $\alpha \in \mathbb{R}$ and $v \in L^p(\mathbb{R}^d)$,

$$C_1(p,\alpha,\|c_0\|_{L^\infty})\|v\|_{L^p(\mathbb{R}^d)} \leq \left(\int v(x)^p\phi(c(x))^\alpha dx\right)^{\frac{1}{p}}, \leq C_2(p,\alpha,\|c_0\|_{L^\infty})\|v\|_{L^p(\mathbb{R}^d)}$$

which comes from the fact that $\phi(c) = \exp\left(\frac{1}{\kappa}\int_0^c \chi(c')dc'\right)$ and $0 \leq c(x,t) \leq \|c_0\|_{L^\infty(\mathbb{R}^d)}$. This way, the conclusion of Theorem 1 follows without difficulty.

Finally, let us mention that when $\chi(c)$ is such that

$$\mu := \frac{1}{2}\inf_{c\geq 0}\{\frac{c\,\chi'}{\chi} + m\} > 0, \quad (44)$$

it has been proved in [10] that system (42) has an energy structure given by

$$\frac{d}{dt}\mathcal{E}(t) \leq -\int_\Omega n\left[\kappa|\nabla\ln(n)|^2 + \mu c^{m-1}|\nabla\Phi(c)|^2\right] \leq 0\,, \quad (45)$$

where

$$\mathcal{E}(t) := \int_\Omega [\frac{1}{2}|\nabla\Phi(c)|^2 + n\ln(n)] \quad \text{and} \quad \Phi'(c) = \sqrt{\frac{\chi}{c^m}}.$$

Note that $\mathcal{E}(t)$ cannot be obtained from the general identity (6) with an appropriate choice of G and F. We obtain $\mathcal{E}(t)$ directly from system (42) in the (n,c) variables using the function $N(c) := -\int_0^t n(\tau)d\tau$. Indeed,

$$\begin{aligned}\frac{d}{dt}\int n\ln n\,dx &= -\kappa\textstyle\int n|\nabla\ln n|^2dx + \int\chi(c)\nabla n\cdot\nabla c\,dx\\ &= -\kappa\textstyle\int n|\nabla\ln n|^2dx - \int\chi(c)c^m\partial_t\left(\frac{|\nabla N|^2}{2}\right)dx\end{aligned}$$

and

$$\frac{d}{dt}\mathcal{E}(t) = \frac{d}{dt}\int[n\ln n + \chi(c)c^m\frac{|\nabla N|^2}{2}]\,dx = -\kappa\int n|\nabla\ln n|^2dx - \int n(\chi(c)c^m)'c^m\frac{|\nabla N|^2}{2}.$$

For initial data with finite energy and $d \geq 2$, the energy (45) allows to prove the existence of weak solutions to (42) where the drift term $n\chi(c)\nabla c$ is well defined. This energy also provides equi-integrability for n, therefore solutions cannot exhibit concentrations. But the question of propagation of smoothness of the solutions is largely open. Our result in Theorem 1 is a contribution in this direction.

Note that the relaxation of the smallness condition in dimension $d \geq 3$ is an open problem. However, in two space dimensions, we have the following:

Theorem 3 *(L^p bound for the angiogenesis system (42) in two dimensions). Assume $d = 2$, $m \geq 1$, χ a positive say continuous differentiable function on $[0, \infty)$ and $\mu > 0$ in (44). Consider some nonnegative initial data (n_0, c_0) with finite energy i.e. $\mathcal{E}(0) < \infty$, such that $\ln(1+|x|)n_0 \in L^1(\mathbb{R}^d)$ and $c_0 \in L^\infty(\mathbb{R}^d)$. Then, there is a weak solution (n, c) of system (42) such that for any fixed $1 \leq p^* < \infty$,*

$$\|n(t)\|_{L^p(\mathbb{R}^2)} \leq C(t, p^*, \|n_0\|_{L^p(\mathbb{R}^2)}), \qquad \forall\ 1 \leq p \leq p^* .$$

See [11] for details on this theorem.

References

[1] N. D. Alikakos. L^p bounds of solutions of reaction-diffusion equations. *Comm. Partial Differential Equations*, **4**, (1979) 827–868.

[2] P. Biler. Global solutions to some parabolic-elliptic systems of chemotaxis. *Adv. Math. Sci. Appl.*, **9**, (1999) 347–359.

[3] P. Biler and T. Nadzieja. A class of nonlocal parabolic problems occurring in statistical mechanics. Colloq. Math., **66**, (1993) 131–145.

