数理解析研究所講究録 1419 巻 2005 年 38-51

Destructible gaps に関する強制概念とその積

依岡 輝幸*† (Teruyuki Yorioka)
神戸大学大学院自然科学研究科
(Graduate School of Science and Technology, Kobe University)

1 Introduction and notation

1.1 Introduction

This note is a part of the paper [23].

In this paper, we deal with destructible gaps. A destructible gap is an (ω_1, ω_1)-gap which can be destroyed by a forcing extension preserving cardinals. A destructible gap has a characterization similar to a Suslin tree ([2]). A Suslin tree is an ω_1-tree having no uncountable chains and antichains. On the other hand, for an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ with the set $a_\alpha \cap b_\alpha$ empty for every $\alpha \in \omega_1$, we say here that α and β in ω_1 are compatible if

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset.$$

Then by the characterization due to Kunen and Todorčević, we notice that an (ω_1, ω_1)-pregap is a destructible gap iff it has no uncountable pairwise compatible and incompatible subsets of ω_1. (We must notice that from results of Farah and Hirschorn [8, 9], the existence of a destructible gap is independent with the existence of a Suslin tree.)

One of differences from an ω_1-tree is that any (ω_1, ω_1)-pregap have never had an uncountable chain and antichain at the same time. We have forcing notions related to an (ω_1, ω_1)-pregap.

Definition 1.1 (E.g. [5, 11, 18, 19]). *Let $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ be an (ω_1, ω_1)-pregap with $a_\alpha \cap b_\alpha = \emptyset$ for every $\alpha \in \omega_1$.*

1. *$\mathcal{F}(\mathcal{A}, \mathcal{B}) := \{\sigma \in [\omega_1]^{<\omega}; \forall \alpha \neq \beta \in \sigma,\ (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset\}$, ordered by reverse inclusion.*

2. *$\mathcal{S}(\mathcal{A}, \mathcal{B}) := \left\{\sigma \in [\omega_1]^{<\omega}; \bigcup_{\alpha\in\sigma} a_\alpha \cap \bigcup_{\alpha\in\sigma} b_\alpha = \emptyset\right\}$, ordered by reverse inclusion.*

*Supported by JSPS Research Fellowships for Young Scientists.
†Supported by Grants-in-Aid for JSPS Fellow, No. 16·3977, Ministry of Education, Culture, Sports, Science and Technology.

We note that $\mathcal{F}(\mathcal{A},\mathcal{B})$ forces $(\mathcal{A},\mathcal{B})$ to be indestructible and $\mathcal{S}(\mathcal{A},\mathcal{B})$ forces $(\mathcal{A},\mathcal{B})$ to be separated. Using these forcing notions, we can express characterizations of being a gap and destructibility.

Theorem 1.2 (E.g. [5, 11, 18, 19]). *Let $(\mathcal{A},\mathcal{B})$ be an (ω_1,ω_1)-pregap. Then;*

1. *$(\mathcal{A},\mathcal{B})$ forms a gap iff $\mathcal{F}(\mathcal{A},\mathcal{B})$ has the countable chain condition.*
2. *$(\mathcal{A},\mathcal{B})$ is destructible (may not be a gap) iff $\mathcal{S}(\mathcal{A},\mathcal{B})$ has the countable chain condition.*

Therefore we say that $(\mathcal{A},\mathcal{B})$ is a destructible gap if both $\mathcal{F}(\mathcal{A},\mathcal{B})$ and $\mathcal{S}(\mathcal{A},\mathcal{B})$ have the ccc. As in the case of a Suslin tree, by the product lemma for forcings, we note that $\mathcal{F}(\mathcal{A},\mathcal{B}) \times \mathcal{S}(\mathcal{A},\mathcal{B})$ does not have the ccc, and we will see that e.g., we may have two destructible gaps $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ so that all variations $\mathcal{X}_0(\mathcal{A},\mathcal{B}) \times \mathcal{X}_1(\mathcal{A},\mathcal{B})$ have the ccc.

In [10], it is proved that for any family $\{(\mathcal{A}_i,\mathcal{B}_i); i \in I\}$ of (ω_1,ω_1)-gaps, the finite support product $\prod_{i\in I} \mathcal{F}(\mathcal{A}_i,\mathcal{B}_i)$ has the countable chain condition. It means that generically making gaps indestructible cannot separate any (ω_1,ω_1)-gap. So we arise a question wether or not the above statement is also true for adding interpolations. We prove that this question cannot be decided from ZFC, i.e.

Theorem 1. *It is consistent with* ZFC *that for any family $\{(\mathcal{A}_i,\mathcal{B}_i); i \in I\}$ of destructible gaps, the product forcing notion $\prod_{i\in I} \mathcal{S}(\mathcal{A}_i,\mathcal{B}_i)$ has the countable chain condition.*

Theorem 2. *It is consistent with* ZFC *that there are two destructible gaps $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ such that the product forcing notion $\mathcal{S}(\mathcal{A},\mathcal{B}) \times \mathcal{S}(\mathcal{C},\mathcal{D})$ does not have the countable chain condition.*

(We note that the statement in Theorem 1 (and the next theorem) is trivially true if there are no destructible gaps. For example, if Martin's Axiom holds, then all (ω_1,ω_1) gaps are indestructible. But it is really consistent with ZFC that the statement in Theorem 1 plus there are many destructible gaps. see the proof of Theorem 1.)

Moreover, we prove the following theorem which is a version of Larson's theorem [14, Theorem 4.6] for a destructible gap.

Theorem 3. *It is consistent with* ZFC *that there exists a destructible gap $(\mathcal{A},\mathcal{B})$ such that $\mathcal{S}(\mathcal{A},\mathcal{B})$ forces that all (ω_1,ω_1)-gaps are indestructible.*

1.2 Notation

A pregap in $\mathcal{P}(\omega)/\text{fin}$ is a pair $(\mathcal{A},\mathcal{B})$ of subsets of $\mathcal{P}(\omega)$ such that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the set $a \cap b$ is finite. For subsets a and b of ω, we say that a is almost contained in b (and denote $a \subseteq^* b$) if $a \setminus l$ is a subset of b for some $l \in \omega$. For a pregap $(\mathcal{A},\mathcal{B})$, both ordered sets $\langle \mathcal{A}, \subseteq^* \rangle$ and $\langle \mathcal{B}, \subseteq^* \rangle$ are well ordered and these order type are κ and λ respectively, then we say that a pregap $(\mathcal{A},\mathcal{B})$ has the type (κ,λ) or a (κ,λ)-pregap. Moreover if $\kappa=\lambda$, we say that the pregap is symmetric. For a pregap $(\mathcal{A},\mathcal{B})$, we say that $(\mathcal{A},\mathcal{B})$ is separated if for some $c\in\mathcal{P}(\omega)$, $a\subseteq^* c$ and the set $c\cap b$ is finite for every $a\in\mathcal{A}$ and $b\in\mathcal{B}$. If a pregap is not separated, we say that it is a gap. Moreover if a gap has the type (κ,λ), it is called a (κ,λ)-gap.

