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Special bounded zero-sets and their applications

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1 Introduction

All spaces considered are completely regular T_1-spaces and all maps are continuous. For a space X, let βX denote the Čech-Stone compactification of X and μX the Dieudonné completion (i.e., the completion with respect to the finest uniformity) of X. For a space X, $C(X)$ denotes the set of all real-valued, continuous functions on X, and a subset A of X is said to be *bounded* (or *relatively pseudocompact*) in X if every $f\in C(X)$ is bounded on A. A *zero-set* in a space X is a set of the form $f^{-1}(0)$ for some $f\in C(X)$. We now call a zero-set Z in X a *full zero-set* if $\mathrm{cl}_{\beta X}Z$ is a zero-set in βX. Such a zero-set was studied by Rudd [15] in the context of rings of continuous functions.

In this paper, we study basic properties of full zero-sets and consider the problem when a bounded, full zero-set is compact. In particular, we prove that:

(i) Let X be a space with a regular G_δ-diagonal. Then every bounded, full zero-set in X is compact and metrizable.

(ii) Let X be a Baire space such that every open cover has a σ-point- finite open refinement. Then, every bounded, full zero-set in X is compact.

(iii) Let X be a Baire space with a σ-point- finite base. Then, every bounded, full zero-set in X is compact and metrizable.

The above results are generalizations of the following theorems:

(1) (McArthur [11]) Every pseudocompact space with a regular G_δ-diagonal is metrizable.

(2) (Uspenskii [16]) Every pseudocompact space such that every open cover has a σ-point-finite open refinement is compact.

(3) (Uspenskii [16]) Every pseudocompact space with a σ-point-finite base is metrizable.

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, ω denotes the first infinite ordinal, and ω_1 denotes the first uncountable ordinal. As usual, an ordinal is identified with the set of smaller oridnals. For a set A, $|A|$ denotes the cardinality of A. For $f \in C(X)$ and $n \in \mathbb{N}$, we write $U(f,n) = \{x \in X : |f(x)| < 1/n\}$ and $Z(f) = \{x \in X : f(x) = 0\}$. A *cozero-set* is a set which is the complement of a zero-set.

2 Full zero-sets

In this section, we study basic properties of full zero-sets. The following two lemmas will be often used in our discussion. In the first one, the equivalence of (1) and (2) was proved by Morita [13, Lemma 2.5] and that of (1) and (3) is well-known and easily proved.

Lemma 2.1 *For a subset A of a space X, the following conditions are equivalent:*

(1) A is bounded in X;

(2) $\mathrm{cl}_{\beta X} A \subseteq \mu X$;

(3) there exists no locally finite family $\{G_n : n \in \mathbb{N}\}$ of non-empty open sets in X such that $G_n \cap A \neq \emptyset$ for each $n \in \mathbb{N}$.

Lemma 2.2 (Rudd [15]) *A subset Z of a space X is a full zero-set in X if and only if there exists $f \in C(X)$ such that $Z = Z(f)$ and (*) for every cozero-set V in X with $Z \subseteq V$, there exists $n \in \mathbb{N}$ such that $U(f,n) \subseteq V$.*

By Lemma 2.2, every open-closed set in a space X is a full zero-set in X. If $Z = Z(f)$ is a full zero-set in X and f satisfies (*), then $\{U(f,n) : n < \mathbb{N}\}$ is a neighborhood base of Z proivided that either X is normal or Z is compact.

Proposition 2.3 *The union of finitely many full zero-sets is a full zero-set. The intersection of finitely many full zero-sets is a full zero-set.*

Example 2.4 *The intersection of countably many full zero-sets is not necessarily a full zero-set. To show this, let S_ω be the sequential fan, i.e., the quotient space obtained from the product space $X = (\omega + 1) \times \omega$ by collapsing the set $\{\omega\} \times \omega$ to a point $p \in S_\omega$. Let $\varphi : X \to S_\omega$ be the quotient map and put*

$$F_n = \varphi[\{\alpha : n < \alpha \leq \omega\} \times \omega]$$

for each $n < \omega$. Then each F_n is a full zero-set in S_ω since it is open-closed in S_ω, but $\{p\} = \bigcap_{n<\omega} F_n$ is not a full zero-set in S_ω since it has no countable neighborhood base.

