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Topological pressure of Cantor minimal systems

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Abstract

This is a survey article of the paper [S4]: F. Sugisaki, Topological pressure of cantor minimal systems within a strong orbit equivalence class.

1 Introduction

For a topological dynamical system (X,T), denote $\mathcal{M}(X)$ by the set of Borel probability measures on X and $\mathcal{M}(X,T)$ by the set of T-invariant Borel probability measures on X. The main theorem is the following.

Theorem 1.1. *Suppose that (X,φ) is a Cantor minimal system and f is a potensial function on X. Choose any α with*

$$\exp\left(\sup\left\{\int f d\mu \;\middle|\; \mu\in\mathcal{M}(X,\phi)\right\}\right)\leq\alpha\leq\infty \tag{1.1}$$

and fix it. Then there exists a Cantor minimal system (Y,ψ) strongly orbit equivalent to (X,φ) such that

$$P(\psi,f\circ\theta^{-1})=\log\alpha,$$

where $P(\psi,\cdot)$ is the topological pressure of ψ and $\theta:X\to Y$ is strong orbit equivalence map. If α is finite, we can take ψ as an expansive homeomorphism.

Remark 1.2. (1) For a topological dynamical system (X,T) and a potential function $f\in C(X,\mathbb{R})$, the variational principle of topological pressure (see Theorem 9.10 in [W1])

$$P(T,f)=\sup\left\{h_\mu(T)+\int f d\mu \;\middle|\; \mu\in\mathcal{M}(X,T)\right\}$$

implies that

- $P(T,f)\geq\sup\{\int f d\mu\mid\mu\in\mathcal{M}(X,T)\}$,

- $P(T,f) = \sup\{\int f d\mu \mid \mu \in \mathcal{M}(X,T)\}$ iff the topological entropy $h(T) = 0$.

Moreover a strong orbit equivalence map θ sends $\mathcal{M}(Y,\psi)$ onto $\mathcal{M}(X,\phi)$ bijectively. So (1.1) is the best possible inequality which α can take.

(2) If $f = 0$, then $P(\psi, 0)$ is equal to the topological entropy of ψ. So this theorem is generalization of the papers [S1], [S2] and [S3].

Basically, we use notations and definitions in [HPS] and [GPS]. Here we will introduce some notations, definitions and properties of (properly ordered) Bratteli diagrams in this paper.

Notation 1.3. Suppose $\mathcal{B} = (V, E, \geq)$ is a properly ordered (also called simply ordered) Bratteli diagram.

(1) Let $r : E \to V$ denote the range map and $s : E \to V$ denote the source map. Namely, $e \in E_n$ connects between $s(e) \in V_{n-1}$ and $r(e) \in V_n$.

(2) Let $M^{(n)} = [\#r^{-1}(u) \cap s^{-1}(v)]_{u \in V_n, v \in V_{n-1}}$ denote the n-th incidence matrix of $\mathcal{B}$ (i.e., $M^{(n)}_{uv}$ is the number of edges connecting between $u \in V_n$ and $v \in V_{n-1}$). We also write $\mathcal{B} = (V, E, \{M^{(n)}\}, \geq)$. Let $M^{(n)}_u = (M^{(n)}_{uv})_{v \in V_{n-1}}$ denote the u's row vector of $M^{(n)}$ which is called *an incidence vector of* u. For $k \leq n$, let $M^{(n,k)}$ denote the product of incidence matrices $M^{(n)} M^{(n-1)} \cdots M^{(k)}$.

(3) Set $X_{\mathcal{B}} = \{(e_i)_{i \in \mathbb{N}} \mid e_i \in E_i, r(e_i) = s(e_{i+1})\ \forall i \in \mathbb{N}\}$. We call it *the (infinite lengths) path space of* $\mathcal{B}$. For $v \in V_n$, let $\mathcal{P}(v)$ denote the set of all (finite lengths) paths connecting between the top vertex $v_0 \in V_0$ and v. Then $|\mathcal{P}(v)| = M^{(n,1)}_{vv_0}$ holds. Put $\mathcal{P}(V_n) = \cup_{v \in V_n} \mathcal{P}(v)$. The range map is extended to $\mathcal{P}(V_n)$, that is, for $p = (e_1, \ldots, e_n) \in \mathcal{P}(V_n)$ $r(p) = r(e_n)$.

(4) For $p \in \mathcal{P}(V_n)$, set $[p]_{\mathcal{B}} = \{(e_i)_{i \in \mathbb{N}} \in X_{\mathcal{B}} \mid (e_1, e_2, \ldots, e_n) = p\}$. We call it *the cylinder set of* p.

(5) For $v \in V_n$ and $e \in r^{-1}(v)$, let Order(e) denote the order of e in $r^{-1}(v)$. If $p_{\min} = (e_1, e_2, \cdots)$ is the unique minimal path in $X_{\mathcal{B}}$, then Order$(e_n) = 1$ for all $n \in \mathbb{N}$. If $p_{\max} = (f_1, f_2, \cdots)$ is the unique maximal path in $X_{\mathcal{B}}$, then Order$(f_n) = |r^{-1}(v_n)|$ for all $n \in \mathbb{N}$, where $v_n = r(f_n)$. Similarly, Order$(\cdot)$ is defined on $\mathcal{P}(V_n)$. I.e., for $p \in \mathcal{P}(V_n)$, Order(p) is the order of p in $\mathcal{P}(r(p))$.

(6) For $v \in V_n$, we write $r^{-1}(v) = \{e_i \mid \text{Order}(e_i) = i \text{ for } 1 \leq i \leq |r^{-1}(v)|\}$. Define List$(v) = (s(e_1), s(e_2), \cdots, s(e_{|r^{-1}(v)|}))$. We call it *the order list of* v.

(7) For a sequence $t_0 = 0 < t_1 < t_2 < t_3 < \cdots$ in $\mathbb{Z}_+$, we say that a Bratteli diagram $\mathcal{B}' = (V', E', \{M'^{(n)}\})$ is a *telescoping* (or *contraction*) of $\mathcal{B}$ to $\{t_n\}_{n \in \mathbb{Z}_+}$, which we write $\mathcal{B}' = (\mathcal{B}, \{t_n\})$, if $V'_n = V_{t_n}$ and $M'^{(n)} = M^{(t_n, t_{n-1}+1)}$. We call $\{t_n\}$ *a sequence of telescoping depths.* Especially, we define $\mathcal{B}_{\text{odd}}$ as telescoping $\mathcal{B}$ to odd depths $(0, 1, 3, \cdots)$ and define $\mathcal{B}_{\text{even}}$ as telescoping $\mathcal{B}$ to even depths $(0, 2, 4, \cdots)$.

(8) Let $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ denote the Bratteli-Vershik system of $\mathcal{B}$. Namely, $\lambda_{\mathcal{B}} : X_{\mathcal{B}} \to X_{\mathcal{B}}$ is the Vershik (lexicographic) map defined by the order $\geq$ on E.

(9) Define an equivalence relation $\sim$ on Bratteli diagrams as follows. $\mathcal{B} \sim \mathcal{B}'$ if there exists a Bratteli diagram $\tilde{\mathcal{B}}$ such that $\tilde{\mathcal{B}}_{\mathrm{odd}}$ yields a telescoping either $\mathcal{B}$ or $\mathcal{B}'$, and $\tilde{\mathcal{B}}_{\mathrm{even}}$ yields a telescoping of the other.

Remark 1.4. (1) In [HPS], Herman, Putnam and Skau showed that the family of Cantor minimal systems coincides with the family of Bratteli-Vershik systems up to conjugacy.