For an ordinal α, if we say that $\langle a_\xi, b_\xi;\xi\in\alpha\rangle$ is a pregap, we always assume that

- if $\xi<\eta$ in α, $a_\xi\subseteq^* a_\eta$ and $b_\xi\subseteq^* b_\eta$, and
- for every $\xi\in\alpha$, the set $a_\xi\cap b_\xi$ is empty.

Our other notation is quite standard in set theory. (See [4, 12].)

2 Products of forcing notions adding interpolations

The referee of the paper [10] has proved the following theorem. (For the proof of the following theorem, see the proof of Claim 2.11 in the proof of Lemma 2.10.)

Theorem 2.1 ([10, Theorem 4]). *Let $n\in\omega$ and $(\mathcal{A}_i,\mathcal{B}_i)$ be (ω_1,ω_1)-gaps for $i<n$. Then $\prod_{i<n}\mathcal{F}(\mathcal{A}_i,\mathcal{B}_i)$ has the countable chain condition.*

This theorem says that the forcing a gap to be indestructible cannot force any (ω_1,ω_1)-gap to be separated. But as seen below, we cannot prove from ZFC that the forcing gaps to be separated does not force a gap to be indestructible. The point of the proofs in this section is the homogeneity of the forcing notion $\mathcal{S}(\mathcal{A},\mathcal{B})$ for a destructible gap $(\mathcal{A},\mathcal{B})$ with some property below. For a homogeneity, we give some definitions.

Definition 2.2 ([18, Definition 2]). *We say that pregaps $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ are equivalent if $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ are cofinal each others.*

We notice that if pregaps $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ are equivalent, then $(\mathcal{A},\mathcal{B})$ is a gap iff so is $(\mathcal{C},\mathcal{D})$ and $(\mathcal{A},\mathcal{B})$ is destructible iff so is $(\mathcal{C},\mathcal{D})$. We note that any (ω_1,ω_1)-pregap has an equivalent pregap $(\mathcal{A},\mathcal{B})$ such that $\mathcal{S}(\mathcal{A},\mathcal{B})$ is homogeneous. The similar property of the following one is appeared in the proof of [6, Proposition 2.5].

Definition 2.3 ([22]). *We say that a pregap $(\mathcal{A},\mathcal{B})=\langle a_\alpha,b_\alpha;\alpha\in\omega_1\rangle$ admits finite changes if for all $\alpha<\omega_1$, $a_\alpha\cap b_\alpha$ is empty and the set $\omega\setminus(a_\alpha\cup b_\alpha)$ is infinite, and for any $\beta<\alpha$ with $\beta=\eta+k$ for some $\eta\in\mathrm{Lim}\cap\alpha$ and $k\in\omega$, $H,J\in[\omega]^{<\omega}$ with $H\cap J=\emptyset$ and $i>\max(H\cup J)$ there exists $n\in\omega$ so that*

$$a_{\eta+n}\cap i=H,\ a_{\eta+n}\setminus i=a_\beta\setminus i,\ b_{\eta+n}\cap i=J,\ \textit{and}\ b_{\eta+n}\setminus i=b_\beta\setminus i.$$

For a homogeneity, we need a little strong property of the admission of finite changes.

Definition 2.4. *We say that a pregap* $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ *strictly admits finite changes if it admits finite changes and for all* $\alpha \neq \beta$ *in* ω_1, $\langle a_\alpha, b_\alpha \rangle \neq \langle a_\beta, b_\beta \rangle$.

We note that any symmetric gap has an equivalent gap which strictly admits finite changes. So the rest of this paper, we consider only (ω_1, ω_1)-gaps which strictly admits finite changes because of the following propositions.

Proposition 2.5. *Let* $\langle (\mathcal{A}_i, \mathcal{B}_i); i < n \rangle$ *be a finite collection of destructible gaps and* $(\mathcal{C}_i, \mathcal{D}_i)$ *a gap equivalent to* $(\mathcal{A}_i, \mathcal{B}_i)$ *for each* $i < n$. *Then for any combination* $\langle \mathcal{X}_i; i < n \rangle$, *where* $\mathcal{X}_i$ *is either* $\mathcal{F}$ *or* $\mathcal{S}$, *the finite support product* $\prod_{i<n} \mathcal{X}_i(\mathcal{A}_i, \mathcal{B}_i)$ *has the countable chain condition iff* $\prod_{i<n} \mathcal{X}_i(\mathcal{C}_i, \mathcal{D}_i)$ *also has the countable chain condition.*

Proof. Let $(\mathcal{A}_i, \mathcal{B}_i) = \left\langle a^i_\xi, b^i_\xi; \xi \in \omega_1 \right\rangle$ and $(\mathcal{C}_i, \mathcal{D}_i) = \left\langle c^i_\xi, d^i_\xi; \xi \in \omega_1 \right\rangle$. It suffices to show that if $\prod_{i<n} \mathcal{X}_i(\mathcal{A}_i, \mathcal{B}_i)$ has the countable chain condition, then $\prod_{i<n} \mathcal{X}_i(\mathcal{C}_i, \mathcal{D}_i)$ also has the countable chain condition.

