数理解析研究所講究録 1418 巻 2005 年 51-54

APPLICATIONS OF EXACTNESS

NATHANIAL P. BROWN

ABSTRACT. The theory of exact C*-algebras rose to prominence following the ground-breaking work of Eberhard Kirchberg over a decade ago. In this survey we look at a few unexpected applications of this theory.

1. INTRODUCTION

This article is a brief survey of some recent applications of the theory of exact C*-algebras. Following Kirchberg's seminal work in the late 1980's and early 1990's, many operator algebraists became casually acquainted with the theory. (See [20] for a nice treatment of exactness.) Now it seems that familiarity with a few of the fundamental theorems in the subject is almost a prerequisite for students wishing to study C*-algebras.

Several researchers in the operator space community have made significant contributions to our understanding of this rather elusive concept. Moreover, interesting connections with amenable actions and the Novikov conjecture have also been discovered. In this note I will not discuss these interactions as the former has already been thoroughly treated in the wonderful book of Pisier [16] while the latter has not yet produced any new examples of groups satisfying the Novikov conjecture.

Instead, I will discuss three non-trivial applications of exactness (which haven't yet appeared in a book). The first will be to a problem in single operator theory, the second falls in the realm of free probability and the third – revolutionary work of Narutaka Ozawa – is to the structure theory of type II_1 factors. I will not reproduce any proofs, though I will refer the reader to the appropriate papers in the literature.

2. OPERATOR THEORY

In the 1970's Paul Halmos introduced the notion of a *quasidiagonal* operator on a separable Hilbert space (cf. [9]).

Definition 2.1. A bounded, linear operator $T \in B(H)$ is called *quasidiagonal* if there exists an increasing sequence of finite rank, orthogonal projections $P_1 \leq P_2 \leq P_3 \leq \cdots$ such that

(1) $\|P_n(v) - v\| \to 0$, as $n \to \infty$, for all $v \in H$ (i.e. $P_n \to 1_H$ in the strong operator topology) and
(2) $\|[P_n, T]\| = \|P_nT - TP_n\| \to 0$ as $n \to \infty$.

We denote the set of all such operators by $QD(H)$.

This definition is a natural generalization of the notion of *block diagonal* operator – i.e. (infinite) direct sums of finite dimensional matrices.

Definition 2.2. $S \in B(H)$ is called *block diagonal* if there exist finite rank projections $P_n \leq P_{n+1}$ which converge (s.o.t.) to the identity and such that $[T, P_n] = 0$ for all $n \in \mathbb{N}$. *We denote the set of all such operators by $BD(H)$.*

Keep in mind the matrix picture of block diagonal operators. If we write

$$H = P_1H \oplus (P_2 - P_1)H \oplus (P_3 - P_2)H \oplus \cdots$$

then the matrix of S w.r.t. this decomposition is really "block diagonal" with each block being finite dimensional. Here is a natural subset of the block diagonal operators.

Definition 2.3. $S \in B(H)$ is called *block diagonal with bounded blocks* if there exist finite rank projections $P_n \leq P_{n+1}$ which converge (s.o.t.) to the identity, $[T, P_n] = 0$ for all $n \in \mathbb{N}$ and

$$\sup_n rank(P_n - P_{n-1}) < \infty.$$

We denote the set of all such operators by $BD_{bdd}(H)$.

This survey was written during a year-long visit to The University of Tokyo. I thank this institution, and especially Yasuyuki Kawahigashi and Narutaka Ozawa, for their hospitality.

Again, thinking of block matrices and the decomposition

$$H = P_1H \oplus (P_2 - P_1)H \oplus (P_3 - P_2)H \oplus \cdots$$

this terminology should be clear.

Halmos first observed that every quasidiagonal operator is the norm limit of a sequence of block diagonal operators - i.e. $QD(H) = \overline{BD(H)}$. Examples suggested that the norm closures of $BD(H)$ and $BD_{bdd}(H)$ may coincide (i.e. for awhile, all the known examples of quasidiagonal operators could be shown to come from the norm closure of $BD_{bdd}(H)$). It was Herrero who first formulated the following question:

Is every quasidiagonal operator in the norm closure of $BD_{bdd}(H)$?