(2) Let (X, T) denote a Cantor minimal system. In [P], Putnam showed that $K^0(X, T)$ with positive cone $K^0(X, T)^+$ is a simple, acyclic (i.e. $K^0(X, T) \ncong \mathbb{Z}$) dimension group with (canonical) distinguished order unit $[1]$, where $1 = 1_X$ is the constant function 1.

(3) Herman, Putnam and Skau showed in [HPS] that $K^0(X, T) \cong K_0(V, E)$ ($\cong$ means two dimension groups are unital order isomorphic), where (V, E) is a Bratteli-Vershik representation of (X, T), and that all (acyclic) simple dimension groups can be obtained in this (dynamical) way.

(4) It is easy to see that $(V, E) \sim (V', E')$ if and only if $K_0(V, E) \cong K_0(V', E)$.

(5) Giordano, Putnam and Skau showed in [GPS] that Bratteli-Vershik systems $(X_{\mathcal{B}_1}, \lambda_{\mathcal{B}_1})$ and $(X_{\mathcal{B}_2}, \lambda_{\mathcal{B}_2})$ are strongly orbit equivalent if and only if $\mathcal{B}_1 \sim \mathcal{B}_2$.

Definition 1.5 (distinct order list). We say V_n has *distinct order lists* if for $v, v' \in V_n$, $\mathrm{List}(v) = \mathrm{List}(v')$ implies $v = v'$.

Definition 1.6 (The minimal/maximal vertex property). Suppose $\mathcal{B} = (V, E, \geq)$ is a properly ordered Bratteli diagram and for $n \in \mathbb{N}$, $v^n_{\mathrm{min}} \in V_n$ ($v^n_{\mathrm{max}} \in V_n$, resp.) is the vertex which unique minimal path (maximal path, resp) in $X_{\mathcal{B}}$ goes through. We say E_n has *the minimal/maximal vertex property* if for any $e, f \in E_n$ with $\mathrm{Order}(e) = 1$ and $\mathrm{Order}(f) = |r^{-1}r(f)|$, then $s(e) = v^{n-1}_{\mathrm{min}}$ and $s(f) = v^{n-1}_{\mathrm{max}}$ hold. ($v^0_{\mathrm{min}} = v^0_{\mathrm{max}} = v_0 \in V_0$.)

The following is the conditions which a Bratteli-Vershik system of (Y, ψ) satisfies.

Property 1.7. We consider a properly ordered Bratteli diagram $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{(n)}\}, \tilde{\geq})$ satisfying the following properties for any $n \in \mathbb{N}$:

(1) $\tilde{M}^{(n)}$ is a positive matrix (i.e. $\tilde{M}^{(n)}_{u,v} \geq 1$ for all u and v),

(2) $|\tilde{V}_n| \geq 3$ and $v^n_{\mathrm{min}} \neq v^n_{\mathrm{max}}$,

(3) for each $v \in \tilde{V}_n$, $\tilde{M}^{(n)}_{vv^{n-1}_{\mathrm{min}}} = \tilde{M}^{(n)}_{vv^{n-1}_{\mathrm{max}}} = 1$,

(4) $\tilde{E}_n$ has the minimal/maximal vertex property,

(5) $\tilde{V}_n$ has distinct order lists. (In the case of $n = 1$, we ignore this property.)

2 Conjugacy between Cantor minimal system and Subshift

In this section we consider a properly ordered Bratteli diagram $\tilde{\mathcal{B}}$ satisfying Property 1.7. We will show that there is a topological conjugacy between the Bratteli-Vershik system $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ and a subshift. The details of shift spaces and its topology, see [LM] in §1 and §6.

Definition 2.1 (subshift). Suppose that A is a finite set, which will be called an *alphabet.* Let $A^{\mathbb{Z}}$ be the set of all biseqences $x = \ldots x_{-1}x_0x_1\ldots$ (with each x_i in A) equipped with the product topology. Then $A^{\mathbb{Z}}$ is a compact metrizable totally disconnected space, and shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $(\sigma x)_i = x_{i+1}$ is a homeomorphism. The restriction of σ to a closed invariant subset X of $A^{\mathbb{Z}}$ is called a *subshift* and such X is called a *shift space.* For $x \in A^{\mathbb{Z}}$ and $i, j \in \mathbb{Z}$ with $i \geq j$, set

$$x_{[i,j]} = x_i x_{i+1} \cdots x_j, \quad x_{[i,j)} = x_i x_{i+1} \cdots x_{j-1},$$

which are called *blocks* of x. Moreover set

$$B_n(X) = \{x_{[0,n-1)} \mid x \in X\}, \quad B(X) = \cup_{n \in \mathbb{N}} B_n(X).$$

Since X is shift invariant, we see that $B_n(X) = \{x_{[i,j)} \mid x \in X,\ j - i = n\}$ and hence $B_n(X)$ is the set of all (length) n-blocks that occur in points in X. We call $B(X)$ *the language of* X. For $B \in B_n(X)$ and i, j with $j - i + 1 = n$, put

$$[B]_i^j = \{x \in X \mid x_{[i,j]} = B\}.$$

Definition 2.2 (Subshift associated with $\tilde{\mathcal{B}}$). Suppose $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram.

(1) Let $\tau_k : X_{\tilde{\mathcal{B}}} \to \mathcal{P}(\tilde{V}_k)$ denote a truncation map, that is, $\tau_k x = (x_1, x_2, \cdots, x_k)$ where $x = (x_1, x_2, \cdots)$. Define a shift invariant subset $X_k \subset \mathcal{P}(\tilde{V}_k)^{\mathbb{Z}}$ to be

$$X_k = \left\{(\tau_k \lambda_{\tilde{\mathcal{B}}}^n x)_{n \in \mathbb{Z}} \;\middle|\; x \in X_{\tilde{\mathcal{B}}}\right\}.$$

One can show that X_k is a compact set. Let σ_k denote the restriction of shift to X_k.

(2) Put $\mathcal{P}(\tilde{V}_k)^* = \{p_1 p_2 \ldots p_n \mid n \in \mathbb{N}, p_1, p_2, \ldots, p_n \in \mathcal{P}(\tilde{V}_k)\}$. Define a concatenation map $\mathrm{Con}_k : \tilde{V} \setminus \cup_{i=0}^{k-1} \tilde{V}_i \to \mathcal{P}(\tilde{V}_k)^*$ to be

$$\mathrm{Con}_k(v) = (\tau_k q_1)(\tau_k q_2) \cdots (\tau_k q_{|\mathcal{P}(v)|}),$$

where $\{q_i\} = \mathcal{P}(v)$ satisfies $q_1 < q_2 < \cdots < q_{|\mathcal{P}(v)|}$ with respect to the order on $\mathcal{P}(v)$ arising from $\tilde{\geq}$. For $t \in \mathbb{Z}_+$, define a shift invariant subset $X_{k,t} \subset \mathcal{P}(\tilde{V}_k)^{\mathbb{Z}}$ to be

$$X_{k,t} = \{(p_n) \mid \exists \{n_i\}_{i \in \mathbb{Z}} \subset \mathbb{Z}\ \exists \{v_i\}_{i \in \mathbb{Z}} \subset \tilde{V}_{k+t} \text{ s.t. } p_{[n_i, n_{i+1})} = \mathrm{Con}_k(v_i)\ \forall i \in \mathbb{Z}\}.$$

Also one can show that $X_{k,t}$ is a compact set. Let $\sigma_{k,t}$ denote the restriction of the shift to $X_{k,t}$.