Let $\{p_\alpha; \alpha \in \omega_1\}$ be a family of conditions in $\prod_{i<n} \mathcal{X}_i(\mathcal{C}_i, \mathcal{D}_i)$. Without loss of generality, we may assume that

- the set $\{p_\alpha(i); \alpha \in \omega_1\}$ forms a Δ-system with a root σ_i for each $i < n$,
- all $p_\alpha(i) \setminus \sigma_i$ have the same size k_i for each $i < n$ and
- for any $\alpha < \beta$ in ω_1 and $i < n$,

$$\max(\sigma_i) < \min\left(p_\alpha(i) \setminus \sigma_i\right) \quad \text{and} \quad \max\left(p_\alpha(i) \setminus \sigma_i\right) < \min\left(p_\beta(i) \setminus \sigma_i\right).$$

Moreover, we may assume that there exists a family $\{q_\alpha; \alpha \in \omega_1\}$ of conditions in $\prod_{i<n} \mathcal{X}_i(\mathcal{A}_i, \mathcal{B}_i)$ and a natural numbers m_i for each $i < n$ such that

- for any $\alpha < \beta$ in ω_1 and $i < n$,

$$\max\left(p_\alpha(i) \setminus \sigma_i\right) < \min(q_\alpha(i)) \leq \max(q_\alpha(i)) < \min\left(p_\beta(i) \setminus \sigma_i\right),$$

- for each $i < n$,
  - if $\mathcal{X}_i = \mathcal{F}$, then for any $\alpha \in \omega_1$, $q_\alpha(i)$ has the size k_i and for each $\xi \in p_\alpha(i) \setminus \sigma_i$, there is $\eta \in q_\alpha(i)$ such that

$$a^i_\eta \setminus m_i \subseteq c^i_\xi \text{ and } b^i_\eta \setminus m_i \subseteq d^i_\xi,$$

  - if $\mathcal{X}_i = \mathcal{S}$, then for any $\alpha \in \omega_1$, $q_\alpha(i) = \left\{\gamma^i_\alpha\right\}$ and

$$\bigcup_{\xi \in p(\alpha)} c^i_\xi \setminus m_i \subseteq a_{\gamma^i_\alpha} \text{ and } \bigcup_{\xi \in p(\alpha)} d^i_\xi \setminus m_i \subseteq b_{\gamma^i_\alpha},$$

and

- for any $\alpha, \beta \in \omega_1$,

$$\bigcup_{\xi \in p(\alpha)} c^i_\xi \cap m_i = \bigcup_{\xi \in p(\beta)} c^i_\xi \cap m_i \quad \text{and} \quad \bigcup_{\xi \in p(\alpha)} d^i_\xi \cap m_i = \bigcup_{\xi \in p(\beta)} d^i_\xi \cap m_i.$$

By the ccc-ness of $\prod_{i<n} \mathcal{X}_i(\mathcal{A}_i, \mathcal{B}_i)$, we can find different ordinals α and β in ω_1 such that q_α and q_β are compatible in $\prod_{i<n} \mathcal{X}_i(\mathcal{A}_i, \mathcal{B}_i)$. Then we notice that p_α and p_β are compatible in $\prod_{i<n} \mathcal{X}_i(\mathcal{C}_i, \mathcal{D}_i)$. □

Lemma 2.6. *If $(\mathcal{A}, \mathcal{B})$ strictly admits finite changes, then $\mathcal{S}(\mathcal{A}, \mathcal{B})$ is homogeneous as a forcing notion, i.e. for every $\sigma, \tau \in \mathcal{S}(\mathcal{A}, \mathcal{B})$ there are extensions σ' and τ' of σ and τ respectively such that $\mathcal{S}(\mathcal{A}, \mathcal{B}) \restriction \sigma'$ and $\mathcal{S}(\mathcal{A}, \mathcal{B}) \restriction \tau'$ are isomorphic.*

Proof. Now we fix $\sigma, \tau \in \mathcal{S}(\mathcal{A}, \mathcal{B})$. By strict admission of finite changes of $(\mathcal{A}, \mathcal{B})$, we can find extensions σ' and τ' of σ and τ respectively such that

(i) $\max\{\alpha \in \omega_1 \cap \mathsf{Lim}; \exists k \in \omega\ (\alpha + k \in \sigma')\} = \max\{\alpha \in \omega_1 \cap \mathsf{Lim}; \exists k \in \omega\ (\alpha + k \in \tau')\}$ and

(ii) there exists $N \in \omega$ such that

- for any $\alpha < \beta \in \sigma'$, $a_\alpha \setminus N \subseteq a_\beta \setminus N$ and $b_\alpha \setminus N \subseteq b_\beta \setminus N$,
- for any $\alpha < \beta \in \tau'$, $a_\alpha \setminus N \subseteq a_\beta \setminus N$ and $b_\alpha \setminus N \subseteq b_\beta \setminus N$, and
- $\bigcup_{\alpha \in \sigma'} (a_\alpha \cap N) \cup \bigcup_{\alpha \in \sigma'} (b_\alpha \cap N) = \bigcup_{\alpha \in \tau'} (a_\alpha \cap N) \cup \bigcup_{\alpha \in \tau'} (b_\alpha \cap N) = N.$

Then we note that

$$\bigcup_{\alpha \in \sigma'} (a_\alpha \setminus N) = \bigcup_{\alpha \in \tau'} (a_\alpha \setminus N) \quad \text{and} \quad \bigcup_{\alpha \in \sigma'} (b_\alpha \setminus N) = \bigcup_{\alpha \in \tau'} (b_\alpha \setminus N)$$

We note that if $\gamma \in \omega_1$ is such that $\sigma' \cup \{\gamma\}$ is also a condition in $\mathcal{S}(\mathcal{A}, \mathcal{B})$, then

$$a_\gamma \cap n \subseteq \bigcup_{\alpha \in \sigma'} (a_\alpha \cap n), \quad b_\gamma \cap n \subseteq \bigcup_{\alpha \in \sigma'} (b_\alpha \cap n)$$

and

$$\left((a_\gamma \setminus n) \cap \left(\bigcup_{\alpha \in \sigma'} (b_\alpha \setminus n) \right) \right) \cup \left((b_\gamma \setminus n) \cap \left(\bigcup_{\alpha \in \sigma'} (a_\alpha \setminus n) \right) \right) = \emptyset.$$