Szarek first showed the existence of counterexamples to Herrero's question (cf. [17]) but his argument was probabalistic and did not provide concrete examples. Voiculescu provided the first explicit examples of quasidiagonal operators which do not belong to the norm closure of $BD_{bdd}(H)$ (cf. [18]). More importantly, he recognized what the right operator *algebraic* obstruction was - exactness! More precisely, Voiculescu noticed that if $T \in B(H)$ belongs to the norm closure of $BD_{bdd}(H)$ then the C*-algebra generated by T, $C^*(T)$, must be nuclearly embeddable and hence exact. Voiculescu's counterexample amounts to producing a quasidiagonal operator which does not generate an exact C*-algebra.

It thus became natural to try and understand precisely what the norm closure of $BD_{bdd}(H)$ is. The answer, at least in operator algebraic terms, was already hinted at in Voiculescu's work. To give a purely operator theoretic formulation we need one more well-known concept.

Definition 2.4. $S \in B(H)$ is called *banded* if there exists an orthonormal basis $\{v_i\}$ of H such that the matrix of S w.r.t. $\{v_i\}$ is banded (meaning only a finite number of non-zero diagonals). We let $Band(H)$ denote the set of banded operators on H.

We can now give a precise description of the norm closure of $BD_{bdd}(H)$.

Theorem 2.5. $\overline{BD_{bdd}(H)} = QD(H) \cap \overline{Band(H)}$. *In other words, if there exists a banded sequence S_n such that $\|T - S_n\| \to 0$ and a block diagonal sequence U_n such that $\|T - U_n\| \to 0$ then there must exist a sequence X_n which is simultaneously banded and block diagonal such that $\|T - X_n\| \to 0$.*

Though this result is formulated only in operator theoretic terms, the proof has virtually nothing to do with single operator theory - it is all about exact, quasidiagonal C*-algebras. Indeed, the real theorem behind everything is the following result.

Theorem 2.6. *Let A be an exact C*-algebra and $\pi : A \to B(H)$ be a quasidiagonal representation. Then for each finite set $\mathfrak{F} \subset A$ and $\epsilon > 0$ there exists a finite dimensional C*-subalgebra $B \subset B(H)$ such that*

$$\pi(\mathfrak{F}) \subset^{\epsilon} B.$$

In the case that $\pi(A)$ contains no non-zero compact operators this theorem is due to Marius Dadarlat (cf. [4]). Passage to the general case was achieved in [1] where trickery, together with Kirchberg's remarkable theorem that exactness implies local reflexivity (cf. [10]), was employed to reduce to Dadarlat's result.

I want to emphasize that it is far from obvious how an operator theorist would go about proving Theorem 2.5. As I mentioned, local reflexivity of exact C*-algebras plays a crucial role in the proof. This, in turn, depends on Connes' uniqueness theorem for the injective II_1 factor (cf. [2]) together with modular theory and even direct integral theory. In other words, operator theorists would have to find a way around Kirchberg's deep work on exact C*-algebras and the remarkable structure theory for von Neumann algebras developed in the 1970's.

3. Free Probability

In this section I will describe a recent application to a natural problem in free probability theory. In Gaboriau's striking work on cost of equivalence relations (cf. [5]) he shows that if $A \subset M$ is a Cartan subalgebra of a von Neumann algebra M and the cost of the corresponding equivalence relation is attained (i.e. M has a set of generators of 'minimal support') then M is isomorphic to a free product of von Neumann algebras with amalgamation over A. (Many thanks to Yoshimichi Ueda for a wonderful series of lectures, at The University of Tokyo, on Gaboriau's work.)

It follows from work of Voiculescu (see [19] for a nice survey) that if the free entropy of an n-tuple of self-adjoint operators in a tracial von Neumann algebra, $\{X_1, \ldots, X_n\} \subset M$, is finite (i.e. $\chi(X_1, \ldots, X_n) > -\infty$) then the free entropy dimension must be as large as possible: $\delta(X_1, \ldots, X_n) = n$. On the other hand, if it turns out that δ is a W*-invariant and one knows $\delta(X_1, \ldots, X_n) = n$ then the set $\{X_1, \ldots, X_n\}$ must be a minimal set of generators (meaning that no set of $n-1$ self-adjoints can generate M). Hence the following question of Dima Shlyakhtenko is a natural analogue of Gaboriau's theorem mentioned earlier.

If $\chi(X_1,\dots,X_n) > -\infty$ is $W^*(X_1,\dots,X_n)$ necessarily a free group factor?