The relationship between $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ and (X_k, σ_k) is the following.

Theorem 2.3 ([S4] Theorem 2.3). *Suppose $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram satisfying Property 1.7. Then $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is topologically conjugate to (X_k, σ_k) for any $k \in \mathbb{N}$. The conjugacy $\pi_k : X_{\tilde{\mathcal{B}}} \to X_k$ is defined by*

$$\pi_k x = (\tau_k \lambda_{\tilde{\mathcal{B}}}^n x)_{n \in \mathbb{Z}}.$$

The relationship between $(X_{k+t,0}, \sigma_{k+t,0})$ and $(X_{k,t}, \sigma_{k,t})$ is the following theorem which is important so as to calculate the topological pressure of $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$.

Theorem 2.4 ([S4]: Theorem 2.6). *Suppose $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram satisfying Property 1.7. Then for any $k \in \mathbb{N}$ and $t \in \mathbb{Z}_+$, $(X_{k+t,0}, \sigma_{k+t,0})$ and $(X_{k,t}, \sigma_{k,t})$ are topologically conjugate. The conjugacy $\pi_{k,t} : X_{k+t,0} \to X_{k,t}$ is defined by*

$$\pi_{k,t}(\cdots x_{-1}.x_0 x_1 \cdots) = (\cdots (\tau_k x_{-1}).(\tau_k x_0)(\tau_k x_1) \cdots).$$

3 Calculation of topological pressure

The aim of this section is to calculate the topological pressure of a Bratteli-Vershik system in a special case. First we introduce the definition of topological pressure. The details of definitions and notations are written in [W1] or [W2].

3.1 Definitions and properties of topological pressure

Definition 3.1. Let (X, T) be a topological dynamical system. (I.e. X is a compact metric spase and T is a continuous transformation on X.) For $f \in C(X, \mathbb{R})$ and $n \in \mathbb{N}$, put $(S_n f)(x) = \sum_{i=0}^{n-1} f(T^i x)$. For $\varepsilon > 0$, put

$$\begin{aligned} Q_n(T, f, \varepsilon) &= \inf \left\{ \sum_{x \in F} e^{(S_n f)(x)} \mid F \text{ is a } (n, \varepsilon) \text{ spanning set for } X \right\}, \\ Q(T, f, \varepsilon) &= \limsup_{n \to \infty} \frac{1}{n} \log Q_n(T, f, \varepsilon), \\ P(T, f) &= \lim_{\varepsilon \to 0} Q(T, f, \varepsilon). \end{aligned}$$

Then it is easy to see that $P(T, f)$ exists but could be ∞. The map $P(T, \cdot) : C(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is called the *topological pressure of* T.

When T is expansive homeomorphism, we can calculate $P(T, f)$ as the following way. A finite open cover α of X is a *generator* for T if for every bisequence $\{A_n\}_{n=-\infty}^{\infty}$ of members of α, the set $\cap_{n=-\infty}^{\infty} T^{-n} \overline{A_n}$ contains at most one point of X. For an open cover α of X, $n \in \mathbb{N}$ and $f \in C(X, \mathbb{R})$, define

$$p_n(T, f, \alpha) = \inf \left\{ \sum_{A \in \beta} \sup_{x \in A} e^{(S_n f)(x)} \mid \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} T^{-i} \alpha \right\}.$$

Theorem 3.2 ([W1]: Lemma 9.3, Theorem 9.6). *Let T be an expansive homeomorphism of X. If α is a generator for T, then*

$$P(T,f) = \lim_{n\to\infty} \frac{1}{n} \log p_n(T,f,\alpha) = \inf_{N\in\mathbb{N}} \frac{1}{N} \log p_N(T,f,\alpha).$$

In the case of a subshift (X,σ) with alphabet A, $\alpha = \{[a]_0^0 \mid a \in A\}$ is generator for σ. Moreover we see that

- $\vee_{i=0}^{n-1}\sigma^{-i}\alpha = \{[B]_0^{n-1} \mid B \in B_n(X)\}$ and hence $\vee_{i=0}^{n-1}\sigma^{-i}\alpha$ is a finite cover of X,
- $\{[B]_0^{n-1} \mid B \in B_n(X)\}$ has no proper subcover.

So by Theorem 3.2 we have the following.

Proposition 3.3. *Suppose that (X,σ) is a subshift and $f \in C(X,\mathbb{R})$ is potential function. Then*

$$P(\sigma,f) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{B\in B_n(X)}\sup_{x\in[B]_0^{n-1}} e^{(S_nf)(x)}\right) = \inf_{N\in\mathbb{N}}\frac{1}{N}\log\left(\sum_{B\in B_N(X)}\sup_{x\in[B]_0^{N-1}} e^{(S_Nf)(x)}\right).$$

3.2 Topological pressre of Bratteli-Vershik systems

In this subsection we assume that $\tilde{\mathcal{B}}$ satisfies Property 1.7. First we calculate the topological pressure of $(X_{k,0},\sigma_{k,0})$ with respect to some special potential functions.

Definition 3.4. Suppose $\mathcal{B}$ is a properly ordered Bratteli diagram. We say that f is a *simple function on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_n)$* if for any $x, x' \in X_{\mathcal{B}}$ with $\tau_n x = \tau_n x'$, $f(x) = f(x')$ holds. Then for $p \in \mathcal{P}(V_n)$ we can define $f[p]_{\mathcal{B}} = f(x)$ if $x \in [p]_{\mathcal{B}}$. Since each cylinder set $[p]_{\mathcal{B}}$ is a clopen set, f is a continuous function.

For $g \in C(X_{\tilde{\mathcal{B}}},\mathbb{R})$ and $k \in \mathbb{N}$, let g_k denote a simple function based on $\mathcal{P}(\tilde{V}_k)$ satisfying $\lim_{k\to\infty} g_k = g$ as the supremum norm. We can extend g_k as a continuous function $\tilde{g}_k$ on $X_{k,0}$ to be

$$\tilde{g}_k(x) = g_k[x_0]_{\tilde{\mathcal{B}}},$$

where $x = (x_n) \in X_{k,0}$ and hence $\tilde{g}_k$ is a simple function on $X_{k,0}$.

Before we calculate the topological pressure, we will prepare the following lemmas.

Lemma 3.5 ([S4]: Lemma 3.6). *In the situation above, we have*

$$P(\sigma_{k,0},\tilde{g}_k) = \log\alpha_k,$$

where α_k is the maximum positive solution of the equation for x given by

$$\sum_{v\in\tilde{V}_k}\frac{\Gamma(v)}{x^{|\mathcal{P}(v)|}} = 1, \quad \textit{where } \Gamma(v) = \exp\left(\sum_{p\in\mathcal{P}(v)} g_k[p]_{\tilde{\mathcal{B}}}\right).$$

Lemma 3.6 ([S4]: Lemma 3.7).

$$P(\sigma_k, g \circ \pi_k^{-1}) = \lim_{t\to\infty} P(\sigma_{k,t}, \tilde{g}_{k+t} \circ \pi_{k,t}^{-1}).$$

Theorem 3.7 ([S4]: Thorem 3.8). *Suppose that $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram satisfying Property 1.7, g is a potential function on $X_{\tilde{\mathcal{B}}}$ and $\{g_n\}$ is a sequence of simple functions on $X_{\tilde{\mathcal{B}}}$ based on $\mathcal{P}(\tilde{V}_n)$ for each n satisfying $\lim_{n\to\infty} ||g - g_n|| = 0$. Suppose α_n is the unique positive solution of the equation for x given by*

$$\sum_{v\in\tilde{V}_n} \frac{\Gamma_n(v)}{x^{|\mathcal{P}(v)|}} = 1, \quad \textit{where } \Gamma_n(v) = \exp\left(\sum_{p\in\mathcal{P}(v)} g_n[p]_{\tilde{\mathcal{B}}}\right)$$

and $\lim_{n\to\infty} \alpha_n = \alpha$ exists. Then $P(\lambda_{\tilde{\mathcal{B}}}, g) = \log\alpha$.