We pick any bijection π from

$$\mathcal{P}\left(\bigcup_{\alpha \in \sigma'} a_\alpha \cap n \right) \times \mathcal{P}\left(\bigcup_{\alpha \in \sigma'} b_\alpha \cap n \right)$$

onto

$$\mathcal{P}\left(\bigcup_{\alpha\in\tau'} a_\alpha \cap n\right) \times \mathcal{P}\left(\bigcup_{\alpha\in\tau'} b_\alpha \cap n\right)$$

and let π_1 and π_2 represent the first and second coordinates of the value of π respectively. We define an isomorphism ψ from $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\sigma'$ onto $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\tau'$ as follow. Let ρ be an extension of σ' and $\beta \in \rho\setminus\sigma'$, say $\beta = \alpha + k$ for $\alpha \in \omega_1 \cap \mathsf{Lim}$ and $k \in \omega$, $a_\beta = H \cup (a_\alpha \setminus N)$ and $b_\beta = K \cup (b_\alpha \setminus N)$, where H and K are subsets of N. Then we let k° be the unique number such that

$$a_{\alpha+k^\circ} = \pi_1(H,K) \cup (a_\beta \setminus N)$$

and

$$b_{\alpha+k^\circ} = \pi_2(H,K) \cup (b_\beta \setminus N).$$

Then we define $\beta^\circ := \alpha + k^\circ$ and

$$\psi(\rho) := \tau' \cup \{\beta^\circ; \beta \in \rho \setminus \sigma'\}.$$

By the above note, this is well defined and certainly an isomorphism. □

Lemma 2.6 says that the theory in the extension with $\mathcal{S}(\mathcal{A},\mathcal{B})$ can calculate in the ground model when $(\mathcal{A},\mathcal{B})$ strictly admits finite changes, that is, if some condition in $\mathcal{S}(\mathcal{A},\mathcal{B})$ can force the statement about elements of the ground model, then the statement holds in any extension with $\mathcal{S}(\mathcal{A},\mathcal{B})$.

Assume that $(\mathcal{A},\mathcal{B})$ is a destructible gap and strictly admits finite changes and that σ and τ are conditions in $\mathcal{S}(\mathcal{A},\mathcal{B})$. By strengthening σ and τ if need, we may assume that σ and τ satisfy the conditions (i) and (ii). When σ, τ and N satisfies above conditions, we say that $\langle\sigma,\tau,N\rangle$ is a good sequence. If $\langle\sigma,\tau,N\rangle$ is a good sequence, as seen in above lemma, $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\sigma$ and $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\tau$ are isomorphic and a finite bijection π from

$$\mathcal{P}\left(\bigcup_{\xi\in\sigma} a_\xi \cap N\right) \times \mathcal{P}\left(\bigcup_{\xi\in\sigma} b_\xi \cap N\right)$$

onto

$$\mathcal{P}\left(\bigcup_{\xi\in\tau} a_\xi \cap N\right) \times \mathcal{P}\left(\bigcup_{\xi\in\tau} b_\xi \cap N\right)$$

induces an isomorphism ψ from $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\sigma$ onto $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\tau$. We say that ψ is an isomorphism induced by π.

Let $\{(\mathcal{A}_i,\mathcal{B}_i); i \in I\}$ be a family of destructible gaps which strictly admits finite changes and $p = \langle\sigma_i; i \in I\rangle$ and $p' = \langle\sigma'_i; i \in I\rangle$ are conditions in the finite support product $\prod_{i\in I}\mathcal{S}(\mathcal{A}_i,\mathcal{B}_i)$. Then by strengthening conditions, we can find a sequence $\langle N_i : i \in I\rangle$ of natural numbers with the property that the supports of two conditions are same and for any $i \in I \cap \mathrm{supp}(p)$, $\langle\sigma_i,\sigma'_i,N_i\rangle$ is a good sequence, then we have an isomorphism between $\prod_{i\in I}\mathcal{S}(\mathcal{A}_i,\mathcal{B}_i)\restriction\langle\sigma_i; i\in I\rangle$ and $\prod_{i\in I}\mathcal{S}(\mathcal{A}_i,\mathcal{B}_i)\restriction\langle\sigma_i'; i\in I\rangle$ induced by finitely many finite bijections. That is, we have

Lemma 2.7. *Let $\{(\mathcal{A}_i,\mathcal{B}_i); i\in I\}$ be a family of destructible gaps which strictly admits finite changes. Then the product forcing $\prod_{i\in I}\mathcal{S}(\mathcal{A}_i,\mathcal{B}_i)$ with a finite support is homogeneous.* □

Moreover assume all $(\mathcal{A}_i,\mathcal{B}_i)$ are the same gap $(\mathcal{A},\mathcal{B})$. By strengthening each σ_i, we have $N\in\omega$ such that for any $i\neq j$ in $I\cap\mathrm{supp}(p)$, $\langle\sigma_i,\sigma_j,N\rangle$ is a good sequence. Then we have the collection of isomorphisms $\psi_{i,j}$ for each $i,j\in I\cap\mathrm{supp}(p)$ from $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\sigma_i$ onto $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\sigma_j$ which are commutative, by taking finite bijections suitably.

The following lemma is to show Theorem 1.

Lemma 2.8. *Let $\mathbb{P}$ is a homogeneous forcing notion with the countable chain condition and $(\mathcal{C},\mathcal{D})$ an (ω_1,ω_1)-pregap. Then the following statements hold.*

1. *If the product forcing $\mathbb{P}\times\mathcal{S}(\mathcal{C},\mathcal{D})$ does not have the countable chain condition, then the product $\mathbb{P}\times\mathcal{F}(\mathcal{C},\mathcal{D})$ has the countable chain condition.*
2. *If the product forcing $\mathbb{P}\times\mathcal{F}(\mathcal{C},\mathcal{D})$ does not have the countable chain condition, then the product $\mathbb{P}\times\mathcal{S}(\mathcal{C},\mathcal{D})$ has the countable chain condition.*

Proof. Both statements follow from the ccc-ness and the homogeneity of $\mathbb{P}$ and the fact that

1. if $\mathcal{S}(\mathcal{C},\mathcal{D})$ does not have the ccc, then $\mathcal{F}(\mathcal{C},\mathcal{F})$ has the ccc, and
2. if $\mathcal{F}(\mathcal{C},\mathcal{D})$ does not have the ccc, then $\mathcal{S}(\mathcal{C},\mathcal{F})$ has the ccc

respectively. □

Proof of Theorem 1. This theorem is true in the model where there are no destructible gaps. We will build a model for the theorem containing a destructible gap by an iteration with a finite support as follows.