Lemma 2.5 *Let $F = Z(f)$ be a full zero-set in a space X such that f satisfies $(^*)$ in Lemma 2.2. Then, there exists no locally finite family $\{G_n : n \in \mathbb{N}\}$ of non-empty open sets in X such that $G_n \subseteq U(f,n) \setminus F$ for each $n \in \mathbb{N}$. If F is bounded in X in addition, then there exists no locally finite family $\{G_n : n \in \mathbb{N}\}$ of non-empty open sets in X such that $G_n \subseteq U(f,n)$ for each $n \in \mathbb{N}$.*

Lemma 2.6 *Let $F = Z(f)$ be a bounded, full zero-set in a space X such that f satisfies $(^*)$ in Lemma 2.2. Let Z be a zero-set in X and let $\{G_n : n \in \mathbb{N}\}$ be a decreasing sequence of open sets in X such that $Z = \bigcap_{n\in\mathbb{N}} \mathrm{cl}_X G_n$ and $Z \subseteq G_n \subseteq U(f,n)$ for each $n \in \mathbb{N}$. Then, for each cozero-set V in X with $Z \subseteq V$, there exists $n \in \mathbb{N}$ such that $G_n \subseteq V$. Moreover, if Z is compact, then $\{G_n : n \in \mathbb{N}\}$ is a neighborhood base of Z in X.*

Proposition 2.7 *Let F be a bounded, full zero-set in a space X and Z a zero-set in X. Then $F \cap Z$ is a full zero-set in X.*

For every space X, the subspace $X \cup (\beta X \setminus \mu X)$ of βX is pseudocompact, because it is G_δ-dense in βX (i.e., it intersects every non-empty G_δ-set in βX). The following proposition shows that a bounded, full zero-set is precisely a zero-set in some pseudocompact space.

Proposition 2.8 *Every zero-set in a pseudocompact space X is a bounded, full zero-set in X. Conversely, every bounded, full zero-set in a space X is a zero-set in the pseudocompact space $Y = X \cup (\beta X \setminus \mu X)$.*

A full zero-set in a space X is not necessarily bounded in X since every space is a full zero-set of itself. We, however, have the following result.

Proposition 2.9 *The boundary of a full zero-set in a space X is bounded in X.*

Remark 2.10 *One might ask if a bounded (or pseudocompact) zero-set is a full zero-set. The answer is negative, because Example 2.4 shows that a compact zero-set need not be a full zero-set.*

We conclude this section by considering small, bounded, full zero-sets. Before stating the results, let us agree on some terminology from [3]. Let ${}^\omega\omega$ be the set of all functions from ω to ω. For $s, t \in {}^\omega\omega$, we write $s \leq^* t$ if $s(n) \leq t(n)$ for all but finitely many $n < \omega$. Recall from [3] that a subset A of ${}^\omega\omega$ is *unbounded* if there is no upper bound of A in $\langle {}^\omega\omega, \leq^* \rangle$, and is *dominating* if it is cofinal in $\langle {}^\omega\omega, \leq^* \rangle$. Following [3], let $\mathfrak{b} = \min\{|B| : B \text{ is an unbounded subset of } {}^\omega\omega\}$, and $\mathfrak{d} = \min\{|D| : D \text{ is a dominating subset of } {}^\omega\omega\}$. Clearly, $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c} (= |{}^\omega\omega|)$. A space X is called $[\mathfrak{b}, \mathfrak{d}]$-*compact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \mathfrak{d}$ has a subcover $\mathcal{V}$ with $|\mathcal{V}| < \mathfrak{b}$. In particular, X is $[\mathfrak{b}, \mathfrak{d}]$-compact if $|X| < \mathfrak{b}$. We now call a space X *nearly countably compact* if there is no infinite, locally finite family of non-empty zero-sets in X.

Proposition 2.11 *Let F be a $[\mathfrak{b}, \mathfrak{d}]$-compact, bounded, full zero-set in a space X. Then, F is nearly countably compact.*

Remark 2.12 *All countably compact spaces are nearly countably compact and all nearly countably compact spaces are pseudocompact, but both converses do not hold in general. For example, it is easily checked that the Tychonoff plank $((\omega_1 + 1) \times (\omega + 1)) \setminus \{\langle \omega_1, \omega \rangle\}$ is nearly countably compact but not countably compact and the space Ψ in [7, 5I] is pseudocompact but not nearly countably compact.*

Corollary 2.13 *Let X be a space in which every point is a G_δ-set and F a bounded, full zero-set in X with $|F| < \mathfrak{b}$. Then F is countably compact.*