Proof. By Theorem 2.3, $\lambda_{\tilde{\mathcal{B}}}$ and σ_k are conjugate and hence $P(\lambda_{\tilde{\mathcal{B}}}, g) = P(\sigma_k, g\circ\pi_k^{-1})$. By Theorem 2.4, $\sigma_{k+t,0}$ and $\sigma_{k,t}$ are conjugate and hence $P(\sigma_{k+t,0}, \tilde{g}_{k+t}) = P(\sigma_{k,t}, \tilde{g}_{k+t}\circ\pi_{k,t}^{-1})$. Therefore by Lemma 3.5 and 3.6 we have

$$P(\lambda_{\tilde{\mathcal{B}}}, g) = \lim_{t\to\infty} P(\sigma_{k,t}, \tilde{g}_{k+t}\circ\pi_{k,t}^{-1}) = \lim_{t\to\infty} P(\sigma_{k+t,0}, \tilde{g}_{k+t}) = \lim_{t\to\infty} \log\alpha_{k+t} = \log\alpha.$$

□

4 The modification of simple Bratteli diagram preserving equivalence relation

In this section, we give two modification propositions within equivalence relation of Bratteli diagrams. The first modification proposition is useful for the construction of a based diagram $\mathcal{C}$ in the main theorem. Using a given simple Bratteli diagram $\mathcal{B} = (V, E, \{M^{(n)}\})$ and a sequence of telescoping depths $\{t_n\}_{n\in\mathbb{Z}_+}$, $\mathcal{C} = (W, F, \{N^{(n)}\})$ is constructed by the following: (We call this construction *the vertex amalgamation.*)

The vertex amalgamation construction of $\mathcal{C}$. Define an equivalence relation $\sim$ on vertices of $(\mathcal{B}, \{t_n\})$ as

$$u \sim v \ (u, v \in V_{t_n}) \qquad \text{if} \qquad M_u^{(t_n, t_{n-1}+1)} = M_v^{(t_n, t_{n-1}+1)}.$$

We amalgamate V using $\sim$ and construct W. For $n \in \mathbb{N}$, we put

$$W_n = V_{t_n}/\sim.$$

For $x \in W_{n-1}$ and $w \in W_n$, define $N_{w,x}^{(n)}$ as

$$N_{w,x}^{(n)} = \sum_{v\in x} M_{u,v}^{(t_n, t_{n-1}+1)}, \quad \text{where } u \in w.$$

(In the case of $n = 1$, we put $v_0 \in w_0$ where $W_0 = \{w_0\}$, $V_0 = \{v_0\}$.) Note that this definition is independent of the choice of $u \in w$.

Remark 4.1. (1) We give an example of (stationary) Bratteli diagrams satisfying the conditions above. For any $n \in \mathbb{N}$, set $t_n = n$, $V_n = \{1,2,3,4,5,6\}$ and $W_n = \{w_1, w_2, w_3\}$. Incidence matrices $M^{(n)}$ and $N^{(n)}$ are defined by

$$M^{(1)} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \\ 3 \\ 4 \end{bmatrix}, N^{(1)} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, M^{(n)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 4 & 5 \end{bmatrix}, N^{(n)} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 8 & 10 \\ 3 & 8 & 10 \end{bmatrix} \ (n \geq 2).$$

Then we see that $1,2 \in w_1$, $3,5 \in w_2$ and $4,6 \in w_3$.

(2) In this example, $w_2 \neq w_3$ but $N^{(n)}_{w_2} = N^{(n)}_{w_3}$.

Proposition 4.2 ([S4]: Proposition 4.2). *Suppose $\mathcal{B} = (V, E, \{M^{(n)}\}_{n \in \mathbb{N}})$ is a simple Bratteli diagram and $\{t_n\}_{n \in \mathbb{Z}_+}$ is a sequence of telescoping depth satisfying that all $M^{(t_n, t_{n-1}+1)}$'s are positive matrices. Suppose $\mathcal{C}$ is the diagram constructed above. Then the following statements hold:*

(1) for any $n \in \mathbb{N}$ and $s \in \mathbb{N}$, $\#\{w \in W_n \mid |r^{-1}(w)| \leq s\} < 2^s$,

(2) for any $v \in w$, $|\mathcal{P}(v)| = |\mathcal{P}(w)|$,

(3) for any $0 \leq r < 1$, there exists $K \in \mathbb{N}$ such that $\sum_{w \in W_n} r^{|\mathcal{P}(w)|} < 1$ for all $n \geq K$,

(4) $\mathcal{B} \sim \mathcal{C}$.

Remark 4.3. (1) Suppose $\mathcal{B}$ and $\mathcal{C}$ are Bratteli diagrams satisfying Proposition 4.2. Then there is an onto map $\Phi : E' \to F$, where $E' = \cup_{n=1}^{\infty} E_{t_n, t_{n-1}+1}$, such that

(a) $\Phi(E_{t_n, t_{n-1}+1}) = F_n$,
(b) for any $e \in E'$, $s(e) \in s(\Phi(e))$ and $r(e) \in r(\Phi(e))$,
(c) for any $v \in V_{t_n}$ and $w \in W_n$ with $v \in w$, Φ is a bijection between $\{e \in E_{t_n, t_{n-1}+1} \mid r(e) = v\}$ and $r^{-1}(w)$,
(d) for any $\tilde{e} \in F_n$ and $e, e' \in \Phi^{-1}(\tilde{e})$, $s(e) = s(e')$.

Define a map $\Phi^n : \mathcal{P}(V_{t_n}) \to \mathcal{P}(W_n)$ as $x_1 x_2 \ldots x_{t_n} \mapsto \Phi(x_{[1,t_1]})\Phi(x_{(t_1,t_2]}) \ldots \Phi(x_{(t_{n-1},t_n]})$. We see that Φ^n is surjective and by Proposition 4.2 (2), the restriction of Φ^n to $\mathcal{P}(v)$ ($v \in V_{t_n}$) is injective and hence bijective. Moreover $\tau_n(\Phi^{n+1}(x_{[1,t_{n+1}]})) = \Phi^n(x_{[1,t_n]})$ holds for any $n \in \mathbb{N}$. Using Φ^n's, we define $\varphi : X_{\mathcal{B}} \to X_{\mathcal{C}}$ as

$$\varphi((x_n)_{n \in \mathbb{N}}) = (y_n)_{n \in \mathbb{N}} \quad \Leftrightarrow \quad \text{for any } n \in \mathbb{N},\ \Phi^n(x_{[1,t_n]}) = y_{[1,n]}.$$

Then we can show that φ is bijective by the following. It is clear that φ is surjective. For any fixed $y \in X_{\mathcal{C}}$, the number of paths in $\mathcal{P}(V_{t_n})$ corresponding to $y_{[1,n]}$ via Φ^n is $|V_{t_n, r(y_n)}|$. However, by the condition (d), sorce vertices of each edge in $E_{t_n+1, t_{n+1}}$ corresponding y_{n+1} via Φ are a same vertex. Therefore we can choose uniquely the path in $\mathcal{P}(V_{t_n})$ corresponding to $y_{[1,n]}$ via Φ^n. This means φ is injective.