When combined with a sufficient number of other known results, the theory of exactness easily yields counterexamples to this question. However, unlike Herrero's approximation problem, it does not make an explicit appearance. Rather, the philosophy of *external approximation* (which underlies exactness) is the necessary tool. Hence this section may be more of a spiritual application of exactness.

More precisely, the following well-known fact is the key observation we will need. (Actually, this fact was used by Voiculescu in the previous section to show that any operator belonging to the norm closure of $BD_{bdd}(H)$ must generate an exact C*-algebra.)

Lemma 3.1. *If $A \subset B(H)$ and there exist exact C*-algebras $A_n \subset B(H)$ with the property that for each $a \in A$ there is a sequence $a_n \in A_n$ such that $\|a - a_n\| \to 0$ then A must also be exact.*

Proof. This is a simple application of Kirchberg's theorem that exactness implies nuclear embeddability. Indeed, if a finite set $\mathfrak{F} \subset A$ and $\epsilon > 0$ are given then we can choose n large enough that $\mathfrak{F}$ is ϵ-contained in A_n. Since A_n is exact we can find u.c.p. maps $\phi : A_n \to M_k$, $\psi : M_k \to B(H)$ which approximate any given finite subset of A_n. By Arveson's Extension Theorem we may assume that ϕ is defined on all of $B(H)$ and hence these maps work for $\mathfrak{F}$ as well (by norm continuity). □

The point here is that the A_n's are approximating A from the 'outside'. They are not subalgebras of A; they need not even intersect A non-trivially. But still exactness is preserved under this 'external approximation'.

In the context of von Neumann algebras this external norm approximation should be replaced with some sort of external approximation in a weaker topology. Moreover, we need to be a little more careful about the 'universe' in which this all takes place. It turns that if we replace $B(H)$ by a II_1 factor (i.e. move to a finite universe) and require that $a_n \to a$ in the 2-norm coming from the unique trace then everything will work smoothly.

Of course, we also need a replacement for the C*-notion of exactness. There are a number of options but perhaps the easiest to apply is the *Haagerup approximation property.*

Definition 3.2. A finite von Neumann algebra (M, τ) is said to have the *Haagerup approximation property* if there exists a sequence of u.c.p. τ-preserving maps $\phi_n : M \to M$ such that (a) $\|\phi_n(x) - x\|_2 \to 0$ for all $x \in M$ and (b) the map induced by ϕ_n on $L^2(M,\tau)$ is a compact operator.

The seminal paper [8] is where Haagerup proved that free group factors have this property.

More precisely, the following lemma is a fairly simple exercise.

Lemma 3.3. *Let M be a II_1 factor with subfactors $N, M_k \subset M$. Assume that for every $n \in N$ we have $E^M_{M_k}(n) \to n$ in 2-norm (i.e. strong operator topology) and that each M_k has the Haagerup approximation property. Then N also has the Haagerup approximation property.*

With this observation in hand we can easily describe the counterexamples mentioned earlier. One begins with any self-adjoint generating set of the group von Neumann algebra of a residually finite property T group; e.g. $\{X_1,\dots,X_n\} \subset L(SL(3,\mathbb{Z}))$. Now let $\{S_1,\dots,S_n\}$ be a family of free semicircular elements which are also free from $\{X_1,\dots,X_n\}$. One then defines

$$X_i^{(\epsilon)} = X_i + \epsilon S_i \text{ and } M^{(\epsilon)} = W^*(X_1^{(\epsilon)},\dots,X_n^{(\epsilon)}).$$

A result of Voiculescu implies that the free entropy of the n-tuple $\{X_1^{(\epsilon)},\dots,X_n^{(\epsilon)}\}$ is finite (cf. [19]) and so if we assume that Shlyakhtenko's question has an affirmative answer then it would follow that each $M^{(\epsilon)}$ is a free group factor. Hence, for each ϵ, $M^{(\epsilon)}$ would enjoy the Haagerup approximation property. Thus, by our external approximation lemma, the von Neumann algebra $W^*(X_1,\dots,X_n) = L(\Gamma)$ would also have the Haagerup approximation property. But, since we started with a property T group Γ, this is not possible by a result of Connes and Jones (cf. [3]) – the desired contradiction.