Corollary 2.14 (van Douwen) *Every first countable, pseudocompact space X with $|X| < \mathfrak{b}$ is countably compact.*

3 Compactness of full zero-sets

In this section, we attempt to generalize the theorems (1) and (2) stated in the introduction to theorems on bounded, full zero-sets in not necessarily pseudocompact spaces. First, we consider the theorem (1) by McArthur. Recall from [8] that a space X has a *regular G_δ-diagonal* if the diagonal $\Delta = \{\langle x, x \rangle : x \in X\}$ is the intersection of the closures of countably many open sets in $X \times X$ including Δ. By [17] a space X has a regular G_δ-diagonal if and only if there is a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X such that if $x, y \in X$ and $x \neq y$, then there exist $n \in \mathbb{N}$ and open neighborhoods U and V of x and y, respectively, such that no member of $\mathcal{G}_n$ intersects both U and V.

Theorem 3.1 *Let X be a space with a regular G_δ-diagonal. Then every bounded, full zero-set in X is compact and metrizable.*

We turn to a generalization of Uspenskii's theorem (2). The following lemma is due to Fletcher and Lindgren [6] (see also [2, Lemma 8.2]). Recall that a space X is a Baire space if the intersection of every sequence $\{U_n : n \in \mathbb{N}\}$ of open dense subsets of X is dense in X.

Lemma 3.2 *Let X be a Baire space. Then, for every point-finite collection $\mathcal{U}$ of open sets in X, the set $\{x \in X : \mathcal{U}$ is locally finite at $x\}$ is dense and open in X.*

Theorem 3.3 *Let X be a Baire space such that every open cover has a σ-point-finite open refinement. Then, every bounded, full zero-set in X is compact.*

Proof. Let $F = Z(f)$ be a bounded, full zero-set in X such that f satisfies (*) in Lemma 2.2. Suppose on the contrary that F is not compact. Then, there exists a family $\mathcal{K}$ of closed sets in F with the finite intersection property such that

$\bigcap\{K : K \in \mathcal{K}\} = \emptyset$. We may assume that $\mathcal{K}$ is closed under finite intersection. By the assumption, there exists a σ-point-finite open cover $\mathcal{V} = \bigcup_{i\in\mathbb{N}} \mathcal{V}_i$ of X such that $\{\mathrm{cl}_X V : V \in \mathcal{V}\}$ refines $\{X \setminus K : K \in \mathcal{K}\}$, where $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$ and $\mathcal{V}_i$ is point-finite for each $i \in \mathbb{N}$. By Lemma 3.2, the set $A_i = \{x \in X : \mathcal{V}_i$ is locally finite at x in $X\}$ is dense and open in X for each $i \in \mathbb{N}$.

By induction, we shall construct a sequence $\{H_n : n \in \mathbb{N}\}$ of non-empty open sets in X such that, for each $n \in \mathbb{N}$, H_n satisfies the following conditions:

(i) $H_n \subseteq U(f, n)$,

(ii) $\{V \in \mathcal{V}_n : H_n \cap V \neq \emptyset\}$ is finite, and

(iii) $H_n \cap \mathrm{cl}_X(\bigcup_{i<n} \mathrm{St}(H_i, \mathcal{V}_i)) = \emptyset$.

For $n = 1$, since A_1 is dense and open in X, we can take a non-empty open set $H_1 \subseteq U(f, 1)$ such that $\{V \in \mathcal{V}_1 : H_1 \cap V \neq \emptyset\}$ is finite. Let $n > 1$ and assume that open sets H_i satisfying (i)–(iii) have been defined for all $i < n$. Since $\{\mathrm{cl}_X V : V \in \mathcal{V}\}$ refines $\{X \setminus K : K \in \mathcal{K}\}$ and $\mathcal{K}$ is closed under finite intersection, it follows from (ii) that

$$\mathrm{cl}_X\left(\bigcup_{i<n} \mathrm{St}(H_i, \mathcal{V}_i)\right) \cap K = \emptyset \text{ for some } K \in \mathcal{K}.$$

Thus, $U(f, n) \setminus \mathrm{cl}_X(\bigcup_{i<n} \mathrm{St}(H_i, \mathcal{V}_i)) \neq \emptyset$. Since A_n is dense and open in X, we can take a non-empty open set H_n such that

$$H_n \subseteq U(f, n) \setminus \mathrm{cl}_X\left(\bigcup_{i<n} \mathrm{St}(H_i, \mathcal{V}_i)\right)$$

and $\{V \in \mathcal{V}_n : H_n \cap V \neq \phi\}$ is finite. The induction is complete.