(2) φ preserves the cofinal relation. I.e,

$$x \neq x' \in X_{\mathcal{B}} \text{ and } \forall n \geq N, x_n = x'_n \quad \Longrightarrow \quad \forall n \geq N,\ \varphi(x)_n = \varphi(x')_n.$$

Therefore, if we assign any proper order $\leq_{\mathcal{B}}$, $\leq_{\mathcal{C}}$ on $\mathcal{B}$, $\mathcal{C}$ respectively, φ is an orbit equivalence map. Moreover if $\leq_{\mathcal{B}}$ and $\leq_{\mathcal{C}}$ satisfies $\varphi(x_{\min}) = y_{\min}$ and $\varphi(x_{\max}) = y_{\max}$, φ is a strong orbit equivalence map.

(3) If f is a simple function on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_{t_{n-1}})$, then $f \circ \varphi^{-1}$ is a simple function on $X_{\mathcal{C}}$ but not based on $\mathcal{P}(W_{n-1})$ in general. Indeed, we can construct f satisfying $f[p]_{\mathcal{B}} \neq f[p']_{\mathcal{B}}$ where $p \neq p' \in \mathcal{P}(V_{t_{n-1}})$ with $\Phi^n(p) = \Phi^n(p')$. However, $f \circ \varphi^{-1}$ is based on $\mathcal{P}(W_n)$. We regard f as a simple function based on $\mathcal{P}(V_{t_n})$ by the following:

$$f(x) = f[\tau_{t_{n-1}} p]_{\mathcal{B}} \quad \text{if } x \in [p]_{\mathcal{B}},\ p \in \mathcal{P}(V_{t_n}).$$

By the condition (d), $\Phi^n(x_{[1,t_n]}) = \Phi^n(x'_{[1,t_n]})$ implies $s(x_{(t_{n-1},t_n]}) = s(x'_{(t_{n-1},t_n]})$ and hence by the condition (c), $x_{[1,t_{n-1}]} = x'_{[1,t_{n-1}]}$. Therfore we have

$$f \circ \varphi^{-1}(y) = f[\tau_{t_{n-1}} p]_{\mathcal{B}} \quad \text{if } y \in [\Phi^n(p)]_{\mathcal{C}}.$$

Here we introduce the "converce" construction of the vertex amalgamation, which are called *the vertex splitting.*

The vertex splitting construction of $\tilde{\mathcal{B}}$. Suppose $\mathcal{C} = (W, F, \{N^{(n)}\}_{n\in\mathbb{N}})$ is a simple Bratteli diagram. We construct $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{(n)}\}_{n\in\mathbb{N}})$ satisfying

- $\tilde{V}_n = \cup_{w\in W_n} \tilde{V}_{n,w}$ as disjoint union. (I.e., we split w into $|\tilde{V}_{n,w}|$ verteces in $\tilde{V}_n$.)
- For any $u, v \in \tilde{V}_{n,w}$, $\tilde{M}^{(n)}_u = \tilde{M}^{(n)}_v$.
- For any $u \in \tilde{V}_{n,w}$, $\sum_{v\in\tilde{V}_{n-1,x}} \tilde{M}^{(n)}_{u,v} = N^{(n)}_{w,x}$.

Remark 4.4. In the case of the vertex amalgamation construction, $\mathcal{C}$ is uniquely determined up to permutations of verteces. However, in the case of the vertex splitting construction, there are ambiguities of a number of veteces and connecting edges and hence $\tilde{\mathcal{B}}$ is not uniquely determined.

Proposition 4.5 ([S4]: Proposition 4.5). $\tilde{\mathcal{B}} \sim \mathcal{C}$.

Remark 4.6. (1) Suppose $\tilde{\mathcal{B}}$ and $\mathcal{C}$ are simple Bratteli diagrams constructed by Proposition 4.5. By similar arguments of Remark 4.3 (1), we have a bijection $\tilde{\varphi} : X_{\tilde{\mathcal{B}}} \to X_{\mathcal{C}}$ preserving the cofinal relation. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are simple Bratteli diagrams constructed by Proposition 4.2 and $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\mathcal{C}$ have proper orders $\leq_{\mathcal{B}}$, $\leq_{\tilde{\mathcal{B}}}$ and $\leq_{\mathcal{C}}$ satisfying

$$\varphi(x_{\min}) = \tilde{\varphi}(\tilde{x}_{\min}) = y_{\min} \text{ and } \varphi(x_{\max}) = \tilde{\varphi}(\tilde{x}_{\max}) = y_{\max}.$$

Then $\tilde{\varphi}^{-1} \circ \varphi$ is a strong orbit equivalence map between $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$.

(2) Let $\tilde{\Phi}^n : \mathcal{P}(\tilde{V}_n) \to \mathcal{P}(W_n)$ be an onto map which provides a conjugacy $\tilde{\varphi}$ (see Remark 4.3 (1)) and h be a simple function on $X_{\mathcal{C}}$ based on $\mathcal{P}(W_n)$. Then we see that for any $\tilde{x}, \tilde{x}' \in X_{\tilde{\mathcal{B}}}$ with $\tilde{\Phi}^n \circ \tau_n(\tilde{x}) = \tilde{\Phi}^n \circ \tau_n(\tilde{x}') = q$,

$$h \circ \tilde{\varphi}(\tilde{x}) = h \circ \tilde{\varphi}(\tilde{x}') = h[q]_{\mathcal{C}}.$$

This implies that for any $v, v' \in \tilde{V}_{n,w}$,

$$\sum_{p \in \mathcal{P}(v)} h \circ \tilde{\varphi}[p]_{\tilde{\mathcal{B}}} = \sum_{p \in \mathcal{P}(v')} h \circ \tilde{\varphi}[p]_{\tilde{\mathcal{B}}} = \sum_{q \in \mathcal{P}(w)} h[q]_{\mathcal{C}}.$$

5 Sketch of proving Theorem 1.1

5.1 Requirements of a simple Bratteli diagram for (Y, ψ).

By Theorem 9.7 in [W1], for a topological dynamical system (X, T) and potential function $f \in C(X, \mathbb{R})$,

$$h(T) + \inf f < P(T, f) < h(T) + \sup f$$

and so $P(T, f) = \infty$ iff $h(T) = \infty$. In the case of $\alpha = \infty$, there exists a Cantor minimal system (Y, ψ) strongly orbit equivalent to (X, ϕ) such that $h(\psi) = \infty$ (see [S2]). This means

$$P(\psi, f \circ \theta^{-1}) = \infty.$$

So we only consider the case where α is finite. Let $\mathcal{B} = (V, E, \{M^{(n)}\}, \geq)$ be a properly ordered Bratteli diagram which is a representation of (X, ϕ). So we identify (X, ϕ) with $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$. From the simplicity of diagram, we may assume that all $M^{(n)}$'s are positive matrices. We only consider within a strong orbit equivalence class of (X, ϕ). So applying Proposition 4.2 to $\mathcal{B}$, we may also assume that

$$\forall n, s \in \mathbb{N},\ \#\{v \in V_n \mid |r^{-1}(v)| \leq s\} \leq 2^s, \tag{5.1}$$

$$0 \leq \forall r < 1,\ \exists K \in \mathbb{N} \text{ s.t.} \forall n \geq K,\ \sum_{v \in V_n} r^{|\mathcal{P}(v)|} < 1. \tag{5.2}$$