Assume that there is a destructible gap, $2^{\aleph_1}=\lambda$ and $\lambda^{<\lambda}=\lambda$. At first we take any family Γ_0 of destructible gaps which strictly admits finite changes with the property that the finite support product $\prod_{(\mathcal{A},\mathcal{B})\in\Gamma_0}\mathcal{S}(\mathcal{A},\mathcal{B})$ has the ccc (which is a weak property of the independence). By recursion on $\alpha\in\omega_2$, we construct Γ_α in the α-th stage of the iteration as follows:

In stage $\alpha+1\in\omega_2$, for a destructible gap $(\mathcal{C},\mathcal{D})$ which strictly admits finite changes (given by a book-keeping map), if $\prod_{(\mathcal{A},\mathcal{B})\in\Gamma_\alpha}\mathcal{S}(\mathcal{A},\mathcal{B})\times\mathcal{S}(\mathcal{C},\mathcal{D})$ has the ccc, then let $\Gamma_{\alpha+1}:=\Gamma\cup\{(\mathcal{C},\mathcal{D})\}$ and does not force in this iterand, otherwise, i.e. $\prod_{(\mathcal{A},\mathcal{B})\in\Gamma_\alpha}\mathcal{S}(\mathcal{A},\mathcal{B})\times\mathcal{S}(\mathcal{C},\mathcal{D})$ does not have the ccc, then let $\Gamma_{\alpha+1}:=\Gamma_\alpha$ and force $\mathcal{F}(\mathcal{C},\mathcal{D})$. By Lemma 2.8, $\prod_{(\mathcal{A},\mathcal{B})\in\Gamma_{\alpha+1}}\mathcal{S}(\mathcal{A},\mathcal{B})$ still has the ccc and by Theorem 2.1, in the extension with $\mathcal{F}(\mathcal{C},\mathcal{D})$, $\mathcal{F}(\mathcal{A},\mathcal{B})$ is still ccc for every

$(\mathcal{A},\mathcal{B}) \in \Gamma$, so every member in $\Gamma_{\alpha+1}$ is still a destructible gap. For a limit ordinal $\alpha \in \omega_2$, let $\Gamma_\alpha := \bigcup_{\beta\in\alpha}\Gamma_\beta$.

We note that in the final model, Γ_λ is the set of all destructible gaps with the admission of finite changes and $\prod_{(\mathcal{A},\mathcal{B})\in\Gamma_\lambda}\mathcal{S}(\mathcal{A},\mathcal{B})$ is ccc. Let Γ be the set of all destructible gaps. Then $\prod_{(\mathcal{A},\mathcal{B})\in\Gamma}\mathcal{S}(\mathcal{A},\mathcal{B})$ also has the ccc and so is $\prod_{(\mathcal{A},\mathcal{B})\in\Gamma'}\mathcal{S}(\mathcal{A},\mathcal{B})$ for every $\Gamma' \subseteq \Gamma$. (We notice that Γ_λ do *not* have to be independent. It follows from ZFC that for any destructible gap $(\mathcal{A},\mathcal{B})$, we can find another destructible gap $(\mathcal{C},\mathcal{D})$ such that $\mathcal{S}(\mathcal{A},\mathcal{B})\times\mathcal{S}(\mathcal{C},\mathcal{D})$ has the ccc but $\mathcal{S}(\mathcal{A},\mathcal{B})\times\mathcal{F}(\mathcal{C},\mathcal{D})$ doesn't have.) □

To prove Theorems 2 and 3, the key lemma is Lemma 2.10. To show this lemma, we need the following lemma due to the referee of the paper [10]. (The following proof is same in [10]. But for a convenience to the reader, I write the proof here.)

Lemma 2.9 ([10, Lemma B.1]). *Let $\langle a_\alpha, b_\alpha; \alpha\in\omega_1\rangle$ be an (ω_1,ω_1)-gap. Then for any uncountable subsets I and J of ω_1, there exist uncountable $I'\subseteq I$ and $J'\subseteq J$ such that for every $\alpha\in I'$ and $\beta\in J'$, $a_\alpha\cap b_\beta\neq\emptyset$.*

Proof. For each $\alpha\in\omega_1$, there is a natural number n_α such that both sets $\{\xi\in\omega_1; a_\alpha\setminus n_\alpha\subseteq a_\xi\}$ and $\{\eta\in\omega_1; b_\alpha\setminus n_\alpha\subseteq b_\eta\}$ are uncountable. We note that the set

$$\bigcup_{\xi\in I}(a_\xi\setminus n_\xi)\cap\bigcup_{\eta\in J}(b_\eta\setminus n_\eta)$$

is not empty because the pregap

$$\langle a_\xi\setminus n_\xi, b_\eta\setminus n_\eta; \xi\in I, \eta\in J\rangle$$

is equivalent to the original one and so is a gap. We take $\alpha\in I$, $\beta\in J$ and $k\in\omega$ such that k is in the set $(a_\alpha\setminus n_\alpha)\cap(b_\beta\setminus n_\beta)$. Let $I' := \{\xi\in I; a_\alpha\setminus n_\alpha\subseteq a_\xi\}$ and $J' := \{\eta\in J; b_\beta\setminus n_\beta\subseteq b_\eta\}$ which are as desired. □

The next lemma is a variation of [14, Corollary 4.3] for a destructible gap which is the key lemma for proofs of Theorems 2 and 3.

Lemma 2.10. *Let $(\mathcal{A},\mathcal{B})$ be a destructible gap and strictly admits finite changes, and $(\dot{\mathcal{C}},\dot{\mathcal{D}})$ be an $\mathcal{S}(\mathcal{A},\mathcal{B})$-name for an (ω_1,ω_1)-gap. Then there exists a ccc forcing notion $\mathbb{P}$ (which is possibly trivial) such that in the extension with $\mathbb{P}$, $(\mathcal{A},\mathcal{B})$ is still a destructible gap and $\mathcal{S}(\mathcal{A},\mathcal{B})$ forces $(\dot{\mathcal{C}},\dot{\mathcal{D}})$ to be indestructible.*

Proof. At first we define a forcing notion $\mathbb{Q}$ as follow.

$$\mathbb{Q} := \left\{p\in\left([\omega_1]^{<\omega}\right)^2; p(0)\in\mathcal{S}(\mathcal{A},\mathcal{B}) \;\&\; p(0)\Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“}\, \check{p(1)}\in\mathcal{S}(\dot{\mathcal{C}},\dot{\mathcal{D}})\,\text{”}\right\},$$

ordered by

$$p\leq_{\mathbb{Q}} q \iff p(0)\supseteq q(0) \;\&\; p(1)\supseteq q(1).$$

If we have an uncountable antichain in $\mathbb{Q}$, we have nothing to do, i.e. what we have to do is that we let $\mathbb{P}$ be the trivial forcing notion.

