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Geometry of finite-dimensional maps (Pasynkov の定理の精密化)

筑波大学・数理物質科学研究科・加藤 久男 (Hisao Kato)

Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan

Abstract. In [2 and 3], Pasynkov proved the following theorem: If $f: X \to Y$ is a map of compacta such that f is a k-dimensional map and $\dim Y = p < \infty$, then the set of maps g in the space $C(X, I^{p+2k+1})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1}$ is an embedding is a G_δ-dense subset of $C(X, I^{p+2k+1})$. In this paper, furthermore we investigate the geometric properties of finite-dimensional maps and finite-to-one maps. We prove that if $f : X \to Y$ is a map as above, then for each $0 \leq i \leq p+k$, the set of maps g in the space $C(X, I^{p+2k+1-i})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1-i}$ is an $(i+1)$-to-1 map is a G_δ-dense subset of $C(X, I^{p+2k+1-i})$. The case $i = 0$ implies the result of Pasynkov. Also, if Y is a one point set, our result implies the following Hurewicz's theorem: If $\dim X = n < \infty$ and $0 \leq i \leq n$, then the set of maps g in the space $C(X, I^{2n+1-i})$ such that $g : X \to I^{2n+1-i}$ is an $(i+1)$-to-1 map is a G_δ-dense subset of $C(X, I^{2n+1-i})$. As a corollary, we have the following representation theorem of finite-dimensional maps: For a map $f : X \to Y$ of compacta such that $0 \leq k < \infty$ and $\dim Y = p < \infty$, f is a k-dimensional map if and only if f can be represented as the composition $f = g_{p+2k+1} \circ \ldots \circ g_{p+k+2} \circ g_{p+k+1} \circ g_{p+k} \circ \ldots \circ g_1$ of maps g_i $(i = 1, 2, .., p+2k+1)$ paralell to the unit interval I such that g_i is an $(i+1)$-to-1 map for each $i = 1, 2, .., p+k$ and g_{p+k+1} is a zero-dimensional map.

$$X = X_0 \xrightarrow{g_1} X_1 \xrightarrow{..} .. \xrightarrow{g_{p+k}} X_{p+k} \xrightarrow{g_{p+k+1}} X_{p+k+1}$$
$$\xrightarrow{g_{p+k+2}} X_{p+k+2} \xrightarrow{....} X_{p+2k} \xrightarrow{g_{p+2k+1}} X_{p+2k+1} = Y$$

1 Introduction.

All spaces considered in this paper are assumed to be separable metric spaces. *Maps* are continuous functions. Let $I = [0, 1]$ be the unit interval. By a *compactum* we mean a nonempty compact metric space. Let X and Y be compacta. Then $C(X, Y)$ denotes the space of all maps $g : X \to Y$ with the usual sup-metric. Note that $C(X, Y)$ is a complete metric space.

A map $f : X \to Y$ is a *k-dimensional map* $(0 \leq k < \infty)$ if for each $y \in Y$ $\dim f^{-1}(y) \leq k$, where $\dim Z$ denotes the topological dimension of a space Z. If a map $f : X \to Y$ is a k-dimensional map, we write $\dim f \leq k$. A map $f : X \to Y$ is a k-to-1 map if for each $y \in Y$, the cardinal number $|f^{-1}(y)|$ of $f^{-1}(y)$ is equal to or less than k.

In [2 and 3], Pasynkov proved that if $f : X \to Y$ is a k-dimensional map from a compactum X to a finite dimensional compactum Y, then there is a map $g : X \to I^k$ such that dim $(f \times g) = 0$. Also, he proved that if $f : X \to Y$ is a map of compacta such that f is a k-dimensional map and $\dim Y = p < \infty$, then the set of maps g in the space $C(X, I^{p+2k+1})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1}$ is an embedding is a G_δ-dense subset of $C(X, I^{p+2k+1})$.

In this paper, furthermore we investigate the geometric properties of finite-dimensional maps and finite-to-one maps. We prove that if $f : X \to Y$ is a map of compacta such that f is a k-dimensional map and $\dim Y = p < \infty$, then for each $0 \le i \le p+k$, the set of maps g in the space $C(X, I^{p+2k+1-i})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1-i}$ is an $(i+1)$-to-1 map is a G_δ-dense subset of $C(X, I^{p+2k+1-i})$. Note that the restriction $g|f^{-1}(y) : f^{-1}(y) \to I^{p+2k+1-i}$ is an $(i+1)$-to-1 map for each $y \in Y$. Also, note that the case $i = 0$ implies the result of Pasynkov, and our proof in this paper is different from the proof of Pasynkov (see [3]). Also, if Y is a one point set, our result implies that if $\dim X = n < \infty$ and $0 \le i \le n$, then the set of maps g in the space $C(X, I^{2n+1-i})$ such that $g : X \to I^{2n+1-i}$ is an $(i+1)$-to-1 map is a G_δ-dense subset of $C(X, I^{2n+1-i})$. As a corollary, we have the following representation theorem of finite-dimensional maps: For a map $f : X \to Y$ of compacta such that $0 \le k < \infty$ and $\dim Y = p < \infty$, f is a k-dimensional map if and only if f can be represented as the composition $f = g_{p+2k+1} \circ \ldots \circ g_{p+k+2} \circ g_{p+k+1} \circ g_{p+k} \circ \ldots \circ g_1$ of maps g_i $(i = 1, 2, .., p+2k+1)$ paralell to the unit interval I (for the definition, see section 3) such that g_i is an $(i+1)$-to-1 map for each $i = 1, 2, .., p+k$ and g_{p+k+1} is a zero-dimensional map.