Choose any sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ satisfying $0 < \frac{1}{3}\varepsilon_n < \varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$ and fix it. Now we will construct a properly ordered Bratteli diagram $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{(n)}\}, \tilde{\geq})$ which is a representation of (Y, ψ). First, applying the vertex amalgamation construction to $(\mathcal{B}, \{t_n\})$ for some suitable telescoping depths $\{t_n\}_{n \in \mathbb{Z}_+}$, we have a based Bratteli diagram $\mathcal{C} = (W, F, \{N^{(n)}\})$ with $\mathcal{C} \sim \mathcal{B}$ (see Proposition 4.2). Second, applying the vertex splitting construction to $\mathcal{C}$, we temporarily have $\tilde{\mathcal{B}}$ with $\tilde{\mathcal{B}} \sim \mathcal{C}$ (see Proposition 4.5 and Remark 4.4). Suppose $\mathcal{C}$ is determined. Define $\varphi : X_{\mathcal{B}} \to X_{\mathcal{C}}$ as Remark 4.3 (1) and $\tilde{\varphi} : X_{\tilde{\mathcal{B}}} \to X_{\mathcal{C}}$ as Remark 4.6 (1). Define a simple function f_n on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_{t_n})$ as

$$f_n(x) = \min\{f(y) \mid y \in [p]_{\mathcal{B}}\} \quad \text{where } x \in [p]_{\mathcal{B}}.$$

(Set $\mathcal{P}(V_0) = \emptyset$ and $[\emptyset]_{\mathcal{B}} = X_{\mathcal{B}}$. Then $f_0(x) = \min\{f(y) \mid y \in X_{\mathcal{B}}\}$.) We see that

- $\{f_n\}$ is monotone increasing and $\lim_{n\to\infty} ||f - f_n|| = 0$,
- $f_{n-1} \circ \varphi^{-1}$ is a simple function on $X_{\mathcal{C}}$ based on $\mathcal{P}(W_n)$,
- for any $v \in w$ ($w \in W_n$),

$$\sum_{p\in\mathcal{P}(v)} f_{n-1}[\tau_{t_{n-1}}p]_{\mathcal{B}} = \sum_{q\in\mathcal{P}(w)} f_{n-1} \circ \varphi^{-1}[q]_{\mathcal{C}} \tag{5.3}$$

(see Remark 4.3 (3)). Define

$$g_n = f_{n-1} \circ \varphi^{-1} \circ \tilde{\varphi} \quad \text{and} \quad g = f \circ \varphi^{-1} \circ \tilde{\varphi}.$$

Then g_n is a simple function on $X_{\tilde{\mathcal{B}}}$ based on $\mathcal{P}(\tilde{V}_n)$ and for any $w \in W_n$ and $v, v' \in \tilde{V}_{n,w}$,

$$\sum_{p\in\mathcal{P}(v)} g_n[p]_{\tilde{\mathcal{B}}} = \sum_{p\in\mathcal{P}(v')} g_n[p]_{\tilde{\mathcal{B}}} = \sum_{q\in\mathcal{P}(w)} f_{n-1} \circ \varphi^{-1}[q]_{\mathcal{C}}$$

(see Remark 4.6 (2)). So we define $\Gamma_n[w]$ as

$$\Gamma_n[w] = \exp\left(\sum_{q\in\mathcal{P}(w)} f_{n-1} \circ \varphi^{-1}[q]_{\mathcal{C}}\right) = \exp\left(\sum_{\tilde{p}\in\mathcal{P}(v)} g_n[\tilde{p}]_{\tilde{\mathcal{B}}}\right) = \Gamma_n(v), \tag{5.4}$$

where $\tilde{v} \in \tilde{V}_{n,w}$ (see Theorem 3.7). We will completely construct $\tilde{\mathcal{B}}$ satisfying the following conditions: For each $n \in \mathbb{N}$,

(1) $\alpha + \varepsilon_n < \alpha_n < \alpha + \varepsilon_{n-1}$ and $\displaystyle\sum_{w\in W_n} \frac{|\tilde{V}_{n,w}|\Gamma_n[w]}{(\alpha_n)^{|\mathcal{P}(w)|}} = 1 \left(\Leftrightarrow \sum_{v\in\tilde{V}_n} \frac{\Gamma_n(v)}{(\alpha_n)^{|\mathcal{P}(v)|}} = 1\right)$,

(2) $\tilde{\mathcal{B}}$ satisfies Property 1.7.

Then (X, ϕ) is strongly orbit equivalent to (Y, ψ) and $\theta = \tilde{\varphi}^{-1} \circ \varphi : X \to Y$ is a strong orbit equivalence map (see Remark 4.6 (1)). Applying Theorem 3.7 to $\tilde{\mathcal{B}}$, we have

$$P(\psi, f \circ \theta^{-1}) = P(\lambda_{\tilde{\mathcal{B}}}, g) = \lim_{n\to\infty} \log \alpha_n = \log \alpha.$$

Finally by Theorem 2.3, (Y, ψ) is topologically conjugate to a subshift.

5.2 Preliminary

In this subsection, we will introduce some lemmas.

Lemma 5.1 ([S4]: Lemma 5.3). *For all $n \in \mathbb{N}$, $\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+2}{e}\right)^{n+2}$.*

Let f be a function of $X_{\mathcal{B}}$. For $x \in X_{\mathcal{B}}$ and $m \in \mathbb{N}$, put

$$S(f,x,m) = \frac{1}{m}\sum_{i=0}^{m-1} f(\lambda_{\mathcal{B}}^i x).$$

Lemma 5.2 ([S4]: Lemma 5.4). *Suppose $\mathcal{B} = (V, E, \geq)$ is a properly ordered Bratteli diagram, f is a simple function on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_N)$. For any $\beta > \exp(\sup\{\int f d\mu \mid \mu \in \mathcal{M}(X_{\mathcal{B}}, \lambda_{\mathcal{B}})\})$, there exists $N' \geq N$ such that for any $n \geq N'$ and $v \in V_n$,*

$$\beta^{|\mathcal{P}(v)|} > \exp\left(\sum_{p\in\mathcal{P}(v)} f[\tau_N p]_{\mathcal{B}}\right).$$

5.3 The construction of a based diagram $\mathcal{C}$.

If $\{t_n\}$ is decided, we can construct $\mathcal{C}$ by the vertex amalgamation construction. Now, we will decide $\{t_n\}$ by induction.

The 1st step. Put $t_0 = 0$. Applying Lemma 5.2 to f_0 and $\mathcal{B}$, there exists $t_1 \in \mathbb{N}$ satisfying

$$\left(\alpha + \frac{1}{3}\varepsilon_1\right)^{|\mathcal{P}(v)|} > \exp\left(\sum_{p\in\mathcal{P}(v)} f_0[\tau_{t_0} p]_{\mathcal{B}}\right), \quad \left(\frac{\alpha+\varepsilon_2}{\alpha+\frac{1}{3}\varepsilon_2}\right)^{|\mathcal{P}(v)|} > 2$$

for all $v \in V_{t_1}$. (The second part of inequality above holds because $\min_{v\in V_{t_1}} |\mathcal{P}(v)|$ is monotone increaseing with respect to t_1.) We fix t_1. Then we can construct W_1 and $N^{(1)}$ of $\mathcal{C}$ by the vertex amalgamation construction. Since $|\mathcal{P}(w)| = |\mathcal{P}(v)|$ holds for $v \in w$, the first part of inequality above is equivalent to $\left(\alpha + \frac{1}{3}\varepsilon_1\right)^{|\mathcal{P}(w)|} > \Gamma_1[w]$ holds for any $w \in W_1$ (see (5.3) and (5.4)). Let $\{A_w^{(1)} \in \mathbb{N} \mid w \in W_1\}$ satisfy

$$A_w^{(1)} > 2 + \max\left\{\frac{(\alpha+\varepsilon_1)^{|\mathcal{P}(w)|}}{\Gamma_1[w]}, |V_{t_1,w}|\right\},$$

where $V_{t_1,w} = \{v \in V_{t_1} \mid v \in w\}$. Then there exists a unique number $\alpha_1 > \alpha + \varepsilon_1$ such that

$$\sum_{w\in W_1} \frac{A_w^{(1)}\Gamma_1[w]}{(\alpha_1)^{|\mathcal{P}(w)|}} = 1.$$

Choose any $\varepsilon_0 > \alpha_1 - \alpha$ and fix it.