Assume that $\mathbb{Q}$ has an uncountable antichain $\{q_\alpha; \alpha \in \omega_1\}$. Without loss of generality, we may assume that the set $\{q_\alpha(1); \alpha \in \omega_1\}$ forms a Δ-system with a root σ and for all $\alpha < \beta$ in ω_1,

$$\max(\sigma) < \min(q_\alpha(1) \setminus \sigma) \quad \text{and} \quad \max(q_\alpha(1) \setminus \sigma) < \min(q_\beta(1) \setminus \sigma).$$

Let $\langle c_\alpha, d_\alpha; \alpha \in \omega_1 \rangle$ the interpretation of $(\dot{\mathcal{C}}, \dot{\mathcal{D}})$ in this extension with $\mathcal{S}(\mathcal{A}, \mathcal{B})$. Then we can find an uncountable subset X of ω_1 such that the set $\{q_\alpha(0); \alpha \in X\}$ is pairwise compatible in $\mathcal{S}(\mathcal{A}, \mathcal{B})$ using an interpolation of $(\mathcal{A}, \mathcal{B})$. Since $\{q_\alpha; \alpha \in \omega_1\}$ is pairwise incompatible in $\mathbb{Q}$, for all $\alpha \neq \beta$ in X,

$$\left(\bigcup_{\xi \in q_\alpha(1) \setminus \sigma} c_\xi \cap \bigcup_{\xi \in q_\beta(1) \setminus \sigma} d_\xi \right) \cup \left(\bigcup_{\xi \in q_\beta(1) \setminus \sigma} c_\xi \cap \bigcup_{\xi \in q_\alpha(1) \setminus \sigma} d_\xi \right) \neq \emptyset.$$

Then by our assumption, the following sequence

$$\left\langle \bigcup_{\xi \in q_\alpha(1) \setminus \sigma} c_\xi, \bigcup_{\xi \in q_\alpha(1) \setminus \sigma} d_\xi; \alpha \in \omega_1 \right\rangle$$

forms a pregap and is an equivalent gap of $\langle c_\alpha, d_\alpha; \alpha \in \omega_1 \rangle$ and so is indestructible. Therefore $\mathcal{S}(\mathcal{A}, \mathcal{B})$ forces $(\dot{\mathcal{C}}, \dot{\mathcal{D}})$ to be indestructible.

Even if $\mathbb{Q}$ has the countable chain condition, we can find a forcing notion $\mathbb{P}$ which adds un uncountable antichain in $\mathbb{Q}$ and preserves the ccc-ness of both $\mathcal{F}(\mathcal{A}, \mathcal{B})$ and $\mathcal{S}(\mathcal{A}, \mathcal{B})$. Let

$$\mathbb{P} := \left\{ P \in [\mathbb{Q}]^{<\omega}; P \text{ is an antichain in } \mathbb{Q} \right\},$$

ordered by reverse inclusion. Since $(\mathcal{A}, \mathcal{B})$ forms a gap, it can be proved that $\mathbb{P}$ has the countable chain condition. Moreover we can show more stronger results. To show them, we use Lemma 2.9. The proof of the following claim is very similar to a proof of Theorem 4 in [10]. And this proof let us know the ccc-ness of $\mathbb{P}$.

Claim 2.11. *$\mathbb{P} \times \mathcal{F}(\mathcal{A}, \mathcal{B})$ has the countable chain condition.*

Proof of Claim 2.11. Assume that $\{\langle P_\alpha, \sigma_\alpha \rangle; \alpha \in \omega_1\}$ is an uncountable collection of conditions in $\mathbb{P} \times \mathcal{F}(\mathcal{A}, \mathcal{B})$. Without loss of generality, we may assume that

- $\{P_\alpha; \alpha \in \omega_1\}$ forms a Δ-system with a root P,
- $\{\sigma_\alpha; \alpha \in \omega_1\}$ forms a Δ-system with a root σ,
- for all $\alpha \in \omega_1$, $P_\alpha \setminus P$ has the same size k, and
- for all $\alpha \in \omega_1$, $\sigma_\alpha \setminus \sigma$ has the same size l.

For $\alpha \in \omega_1$, we let $P^0_\alpha := \{p(0); p \in P_\alpha \setminus P\}$ and denote the i-th member of P^0_α and $\sigma_\alpha \setminus \sigma$ by $P^0_\alpha(i)$ and $\sigma_\alpha(j)$ for all $i < k$ and $j < l$ respectively. Using Lemma 2.9 of $\frac{k(k+1)}{2} + \frac{l(l+1)}{2}$ times, we can find uncountable subsets I_0 and I_1 of ω_1 such that

- for all $\alpha \in I_0$ and $\beta \in I_1$ and $i, j < k$,

$$\bigcup_{\xi \in P^0_\alpha(i)} a_\xi \cap \bigcup_{\xi \in P^0_\beta(j)} b_\xi \neq \emptyset,$$

and

- for all $\alpha \in I_0$ and $\beta \in I_1$ and $i, j < l$,

$$a_{\sigma_\alpha(i)} \cap b_{\sigma_\beta(j)} \neq \emptyset.$$

Then for any $\alpha \in I_0$ and $\beta \in I_1$, $\langle P_\alpha, \sigma_\alpha \rangle$ and $\langle P_\alpha, \sigma_\alpha \rangle$ are compatible in $\mathbb{P} \times \mathcal{F}(\mathcal{A}, \mathcal{B})$. $\dashv$

By the fact that $(\dot{\mathcal{C}}, \dot{\mathcal{D}})$ is an $\mathcal{S}(\mathcal{A}, \mathcal{B})$-name for a gap and the homogeneity of $\mathcal{S}(\mathcal{A}, \mathcal{B})$, we can moreover prove the following claim and this completes the proof.

