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On a singular diffusion equation with a linear source

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1 Introduction

This article is based on a joint work with Peter Takáč of the Rostock University. The aim of this article is to discuss the qualitative behavior of positive solutions for the equation

$$u_t = (\log u)_{xx} + u, \quad x \in \mathbf{R}, \tag{1}$$

under the condition

$$\lim_{x\to-\infty} (\log u)_x = \alpha > 0, \quad \lim_{x\to-\infty} (\log u)_x = -\beta < 0, \tag{2}$$

where α and β are given positive constants.

The equation of the form

$$u_t = \Delta \log u \tag{3}$$

has appeared in various context. It appears as the central limit approximation of Carleman's model of the Boltzman equation [6], the expansion into a vacuum of a thermalized electron cloud [5], Ricci flow on R^2 in differential geometry [2]. It also has some relations with the porous media equation [11] and a biological population model [7]. For the mathematical analysis of (3), see [3, 4, 8, 9, 1].

Equation (3) is rewritten as

$$u_t = \operatorname{div}(\frac{1}{u}\nabla u).$$

This implies that the diffusion becomes very fast if $u > 0$ is small. Therefore, (3) is often referred to as the *very fast diffusion equation.* A remarkable feature of the very fast diffusion equation is that a finite time extinction of solutions may occur. More precisely, if the initial data are positive and decay to 0 as $|x| \to \infty$, then there exists $T < \infty$ such that the solution satisfies

$$\lim_{t\uparrow T} u(x,t) = 0$$

uniformly.

Our question is the following: What happens if we add a source term to the fast diffusion equation? Due to the source term, positive solutions may exist globally in time, and some interesting may be observed by the interplay between the diffusion and the source. This is the motivation to study (1). Here we restrict our attention to qualitative behavior of solutions of (3), and do not pay much attention to the existence problem of solutions for the initial value problem for (1), which will be discussed in our forthcoming paper [10].

2 Traveling solutions

First, let us consider travelling wave solutions for (1). Any traveling solution $u = \varphi(z)$, $z = x - ct$, must satisfy the ordinary differential equation

$$(\log\varphi)_{zz} + c\varphi_z + \varphi = 0, \tag{4}$$

where $c \in \mathbf{R}$ denotes a propagation speed toward the right. If we introduce an auxiliary variable

$$\psi = (\log \varphi)_z,$$

then (4) is equivalent to the system

$$\begin{cases} \varphi_z = \varphi\psi, \\ \psi_z = -c\varphi\psi - \varphi. \end{cases} \tag{5}$$

By the phase plane analysis, we can easily show that for each $\alpha, \beta > 0$, there exists a unique $c = c(\alpha, \beta)$ such that (5) has an orbit connecting $(0, \alpha)$ and $(0, -\beta)$ in the $\varphi - \psi$ plane. In other words, if $c = c(\alpha, \beta)$, then (4) has a solution satisfying

$$(\log \varphi)_z(-\infty) = \alpha, \quad (\log \varphi)_z(+\infty) = -\beta. \tag{6}$$

We note that the propagation speed of the traveling wave is monotone decreasing in α and monotone increasing in β. In particular, $c = 0$ holds if and only if $\alpha = \beta$.

Hereafter, we denote by $\varphi(z)$ a traveling wave solution with the speed c satisfying (6). Let us consider the linearized stability of the traveling wave solution by considering the following linearized eigenvalue problem:

$$\begin{aligned} &\lambda U = (U/\varphi)_{zz} + cU_z + U, \quad z \in \mathbf{R}, \\ &(U/\varphi)_z \to 0 \quad \text{as } z \to \pm\infty. \end{aligned} \tag{7}$$

It is easy to see that $\lambda_0 = 1$ is an eigenvalue and the associated eigenfunction is given by $U_0 = \varphi + c\varphi_z$. Similarly, $\lambda_1 = 0$ is an eigenvalue with the associated eigenfunction $U_0 = \varphi_z$, which corresponds to the spatial translation. Thus the traveling wave solution is necessarily unstable in the linearized sense. However, if the disturbance is in a class of mass conservation, then the traveling solution may be stable.

3 Mass transfer

Integrating (1) over $\mathbf{R}$, we obtain

$$\frac{d}{dt}m(t) = -(\alpha+\beta) + m(t),$$

where $m(t)$ is a total mass defined by

$$m(t) := \int_{\mathbf{R}} u(x,t)dx.$$

This implies the following :

(i) If $m(0) > \alpha+\beta$, then $m(t) \to \infty$ as $t \to \infty$.

(ii) If $m(0) = \alpha+\beta$, then $m(t) \equiv \alpha+\beta$ for all $t > 0$.

(iii) If $0 < m(0) < \alpha+\beta$, then $m(t) = 0$ at some $t = T < \infty$.

Thus , in the case of (iii), the finite time extinction must occur.

Let us consider the case where $m(0) = \alpha+\beta$. Then the mass is conserved for all $t > 0$. In this case, we can show that the most mass moves with the speed $c = c(\alpha,\beta)$. To see this, let $a > 0$ be a sufficiently large number and be fixed. Suppose that the initial data u_0 satisfy the following conditions:

(A1) $u_0(x) - \varphi(x+a)$ changes its sign from negative to positive only once when x varies over $\mathbf{R}$.

(A2) $u_0(x) - \varphi(x-a)$ changes its sign from positive to negative only once when x varies over $\mathbf{R}$.

In this situation, the manner of intersection for $u(x,t)$ and $\varphi(x-ct+a)$ is preserved for all $t > 0$. This implies that the most mass moves rightward with the speed at least c. Similarly, the manner of intersection for $u(x,t)$ and

$\varphi(x - ct - a$ is preserved for all $t > 0$ so that the most mass moves rightward with the speed at least c.

Next, let us consider the case where $m(0) \neq \alpha + \beta$. In this case, we introduce a transformation of variables

$$v(x,s) = \frac{\alpha+\beta}{m(t)} u(x,t), \quad s = \int_0^t \frac{\alpha+\beta}{m(\tau)} d\tau.$$

Then, from (1), we see that v satisfies

$$v_t = (\log v)_{xx} + v, \quad x \in \mathbf{R},$$

and

$$\lim_{x\to-\infty} (\log v)_x = \alpha > 0, \quad \lim_{x\to-\infty} (\log v)_x = -\beta < 0.$$

Moreover, v must satisfy

$$\int_{\mathbf{R}} v(x,t)dx \equiv \alpha + \beta.$$

Thus the unbalanced case can be transformed into the balanced case by the above change of variables. It is interesting to note that if $m(0) < \alpha+\beta$, then s ranges over $[0,\infty)$ while t varies over $[0,T)$, where $T > 0$ is the extinction time of u. We see from the above argument that most mass moves to $x = \infty$ if $\alpha > \beta$ and moves to $x = -\infty$ if $\alpha < \beta$.

Finally, if $m(0) > \alpha + \beta$, then s varies over $[0,S)$, while t ranges over $[0,\infty)$, where $S > 0$ is given by

$$S := \int_0^\infty \frac{\alpha+\beta}{m(\tau)} d\tau < \infty.$$

This implies that the mass stays bounded for all $t > 0$.

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