4. Structure of II_1 factors

In the final section of this note I will briefly discuss the single most surprising, not to mention important, application of exactness to date. This is work of Narutaka Ozawa and its influence has been tremendous. To keep things simple I will not discuss all of Ozawa's work or even give a proper historical account. However, I hope the point will be conveyed; Ozawa's application of exactness was a vast generalization of some important previous work...and the proof was a lot simpler.

We begin with a definition.

Definition 4.1. A II_1 factor is called *prime* if it **can't** be decomposed as a tensor product of II_1 factors.

Every II_1 factor can be decomposed as the tensor product of a finite dimensional matrix algebra and another II_1 factor. However, for many years it was not known whether or not a prime II_1 factor existed. This problem was eventually solved by Liming Ge who used some highly non-trivial free probability techniques to show that the free group factors are prime. (See [6] for a survey.) This was later generalized by Marius Stefan to finite index subfactors of free group factors.

While these were certainly exciting results it turns out that exactness, together with some geometric group theory, provides much, much better theorems. Indeed, in [13] Ozawa proved the following.

Theorem 4.2. *Let Γ be a hyperbolic group (cf. [7]) and $N \subset L(\Gamma)$ be a subfactor of the group von Neumann algebra of Γ. Then either (a) N is a matrix algebra, (b) N is the (non-prime) hyperfinite II_1 factor or (c) N is prime.*

Ozawa's proof is not trivial, but it is quite short, extremely clever and significantly simpler than the free entropy approach (which only works on free group factors). Also, it seems likely that the free entropy approach would never have been able to handle the case of infinite index subfactors, while Ozawa's approach treats all subfactors on equal footing. Moreover, Ozawa's result holds for many other examples of groups (not just hyperbolic) and has led to some exciting (non)isomorphism theorems (cf. [14], [15]). In short, it is a spectacular piece of work: a deep, important and completely unexpected application of exactness.

References

1. N.P. Brown, *Herrero's approximation problem for quasidiagonal operators*, J. Funct. Anal. **186** (2001), 360–365.
2. A. Connes, *Classification of injective factors: cases II_1, II_∞, III_λ, $\lambda \neq 1$*, Ann. Math. **104** (1976), 73–115.
3. A. Connes and V. Jones, *Property T for von Neumann algebras*, Bull. London Math. Soc. **17** (1985), 57–62.
4. M. Dadarlat, *On the approximation of quasidiagonal C^*-algebras*, J. Funct. Anal. **167** (1999), 69–78.
5. D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), 41–98.
6. L. Ge, *Free probability, free entropy and applications to von Neumann algebras.* Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 787–794, Higher Ed. Press, Beijing, 2002.
7. M. Gromov, *Hyperbolic groups*, Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ. 8, Springer, New York, 1987.
8. U. Haagerup, *An example of a nonnuclear C^*-algebra which has the metric approximation property*, Invent. Math. **50** (1978/79), 279–293.
9. P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.
10. E. Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. Reine Angew. Math. **452** (1994), 39–77.
11. E. Kirchberg, *On nonsemisplit extensions, tensor products and exactness of group C^*-algebras*, Invent. Math. **112** (1993), 449 – 489.
12. E. Kirchberg, *Exact C^*-algebras, tensor products and the classification of purely infinite algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1,2 (Zurich, 1994), 943–954.
13. N. Ozawa, *Solid von Neumann algebras*, Acta Math. **192** (2004), 111–117.
14. N. Ozawa, *A Kurosh type theorem for type II_1 factors*, preprint 2004.
15. N. Ozawa and S. Popa, *Some prime factorization results for type II_1 factors*. Invent. Math. **156** (2004), 223–234.
16. G. Pisier, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003.
17. S. Szarek, *An exotic quasidiagonal operator*, J. Funct. Anal. **89** (1990), 274–290.
18. D.V. Voiculescu, *A note on quasidiagonal operators*, Operator Theory: Advances and Applications, Vol. 32, Birkhauser Verlag, Basel, 1988, 265–274.
19. D. Voiculescu, *Free entropy*, Bull. London Math. Soc. **34** (2002), 257–278.
20. S. Wassermann, *Exact C^*-algebras and related topics*, Lecture Notes Series no.19, GARC, Seoul National University, 1994.

Department of Mathematics, Penn State University, State College, PA 16802. Currently visiting The University of Tokyo.

E-mail address: nbrown@math.psu.edu or nbrown@ms.u-tokyo.ac.jp