Claim 2.12. *$\mathbb{P} \times \mathcal{S}(\mathcal{A}, \mathcal{B})$ has the countable chain condition.*

Proof of Claim 2.12. Let $\{\langle P_\alpha, \sigma_\alpha \rangle : \alpha \in \omega_1\}$ be in $\mathbb{P} \times \mathcal{S}(\mathcal{A}, \mathcal{B})$ for all $\alpha \in \omega$. Without loss of generality, we may assume that

- $\{P_\alpha; \alpha \in \omega_1\}$ forms a Δ-system with a root P,
- for all $\alpha \in \omega_1$, $P_\alpha \setminus P$ has the same size m, and
- for any $\alpha < \beta \in \omega_1$,

$$\max\left(\bigcup_{p \in P} p(1)\right) < \min\left(\bigcup_{p \in P_\alpha \setminus P} p(1)\right)$$

and

$$\max\left(\bigcup_{p \in P_\alpha \setminus P} p(1)\right) < \min\left(\bigcup_{p \in P_\beta \setminus P} p(1)\right).$$

Let $\{\langle \tau^i_\alpha, v^i_\alpha \rangle ; i < m\}$ enumerate the set $P_\alpha \setminus P$ and we denote σ_α by τ^m_α to simplify the notation for all $\alpha \in \omega_1$. Since $(\mathcal{A}, \mathcal{B})$ strictly admits finite changes, for every $\alpha \in \omega_1$ and $i \leq m$, there exists $\delta^i_\alpha \in \omega_1$ such that

$$\bigcup_{\xi \in \tau^i_\alpha} a_\xi = a_{\delta^i_\alpha} \quad \text{and} \quad \bigcup_{\xi \in \tau^i_\alpha} b_\xi = b_{\delta^i_\alpha}.$$

Since $\mathcal{S}(\mathcal{A},\mathcal{B})$ has the ccc, for each $i \leq m$, there exists $\rho^i \in \mathcal{S}(\mathcal{A},\mathcal{B})$ such that

$$\rho^i \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \dot{I}^i := \left\{\alpha \in \check{\omega_1}; \check{\tau}^i_\alpha \in \dot{G}\right\} \text{ is uncountable ”}.$$

We note that

$$\rho^i \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \dot{I}^i = \left\{\alpha \in \check{\omega_1}; \left\{\check{\delta^i_\alpha}\right\} \in \dot{G}\right\} \text{ ”}$$

for all $i \leq m$. By strengthening ρ^i's if need, we may assume that there exists $N \in \omega$ such that for all $i \neq j \leq m$, $\langle \rho^i, \rho^j, N\rangle$ is a good sequence. Then without loss of generality again, we may moreover assume that for all $\alpha, \beta \in \omega_1$ and $i \leq m$,

$$a_{\delta^i_\alpha} \cap N = a_{\delta^i_\beta} \cap N \quad \text{and} \quad b_{\delta^i_\alpha} \cap N = b_{\delta^i_\beta} \cap N.$$

We let $\pi_{i,m}$ be a finite bijection for an isomorphism so that

$$\pi_{i,m}\left(a_{\delta^i_\alpha} \cap N, b_{\delta^i_\alpha} \cap N\right) = \left\langle a_{\delta^m_\alpha} \cap N, b_{\delta^m_\alpha} \cap N\right\rangle$$

for each $i < m$ (and some (any) $\alpha \in \omega_1$) and let $\psi_{i,m}$ be the isomorphism from $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\rho^i$ onto $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\rho^m$ induced by $\pi_{i,m}$. We note that for every $i < m$, the calculations of $\psi_{i,m}$ are absolute and if $\left\{\delta^i_\alpha\right\} \cup \rho^i \in \mathcal{S}(\mathcal{A},\mathcal{B})$, then

$$\psi_{i,m}\left(\left\{\delta^i_\alpha\right\} \cup \rho^i\right) = \left\{\delta^m_\alpha\right\} \cup \rho^m$$

for all $\alpha \in \omega_1$. For each $i \neq j \leq m$, we define $\psi_{i,j} := (\psi_{j,m})^{-1} \circ \psi_{i,m}$. We note that for every $i \neq j \leq m$, $\psi_{i,j}\restriction\left(\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\rho^i\right)$ is an isomorphism onto $\mathcal{S}(\mathcal{A},\mathcal{B})\restriction\rho^j$, and if $\left\{\delta^i_\alpha\right\} \cup \rho^i \in \mathcal{S}(\mathcal{A},\mathcal{B})$, then

$$\psi_{i,j}\left(\left\{\delta^i_\alpha\right\} \cup \rho^i\right) = \left\{\delta^j_\alpha\right\} \cup \rho^j$$

for all $\alpha \in \omega_1$. Using Lemma 2.9, since $(\dot{\mathcal{C}}, \dot{\mathcal{D}})$ is a name for a gap, we can define $\mathcal{S}(\mathcal{A},\mathcal{B})$-names $\dot{I}^i_0$ and $\dot{I}^i_1$, for $i < m$, such that for each $i < m$,

- $\rho^i \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})}$ “ both $\dot{I}^i_0$ and $\dot{I}^i_1$ are uncountable subsets of $\dot{I}^i$ ”,
- $\rho^i \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})}$ “ for all $\alpha \in \dot{I}^i_0$ and all $\beta \in \dot{I}^i_1$, $\bigcup_{\xi \in \check{v}^i_\alpha} \dot{c}_\xi \cap \bigcup_{\xi \in \check{v}^i_\beta} \dot{d}_\xi \neq \emptyset$ ”,
- $\rho^0 \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})}$ “ $\dot{I}^0_0 \subseteq \check{\psi}_{m,0}\left(\dot{I}^m\right)$ and $\dot{I}^0_1 \subseteq \check{\psi}_{m,0}\left(\dot{I}^m\right)$ ”,

  and

  $$\rho^{i+1} \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \dot{I}^{i+1}_0 \subseteq \check{\psi}_{i,i+1}\left(\dot{I}^i_0\right) \text{ and } \dot{I}^{i+1}_1 \subseteq \check{\psi}_{i,i+1}\left(\dot{I}^i_1\right) \text{ ”}.$$

This can be done because for every $i \neq j \leq m$, if $\mu \leq \rho^i$ and $\tau \in [\omega_1]^{<\omega}$ such that

$$\mu \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \check{\tau} \in \dot{G} \text{ ”},$$

then $\psi_{i,j}(\mu) \leq \rho^j$ and

$$\psi_{i,j}(\mu) \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \check{\psi}_{i,j}(\check{\tau}) \in \dot{G} \text{ ”}$$

and because of the property of $\psi_{i,j}$'s. (We note that $\mathcal{S}(\mathcal{A},\mathcal{B})$ is not separative.)