$$X = X_0 \xrightarrow{g_1} X_1 \xrightarrow{..} \; .. \xrightarrow{g_{p+k}} X_{p+k} \xrightarrow{g_{p+k+1}} X_{p+k+1}$$
$$\xrightarrow{g_{p+k+2}} X_{p+k+2} \xrightarrow{....} X_{p+2k} \xrightarrow{g_{p+2k+1}} X_{p+2k+1} = Y$$

Note that the maps g_i $(p+k+2 \le i \le p+2k+1)$ are 1-dimensional maps.

2 Main theorem.

A map $h : X \to Y$ is a (p, ϵ)-map $(\epsilon > 0)$ if for each $y \in Y$, there are subsets $A_1, A_2, .., A_p$ of $h^{-1}(y)$ such that $h^{-1}(y) = \bigcup_{i=1}^{p} A_i$ and diam $A_i < \epsilon$ for each i. Let $f : X \to Y$ be a map and $A \subset X$. Then $f|A : A \to Y$ is a *strict embedding* for f if $f|A$ is an embedding and $f^{-1}(f(A)) = A$. Note that $f|A : A \to Y$ is a strict embedding for f if and only if $A \subset \{x \in X|\ f^{-1}(f(x)) = \{x\}\}$.

In this paper, we need the following key lemma of Toruńczyk [4, Lemma 2].

Lemma 2.1. *Let $\epsilon > 0$. Suppose that $f : X \to Y$ is a map of compacta with $\dim f = 0$ and $\dim Y = p < \infty$. For each $i = 1, 2, .., l$, let K_i and L_i be closed* *disjoint subsets of* X. *Then there are open subsets* E_i *of* X *separating* X *between* K_i *and* L_i *such that* $f|(Cl(E_1) \cup ... \cup Cl(E_l))$ *is a* (p, ϵ)*-map.*

The next proposition was proved by Pasynkov in [2] (see also [4, Corollary 1] and [1, p. 48]).

Proposition 2.2. *If* $f : X \to Y$ *is a k-dimensional map from a compactum* X *to a finite dimensional compactum* Y, *then the set of maps* g *in* $C(X, I^k)$ *such that* $\dim\ (f \times g) = 0$ *is a* G_δ*-dense subset of* $C(X, I^k)$.

The following lemma is easily proved.

Lemma 2.3. *Let* X *and* Y *be compacta and* A *a closed subset of* X. *Let* $C(X, Y; A, p)$ *be the set of all maps* $g : X \to Y$ *such that* $g|A$ *is a p-to-1 map. Then* $C(X, Y; A, p)$ *is* G_δ *in* $C(X, Y)$.

Theorem 2.4. *If* $f : X \to Y$ *is a map of compacta such that* f *is a k-dimensional map and* $\dim Y = p < \infty$, *then for each* $0 \leq i \leq p + k$, *the set of maps* g *in the space* $C(X, I^{p+2k+1-i})$ *such that the diagonal product* $f \times g : X \to Y \times I^{p+2k+1-i}$ *is an* $(i+1)$*-to-1 map is a* G_δ*-dense subset of* $C(X, I^{p+2k+1-i})$. *Hence the restriction* $g|f^{-1}(y) : f^{-1}(y) \to I^{p+2k+1-i}$ *is an* $(i + 1)$*-to-1 map for each* $y \in Y$.

3 Finite-dimensional maps and compositions of maps parallel to the unit interval.

A map $f : X \to Y$ is said to be *embedded in a map* $f_0 : X_0 \to Y_0$ (see [2 and 3]) if there exists embeddings $g : X \to X_0$ and $h : Y \to Y_0$ such that $h \circ f = f_0 \circ g$. A map $f : X \to Y$ is *parallel* to the unit interval I (see [2 and 3]) if f can be embedded in the natural projection $p : Y \times I \to Y$. In [2 and 3], Pasynkov proved the following theorem: If $f : X \to Y$ is a map such that $\dim f = k$ and $\dim Y < \infty$, then f can be represented as the composition $f = h_k \circ ...h_1 \circ g$ of a zero-dimensional map g and maps h_i $(i = 1, 2, .., k)$ paralell to the unit interval I (see Proposition 2.2).