[4] P. Biler and T. Nadzieja. Global and exploding solutions in a model of self-gravitating systems. *Rep. Math. Phys.*, **52**, (2003), 205–225.

[5] M. P. Brenner, P. Constantin, L. P. Kadanoff, A. Schenkel, and S. C. Venkataramani. Diffusion, attraction and collapse. *Nonlinearity*, **12**, (1999) 1071–1098.

[6] M. P. Brenner, L. Levitov and E. O. Budrene. Physical mechanisms for chemotactic pattern formation by bacteria. *Biophysical Journal*, **74**, (1995) 1677–1693.

[7] H. Brezis. *Analyse fonctionnelle. Théorie et applications*, Masson, (1983).

[8] M. A. J. Chaplain. Avascular growth, angiogenesis and vascular growth in solid tumors : the mathematical modelling of the stages of tumor development, *Math. Comput. Modelling*, **23**, (1996) 47–87.

[9] M. A. J. Chaplain and L. Preziosi. Macroscopic modelling of the growth and development of tumor masses. Preprint n. 27 (2000), Politecnico di Torino.

[10] L. Corrias, B. Perthame and H. Zaag. A chemotaxis model motivated by angiogenesis. *C. R. Acad. Sci. Paris*, Ser. I , **336**, (2003) 141–146.

[11] L. Corrias, B. Perthame and H. Zaag. Global solutions of some chemotaxis and angiogenesis systems in high space dimensions. *Milan J. Math.* **72** (2004) 1–28.

[12] L. C. Evans. *Partial Differential Equations*, AMS, 1998.

[13] M. A. Fontelos, A. Friedman and B. Hu. Mathematical analysis of a model for the initiation of angiogenesis. *SIAM J. Math. Anal.*, **33**, (2002) 1330–1355.

[14] H. Gajewski and K. Zacharias. Global behaviour of a reaction-diffusion system modelling chemotaxis. *Math. Nachr.*, **195**, (1998) 77–114.

[15] D. Horstmann. Lyapunov functions and L^p estimates for a class of reaction-diffusion systems. *Coll. Math.*, **87**, (2001) 113–127.

[16] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.*, **105** , (2003), 103–165.

[17] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II. *Jahresber. Deutsch. Math.-Verein.*, **106**, (2004), 51–69.

[18] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.* **329**, (1992) 819–824.

[19] E.F. Keller. Assessing the Keller-Segel model : how has it fared ? Biological growth and spread (Proc. Conf., Heidelberg, 1979), Lecture Notes in Biomath., **38**, Springer, Berlin-New York, (1980) 379–387.

[20] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, **26**, (1970) 399–415.

[21] E.F. Keller and L.A. Segel. Model for chemotaxis. *J. Theor. Biol.*, **30**, (1971) 225–234.

[22] H. A. Levine and B. D. Sleeman. A system of reaction diffusion equations arising in the theory of reinforced random walks. *SIAM J. Appl. Math.*, **57**, (1997) 683–730.

[23] H. A. Levine, M. Nilsen-Hamilton and B. D. Sleeman. Mathematical modelling of the onset of capillary formation initiating angiogenesis. *J. Math. Biol.*, **42**, (2001) 195–238.

[24] H. G. Othmer and A. Stevens. Aggregation, blowup and collapse : the ABC's of taxis in reinforced random walks. *SIAM J. Appl. Math.*, **57**, (1997) 1044–1081.

[25] C. S. Patlak. Random walk with persistence and external bias. *Bull. of Math. Biophys.*, **15**, (1953) 311–338.

[26] M. Rascle. On a system of non-linear strongly coupled partial differential equations arising in biology. Conf. on Ordinary and Partial Differential Equation, Lectures Notes in Math. 846, Everitt and Sleeman eds., Springer-Verlag, New-York, 1980, 290–298.

[27] Y. Yang, H. Chen and W. Liu. On existence of global solutions and blow-up to a system of reaction-diffusion equations modelling chemotaxis. *SIAM J. Math. Anal.*, **33**, (2001) 763–785.

[a] Département de Mathématiques,
Université d'Evry Val d'Essonne,
Rue du Père Jarlan, F 91025 Evry Cedex
email: corrias@maths.univ-evry.fr

[b] Département de Mathématiques et Applications,
École Normale Supérieure, CNRS UMR8553 ,
45 rue d'Ulm, F 75230 Paris cedex 05
email: perthame@dma.ens.fr, zaag@dma.ens.fr