The n-th step. For $n \geq 2$, suppose the $(n-1)$-th step data are given by the following: For any $w \in W_{n-1}$,

(D_{n-1}-1) $(\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|} < (A_w^{(n-1)} - 2)\Gamma_{n-1}[w]$,

(D_{n-1}-2) $|V_{t_{n-1},w}| < A_w^{(n-1)} - 2$.

Choose $r_w \in \mathbb{R}$ satisfying (5.5) and fix it.

$$1 < r_w < \min\left(\frac{3}{2}, \frac{(A_w^{(n-1)}-2)\Gamma_{n-1}[w]}{(\alpha+\varepsilon_{n-1})^{|\mathcal{P}(w)|}}\right) \tag{5.5}$$

For any fixed $t_n > t_{n-1}$, we can temporarily construct W_n and $N^{(n)}$ by the vertex amalgamation construction. Define $Q_{x,w} \in \mathbb{N}$ and $R_{x,w} \in \mathbb{Z}_+$ to be the unique numbers such that

$$N_{x,w}^{(n)} - 2 = (A_w^{(n-1)}-2)Q_{x,w} + R_{x,w} \quad \text{and} \quad 0 \leq R_{x,w} < A_w^{(n-1)} - 2. \tag{5.6}$$

Define $B_x, C_{x,w}, D_{x,w}$ as

$$B_x = \frac{((\bar{N}_x^{(n)}-2)/e)^{\bar{N}_x^{(n)}-2}}{\prod_{w\in W_{n-1}}((r_w Q_{x,w}+2)/e)^{N_{x,w}^{(n)}+2A_w^{(n-1)}}},$$

$$C_{x,w} = \left\{ (n_v) \in \mathbb{N}^{|V_{t_{n-1},w}|} \;\middle|\; \sum_{v\in w} n_v = N_{x,w}^{(n)} \right\},$$

$$D_{x,w} = \left\{ (n_i) \in \mathbb{N}^{A_w^{(n-1)}-2} \;\middle|\; \sum_{i=1}^{A_w^{(n-1)}-2} n_i = N_{x,w}^{(n)} - 2,\ 1 \leq n_i < r_w Q_{x,w} \right\}.$$

Now we can show that Claim 5.3 holds for sufficiently large t_n.

Claim 5.3. For any $x \in W_n$,

(1) $\Gamma_n[x] < (\alpha + \frac{1}{3}\varepsilon_n)^{|\mathcal{P}(x)|}$,

(2) $B_x\Gamma_n[x](\alpha+\varepsilon_{n-1})^{-|\mathcal{P}(x)|} > 1$,

(3) for any $w \in W_{n-1}$, $|C_{x,w}| < |D_{x,w}|$,

(4) $\sum_{x\in W_n} \frac{2(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|}}{(\alpha+\varepsilon_{n-1})^{|\mathcal{P}(x)|}} < 1$.

Put t_n satisfying Claim 5.3. Then we can define $A_x^{(n)} \in \mathbb{N}$ as

$$(A_x^{(n)}-3)\Gamma_n[x] \leq (\alpha+\varepsilon_n)^{|\mathcal{P}(x)|} < (A_x^{(n)}-2)\Gamma_n[x] < A_x^{(n)}\Gamma_n[x] < 2(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|} \tag{5.7}$$

becase of Claim 5.3 (1). So we have the n-th step data by the following: For any $x \in W_n$,

(D_n-1) $(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|} < (A_x^{(n)}-2)\Gamma_n[x]$,

(D_n-2) $|V_{t_n,x}| < A_x^{(n)} - 2$.

5.4 The construction of $\tilde{\mathcal{B}}$.

In this subsection we will construct $\tilde{V}_n$, $\tilde{M}^{(n)}$ and an order $\tilde{\geq}$ on $\tilde{E}_n$ satisfying Property 1.7 and check that for each $n \in \mathbb{N}$,

$$\alpha + \varepsilon_n < \alpha_n < \alpha + \varepsilon_{n-1} \quad \text{and} \quad \sum_{x \in W_n} \frac{|\tilde{V}_{n,x}| \Gamma_n[x]}{(\alpha_n)^{|\mathcal{P}(x)|}} = 1. \tag{5.8}$$

<u>**The construction of $\tilde{V}_n$.**</u> For $x \in W_n$, we set

$$|\tilde{V}_{n,x}| = A_x^{(n)}. \tag{5.9}$$

By the condition (D$_n$-2), $|\tilde{V}_{n,w}| \geq 3$ holds. Let $* \in W_n$ ($** \in W_n$ resp.) denote the vertex satisfying that the minimal path $x_{\min} \in X_{\mathcal{B}}$ (the maximal path $x_{\max} \in X_{\mathcal{B}}$ resp.) goes through some vertex in $V_{t_n,*}$ ($V_{t_n,**}$ resp.). We can choose any distinct vertices $v_{\min}^n \in \tilde{V}_{n,*}$ and $v_{\max}^n \in \tilde{V}_{n,**}$ becase of $|\tilde{V}_{n,w}| \geq 3$ and fix them.
<u>**The construction of $\tilde{M}^{(n)}$.**</u> We consider the following conditions with respect to $\tilde{M}^{(n)}$:

(c.0) If $x, x' \in W_n$ with $x \neq x'$, then $\tilde{M}_v^{(n)} \neq \tilde{M}_{v'}^{(n)}$, where $v \in \tilde{V}_{n,x}$ and $v' \in \tilde{V}_{n,x'}$.

(c.1) For any $v, v' \in \tilde{V}_{n,x}$, $\tilde{M}_{n,v}^{(n)} = \tilde{M}_{n,v'}^{(n)}$.

(c.2) For any $v \in \tilde{V}_{n,x}$,

$$(\tilde{M}_{v,u}^{(n)})_{u \in \tilde{V}_{n-1,w}} \in \tilde{D}_{x,w}$$

where $\tilde{D}_{x,w}$ is defined by

$$\tilde{D}_{x,w} = \begin{cases} \left\{ (n_u) \in \mathbb{N}^{|\tilde{V}_{n-1,w}|} \mid \sum_{u \in \tilde{V}_{n-1,w}} n_u = N_{x,w}^{(n)},\ 1 \leq n_u < r_w Q_{x,w} \right\} & \text{if } w \neq *, **, \\ \left\{ (n_u) \in \mathbb{N}^{|\tilde{V}_*|} \mid \sum_{u \in \tilde{V}_*} n_u = N_{x,*}^{(n)} - 1,\ 1 \leq n_u < r_w Q_{x,*} \right\} & \text{if } * \neq ** \text{ and } w = *, \\ \left\{ (n_u) \in \mathbb{N}^{|\tilde{V}_{**}|} \mid \sum_{u \in \tilde{V}_{**}} n_u = N_{x,**}^{(n)} - 1,\ 1 \leq n_u < r_w Q_{x,**} \right\} & \text{if } * \neq ** \text{ and } w = **, \\ \left\{ (n_u) \in \mathbb{N}^{|\tilde{V}_{***}|} \mid \sum_{u \in \tilde{V}_{***}} n_u = N_{x,*}^{(n)} - 2,\ 1 \leq n_u < r_w Q_{x,*} \right\} & \text{if } * = ** \text{ and } w = *, \end{cases}$$

where $\tilde{V}_* = \tilde{V}_{n-1,*} \setminus \{v_{\min}^{n-1}\}$, $\tilde{V}_{**} = \tilde{V}_{n-1,**} \setminus \{v_{\max}^{n-1}\}$ and $\tilde{V}_{***} = \tilde{V}_{n-1,*} \setminus \{v_{\min}^{n-1}, v_{\max}^{n-1}\}$.