To show that $\{H_n : n \in \mathbb{N}\}$ is locally finite in X, let $x \in X$. Then, $x \in V$ for some $V \in \mathcal{V}_k$ and some $k \in \mathbb{N}$. If V intersects some H_n with $n > k$, then $V \subseteq \bigcup_{i<n+1} \mathrm{St}(H_i, \mathcal{V}_i)$ since $\mathcal{V}_k \subseteq \mathcal{V}_n$. Hence, by (iii), $V \cap H_m = \emptyset$ for each $m > n$, which implies that $\{H_n : n \in \mathbb{N}\}$ is locally finite in X. By (i), this contradicts the second statement of Lemma 2.5. Hence, F is compact.

Corollary 3.4 *Let X be a metacompact, Baire space. Then, every bounded, full zero-set in X is compact.*

We do not know if Theorem 3.3 and Corollary 3.4 hold without assuming that X is a Baire space. The following example, however, shows that they fail to be true if 'full' is removed. For ordinals κ and λ, $\kappa\lambda$ denotes the ordinal multiplication of κ and λ; on the other hand, $\kappa \times \lambda$ denotes the product set of κ and λ, where κ and λ are identified with the sets of smaller ordinals.

Example 3.5 *There exists a metacompact, Baire space X, with a σ-point-finite base, which has a non-compact, bounded, zero-set.*

Proof. Let $\mathfrak{c}$ be the cardinality of the continuum and let $\kappa = \mathfrak{c}\omega_1$. Define $X = \kappa \cup D$, where $D = \kappa \times \kappa \times \omega$. For each $\alpha < \kappa$ and $n < \omega$, let $U_n(\alpha) = \{\alpha\} \cup S_n(\alpha) \cup T_n(\alpha)$, where $S_n(\alpha)$ and $T_n(\alpha)$ are subsets of D defined as follows: For each $\alpha < \kappa$, define

$$S_n(\alpha) = \{\alpha\} \times \kappa \times (\omega \setminus n), \quad n < \omega.$$

For each $\mu < \kappa$, let Σ_μ denote the set of all increasing sequences in μ, where by an *increaing sequence* in μ, we mean a map $\sigma : \omega \to \mu$ such that $\sigma(i) < \sigma(j)$ whenever $i < j$. For each $\lambda < \omega_1$ with $\lambda \geq 1$, since $|\Sigma_{\mathfrak{c}\lambda} \times \Sigma_\omega| = \mathfrak{c}$, there exists a bijection $\varphi_\lambda : \mathfrak{c}(\lambda + 1) \setminus \mathfrak{c}\lambda \to \Sigma_{\mathfrak{c}\lambda} \times \Sigma_\omega$. Let $\alpha < \kappa$. Then, $\alpha \in \mathfrak{c}(\lambda+1) \setminus \mathfrak{c}\lambda$ for some $\lambda < \omega_1$. If $\lambda = 0$ (i.e., $\alpha < \mathfrak{c}$), then we define $T_n(\alpha) = \emptyset$ for each $n < \omega$. If $\lambda \geq 1$ and $\varphi_\lambda(\alpha) = \langle \sigma, f \rangle \in \Sigma_{\mathfrak{c}\lambda} \times \Sigma_\omega$, then we define

$$T_n(\alpha) = \{\langle \sigma(i), \alpha, f(i) \rangle : n < i < \omega\}, \quad n < \omega.$$

By the definitions, for each $\alpha, \beta < \kappa$,

$$\text{if } \alpha \neq \beta, \text{ then for each } m, n < \omega, \quad S_m(\alpha) \cap S_n(\beta) = \emptyset \text{ and } T_m(\alpha) \cap T_n(\beta) = \emptyset. \quad (1)$$