In this section, furthermore we study the properties of finite-dimensional maps and compositions of maps parallel to the unit interval. In fact, we show that the zero-dimensional map g as in the above theorem of Pasynkov can be represented as a composition of some special maps parallel to I.

First, we prove the following proposition (Proposition 3.2) which is related to results of Uspenskij [6], Tuncali and Valov [5]. Our proof is similar to the proof of Theorem 2.4. We give the proof which is different from the proofs of Uspenskij, Tuncali and Valov (see [6] and [5]).

Lemma 3.1. *Let* X, Y *and* Z *be compacta and* $0 \leq k < \infty$. *Let* T *be the set of maps* $g = u \times v : X \to Y \times Z$ *in* $C(X, Y \times Z)$ *such that* $\dim\ v(u^{-1}(y)) \leq k$ *for each* $y \in Y$. *Then* T *is a* G_δ*-set of* $C(X, Y \times Z)$.

Proposition 3.2. *Let $f : X \to Y$ be a map of compacta such that f is a k-dimensional map and $\dim Y = p < \infty$. Let T be the set of all maps $h = g \times u : X \to I^k \times I$ in $C(X, I^{k+1})$ such that $\dim h(f^{-1}(y)) \leq k$, $\dim u((f \times g)^{-1}(y,t)) = 0$ for each $y \in Y$, $t \in I^k$, $\dim\ (f \times g) = 0$ and $f \times h$ is a $(p+k+1)$-to-1 map. Then T is a G_δ-dense subset of $C(X, I^{k+1})$.*

Corollary 3.3. *Let $f : X \to Y$ be a map of compacta such that f is a k-dimensional map and $\dim Y = p < \infty$. Let $\tilde{E}(X, I^{p+2k+1})$ be the set of maps g in the space $C(X, I^{p+2k+1})$ such that (1) $f \times g$ is an embedding, (2) for each $1 \leq i \leq p+k$, $f \times (p_i \circ g) : X \to Y \times I^{p+2k+1-i}$ is an $(i+1)$-to-1 map, and (3) for $h = p_{p+k} \circ g = g' \times u : X \to I^k \times I$, $\dim h(f^{-1}(y)) \leq k$, $\dim u((f \times g')^{-1}(y,t)) = 0$ for each $y \in Y$ and $t \in I^k$, and $\dim\ (f \times g') = 0$, where $p_i : I^{p+2k+1} \to I^{p+2k+1-i}$ is the natural projection. Then $\tilde{E}(X, I^{p+2k+1})$ is a G_δ-dense subset of $C(X, I^{p+2k+1})$.*

$$\begin{array}{ccc}
Y \times I^{p+2k+1} & \xleftarrow{f \times g} & X \\
\big\downarrow Pr & & \\
Y \times I^{p+2k+1-i} & & \\
\big\downarrow Pr & & \\
Y \times I^k & \xrightarrow{Pr} & Y
\end{array}$$

Now, we have the following representation theorem of finite-dimensional maps.

Theorem 3.4. *Let $f : X \to Y$ be a map of compacta such that $0 \leq k < \infty$ and $\dim Y = p < \infty$. Then f is a k-dimensional map if and only if f can be represented as the composition*

$$f = g_{p+2k+1} \circ \ldots \circ g_{p+k+2} \circ g_{p+k+1} \circ g_{p+k} \circ \ldots \circ g_1$$

of maps g_i $(i = 1, 2, .., p+2k+1)$ paralell to I such that g_i is an $(i+1)$-to-1 map for each $i = 1, 2, .., p+k$ and g_{p+k+1} is a zero-dimensional map.

$$\begin{array}{l}
X = X_0 \xrightarrow{g_1} X_1 \xrightarrow{..} \ .. \xrightarrow{g_{p+k}} X_{p+k} \xrightarrow{g_{p+k+1}} X_{p+k+1} \\
\xrightarrow{g_{p+k+2}} X_{p+k+2} \xrightarrow{....} X_{p+2k} \xrightarrow{g_{p+2k+1}} X_{p+2k+1} = Y
\end{array}$$

Remark. In the proof of Theorem 3.4, the maps g_i $(i = 1, 2, ..., p+k)$ satisfy the condition that $g_i \circ \ldots \circ g_1$ $(i \leq p+k)$ is an $(i+1)$-to-1 map. In particular, g_i $(i \leq p+k)$ is an $(i+1)$-to-1 map.

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