(c.3) $\tilde{M}_{v,v_{\min}^{n-1}}^{(n)} = \tilde{M}_{v,v_{\max}^{n-1}}^{(n)} = 1$ for any $v \in \tilde{V}_n$.

It is easy to construct $\tilde{M}^{(n)}$ satisfying the conditions (c.1), (c.2) and (c.3) and these conditions imply that $\tilde{\mathcal{B}}$ and $\mathcal{C}$ satisfy the assumptions in Proposition 4.5. Now we will show that we can construct it satisfying also the condition (c.0).

Suppose that $\tilde{M}^{(n)}$ satisfies the conditions (c.1), (c.2) and (c.3). It is clear that if $N_w^{(n)} \neq N_x^{(n)}$, then $\tilde{M}_u^{(n)} \neq \tilde{M}_v^{(n)}$ where $u \in \tilde{V}_{n,w}$ and $v \in \tilde{V}_{n,x}$. In general, $x \neq x' \in W_n$ does not imply $N_x^{(n)} \neq N_{x'}^{(n)}$ (see Remark 4.1 (2)) and so we will show that for any $x \neq x' \in W_n$ with $N_x^{(n)} = N_{x'}^{(n)}$, we can construct $\tilde{M}_v^{(n)}$ and $\tilde{M}_{v'}^{(n)}$ satisfying $\tilde{M}_v^{(n)} \neq \tilde{M}_{v'}^{(n)}$ for $v \in \tilde{V}_{n,x}$ and $v' \in \tilde{V}_{n,x'}$. By the construction of $N^{(n)}$, we see that

$$\# \{s \in W_n \mid N_s^{(n)} = N_x^{(n)}\} \leq \prod_{w \in W_{n-1}} |C_{x,w}|. \tag{5.10}$$

As $\tilde{M}_v^{(n)}$ and $\tilde{M}_{v'}^{(n)}$ satisfy the condition (c.2), by Claim 5.3 (3) and (5.10) we have

$$\# \{s \in W_n \mid N_s^{(n)} = N_x^{(n)}\} \leq \prod_{w \in W_{n-1}} |D_{x,w}| \leq \prod_{w \in W_{n-1}} |\tilde{D}_{x,w}|. \tag{5.11}$$

The right part of the inequality (5.11) means what the maximum possible value for incidence vectors in $\mathbb{N}^{|\tilde{V}_{n-1}|}$ satisfying the condition (c.2) is. Therefore, we can choose incidence vectors satisfying $\tilde{M}_v^{(n)} \neq \tilde{M}_{v'}^{(n)}$.

<u>**The construction of $\tilde{\geq}$.**</u> We will check that we can construct $\tilde{\geq}$ on $\tilde{E}$ with the property that each $\tilde{V}_n$ has distinct order lists (Property 1.7 (5)). For $x \in W_n$, define $\mathrm{Dist}(x) \in \mathbb{N}$ as

$$\mathrm{Dist}(x) = \frac{\left(\sum_{u \in \tilde{V}_{n-1}^*} \tilde{M}_{v,u}^{(n)}\right)!}{\prod_{u \in \tilde{V}_{n-1}^*} \tilde{M}_{v,u}^{(n)}!} = \frac{(\tilde{N}_x^{(n)} - 2)!}{\prod_{u \in \tilde{V}_{n-1}^*} \tilde{M}_{v,u}^{(n)}!},$$

where $v \in \tilde{V}_{n,x}$ and $\tilde{V}_{n-1}^* = \tilde{V}_{n-1}/\{v_{\min}^{n-1}, v_{\max}^{n-1}\}$. $\mathrm{Dist}(x)$ means the maximal possible number of order lists of $v \in \tilde{V}_{n,x}$ satisfying Property 1.7 (4). Suppose $w \neq x$, $u \in \tilde{V}_{n,w}$ and $v \in \tilde{V}_{n,x}$. By the condition (c.0), if we assign any order on $r^{-1}(u)$, $r^{-1}(v)$ respectively, $\mathrm{List}(u) \neq \mathrm{List}(v)$ always holds. Therefore $\tilde{V}_n$ can have distinct order lists if and only if

$$\mathrm{Dist}(x) \geq |\tilde{V}_{n,x}|$$

for any x and hence we check this inequality. Since $\tilde{M}^{(n)}$ satisfies the conditions (c.2) and (c.3), using Claim 5.3 (2) and Lemma 5.1, we have

$$\begin{aligned}\frac{(\mathrm{Dist}(x) - 3)\Gamma_n[x]}{(\alpha + \varepsilon_n)^{|\mathcal{P}(x)|}} &> \frac{\left((\tilde{N}_x^{(n)} - 2)/e\right)^{\tilde{N}_x^{(n)}-2} \Gamma_n[x]}{\prod_{u \in \tilde{V}_{n-1}^*} \left((\tilde{M}_{v,u}^{(n)} + 2)/e\right)^{\tilde{M}_{v,u}^{(n)}+2}} \times (\alpha + \varepsilon_n)^{-|\mathcal{P}(x)|} \\ &> \frac{\left((\tilde{N}_x^{(n)} - 2)/e\right)^{\tilde{N}_x^{(n)}-2} \Gamma_n[x]}{\prod_{w \in W_{n-1}} ((r_w Q_{x,w} + 2)/e)^{N_{x,w}^{(n)}+2|\tilde{V}_{n-1,w}|}} \times (\alpha + \varepsilon_{n-1})^{-|\mathcal{P}(x)|} = B_x \Gamma_n[x] > 1,\end{aligned}$$

where $v \in \tilde{V}_{n,x}$. (We use the fact that if $n \geq 4$, then $n! - 3 > (\frac{n}{e})^n$ holds.) Therefore

$$\mathrm{Dist}(x) > (\alpha + \varepsilon_n)^{|\mathcal{P}(x)|} \Gamma_n[x]^{-1} + 3 \geq |\tilde{V}_{n,x}|$$

because of (5.7) and (5.9).
The check of (5.8). By (5.7), (5.9) and Claim 5.3 (4), we have

$$\sum_{x\in W_n} \frac{|\tilde{V}_{n,x}|\Gamma_n[x]}{(\alpha+\varepsilon_{n-1})^{|\mathcal{P}(x)|}} < 1.$$

The n-th step data (D_n-1) implies that $(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|} < |\tilde{V}_{n,x}|\Gamma_n[x]$. Therefore there exists unique α_n with $\alpha+\varepsilon_n < \alpha_n < \alpha+\varepsilon_{n-1}$ such that

$$\sum_{x\in W_n} \frac{|\tilde{V}_{n,x}|\Gamma_n[x]}{(\alpha_n)^{|\mathcal{P}(x)|}} = 1.$$

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