Now, we topologize X by letting sets of the form $U_n(\alpha)$ be basic open neighborhoods of $\alpha \in \kappa$ and declaring points of D to be isolated. Then, X is a Baire space since every dense open set in X indcludes D. For each $n < \omega$, if we put $\mathcal{B}_n = \{U_n(\alpha) : \alpha < \kappa\}$, then $\mathcal{B}_n$ is point-finite by (1). Thus, $\bigcup_{n<\omega} \mathcal{B}_n \cup \{\{p\} : p \in D\}$ is a σ-point-finite base of X. Similarly, we can show that X is metacompact by (1). The set $\kappa \subseteq X$ is a zero-set in X since it is the intersection of countably many open-closed sets $\kappa \cup (\kappa \times \kappa \times (\omega \setminus n))$, $n < \omega$, of X. It remains to show that $\kappa \subseteq X$ is bounded in X. If κ is not bounded, then there exist $\sigma \in \Sigma_\kappa$ and $f \in \Sigma_\omega$ such that $\{U_{f(n)}(\sigma(n)) : n < \omega\}$ is discrete in X. By choosing $\lambda < \omega_1$ with $\sup_{n<\omega} \sigma(n) < \mathfrak{c}\lambda$, we can consider $\sigma \in \Sigma_{\mathfrak{c}\lambda}$, and hence, $\varphi_\lambda(\alpha) = \langle \sigma, f \rangle$ for some $\alpha \in \mathfrak{c}(\lambda+1) \setminus \mathfrak{c}\lambda$. Then, $\{U_{f(n)}(\sigma(n)) : n < \omega\}$ accumulates the point α since each $T_m(\alpha)$ intersects infinitely many $S_{f(n)}(\sigma(n))$'s, which is a contradiction. Hence, κ is a non-compact, bounded, zero-set in X.

4 M'-spaces

M'-spaces were first studied by Isiwata [10] as a generalization of pseudocompact spaces and M-spaces. A space X is called an *M'-space* if there is a normal sequence $\{\mathcal{U}_i\}_{i \in \mathbb{N}}$ of open covers of X satisfying the following condition: if $\{K_i\}_{i\in\mathbb{N}}$ is a decreasing sequence of non-empty zero-set of X such that $K_i \subseteq \mathrm{St}(x, \mathcal{U}_i)$ for each $i \in \mathbb{N}$ and for a fixed point x in X, then $\bigcap_{i\in\mathbb{N}} K_i \neq \emptyset$.

We apply the results in the previous sections to generalize McArthur's theorem (1) and Uspenskii's theorem (3) stated in the introduction to theorems on an M'-space X. It is known that X is an M'-space if and only if μX is a paracompact

M-space [12]. A space X is a *paracompact M-space* (or equivalently, *paracompact p-space*) if there exists a perfect map from X to a metric space.

A map $f : X \to Y$ is said to be *z-closed* if $f(Z)$ is closed in Y for every zero-set Z in X.

Theorem 4.1 (Morita) *X is an M'-space if and only if there exists a z-closed map f from X to a metric space M such that $f^{-1}(t)$ is bounded in X for each $t \in M$.*

Lemma 4.2 *Let X be an M'-space and let $f : X \to M$ be a map stated in Theorem 4.1. Then, $f^{-1}(t)$ is a full zero-set in X for each $t \in M$. Moreover, if $f^{-1}(t)$ is compact for each $t \in M$, then f is a perfect map, and hence, X is a paracompact M-space.*

Proof. To prove the first statement, fix $t \in M$ and put $Z = f^{-1}(t)$. Let d be the metric on the metric space M and define $g \in C(X)$ by $g(x) = d(t, f(x))$ for $x \in X$. Then, g and Z satisfy the condition (*) in Lemma 2.2, because f is z-closed. Hence, Z is a full zero-set in X. The second statement follows from the facts that f is z-closed and every open set containing a compact $f^{-1}(t)$ includes a cozero-set containing $f^{-1}(t)$.

Since every M-space with a G_δ-diagonal is metrizable (see [8, Corollary 3.8]), the following theorem, which is a generalization of McArthur's theorem, immediately follows from Theorem 3.1 and Lemma 4.2.

Theorem 4.3 *M'-space with a regular G_δ-diagonal is metrizable.*

Theorem 4.4 *Let X be a Baire space. If X is an M'-space with a σ-point-finite base, then X is metrizable.*

Proof. By Theorem 3.3 and Lemma 4.2, X is a paracompact M-space. Since every M-space with a point-countable base is metrizable (see [8, Corollary 7.11]), X is metrizable.

Since every pseudocompact space is a Baire space, Theorem 4.4 can be seen as a generalization of Uspenskii's theorem. We do not know if Theorem 4.4 remains true if the assumption that X is a Baire space is removed.

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