We take any $\rho \le \rho^{m-1}$ and $\alpha, \beta \in \omega_1$ such that

$$\rho \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \check{\alpha} \in \dot{I}_0^{m-1} \text{and } \check{\beta} \in \dot{I}_1^{m-1} \text{ ”}.$$

Then by the conditions of $\dot{I}_0^i$ and $\dot{I}_1^i$, we note that for each $i < m-1$,

$$\psi_{m-1,i}(\rho) \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \check{\alpha} \in \dot{I}_0^i \text{ and } \check{\beta} \in \dot{I}_1^i \text{ ”}.$$

This means that for every $i \le m$, $\rho \cup \tau_\alpha^i \cup \tau_\beta^i$ is a condition in $\mathcal{S}(\mathcal{A},\mathcal{B})$ and for every $i < m$,

$$\rho \cup \tau_\alpha^i \cup \tau_\beta^i \Vdash_{\mathcal{S}(\mathcal{A},\mathcal{B})} \text{“ } \check{v}_\alpha^i \text{ and } \check{v}_\beta^i \text{ are incompatible in } \mathcal{S}(\dot{\mathcal{C}},\dot{\mathcal{D}}) \text{ ”}.$$

This implies that $P_\alpha \cup P_\beta$ is pairwise incompatible in $\mathbb{Q}$ and σ_α and σ_β are compatible in $\mathcal{S}(\mathcal{A},\mathcal{B})$, hence $\langle P_\alpha, \sigma_\alpha \rangle$ and $\langle P_\beta, \sigma_\beta \rangle$ are compatible in $\mathbb{P} \times \mathcal{S}(\mathcal{A},\mathcal{B})$, which completes the proof of the claim. $\dashv$ $\square$

Proof of Theorem 2. Without loss of generality, we may assume that there are two independent destructible gaps $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ both of which strictly admit finite changes. Since $\mathcal{S}(\mathcal{A},\mathcal{B}) \times \mathcal{F}(\mathcal{C},\mathcal{D})$ is ccc and $\mathcal{S}(\mathcal{A},\mathcal{B})$ is homogeneous, we can consider $(\mathcal{C},\mathcal{D})$ as an $\mathcal{S}(\mathcal{A},\mathcal{B})$-name for a gap. As in the proof of Lemma 2.10, let $\mathbb{P}$ be a forcing notion adding an uncountable antichain in $\mathcal{S}(\mathcal{A},\mathcal{B}) \times \mathcal{S}(\mathcal{C},\mathcal{D})$ by finite approximations. Then not only $\mathbb{P} \times \mathcal{F}(\mathcal{A},\mathcal{B})$ and $\mathbb{P} \times \mathcal{S}(\mathcal{A},\mathcal{B})$, but also $\mathbb{P} \times \mathcal{F}(\mathcal{C},\mathcal{D})$ and $\mathbb{P} \times \mathcal{S}(\mathcal{C},\mathcal{D})$ have the ccc. So in the extension with $\mathbb{P}$, both $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ are still destructible gaps and $\mathcal{S}(\mathcal{A},\mathcal{B}) \times \mathcal{S}(\mathcal{C},\mathcal{D})$ does not have the countable chain condition. $\square$

Proof of Theorem 3. This is just a corollary of Lemma 2.10. We fix one destructible gap which strictly admits finite changes, and then by an iteration with a finite support, we can force the desired statement. We note it is upward closed that the forcing notion $\mathbb{Q}$ as in Lemma 2.10 has an uncountable antichain. We notice that the continuum can be large. $\square$

References

[1] U. Abraham and S. Shelah. *A Δ_2^2 well-order of the reals and incompactness of $L(Q^{\mathrm{MM}})$*, Annals of Pure and Applied Logic, 59 (1993), no. 1, 1–32.

[2] U. Abraham and S. Todorčević. *Partition properties of ω_1 compatible with CH*, Fundamenta Mathematicae, 152 (1997), 165–180.

[3] J. Bagaria and H. Woodin. *$\underset{\sim}{\Delta}_n^1$ sets of reals.*, Journal of Symbolic Logic, 62 (1997), no. 4, 1379–1428.

[4] T. Bartoszyński and H. Judah. *Set Theory: On the structure of the real line,* A.K.Peters, Wellesley, Massachusetts, 1995.

[5] H. Dales and H. Woodin. *An introduction to independence for analysts*, London Mathematical Society Lecture Note Series, 115.

[6] A. Dow. *More set-theory for topologists*, Topology and its Applications, 64 (1995), no. 3, 243–300.

[7] I. Farah. *Embedding partially ordered sets into* ω^ω, Fundamenta Mathematicae, 151, (1996), 53-95.

[8] I. Farah. *OCA and towers in* $\mathcal{P}(\mathbb{N})/fin$, Commentationes Mathematicae Universitatis Carolinae, 37, (1996), 861-866.

[9] J. Hirschorn. *Summable gaps*, Annals of Pure and Applied Logic, 120 (2003), 1-63.

[10] S. Kamo. *Almost coinciding families and gaps in* $\mathcal{P}(\omega)$, Journal of the Mathematical Society of Japan, 45 (1993), no. 2, 357–368.

[11] K. Kunen. (κ, λ^*)*-gaps under MA*, handwritten note, 1976.

[12] K. Kunen. *Set Theory: An Introduction to Independence Proofs*, volume 102 of *Studies in Logic*, North Holland, 1980.

[13] K. Kunen and F. Tall. *Between Martin's axiom and Souslin's hypothesis*, Fundamenta Mathematicae, 102 (1979), no. 3, 173–181.

[14] P. Larson. *An* $\mathbb{S}_{\max}$ *variation for one Souslin tree*, Journal of Symbolic Logic, 64 (1999), no. 1, 81–98.

[15] R. Laver. *Linear orders in* $(\omega)^\omega$ *under eventual dominance*, Logic Colloquium '78, North-Holland, 299-302, 1979.

[16] J. Moore, M. Hrušák and M. Džamonja. *Parametrized* $\diamondsuit$ *principles*, Transactions of American Mathematical Society, 356 (2004), 2281-2306.

[17] M. Rabus. *Tight gaps in* $\mathcal{P}(\omega)$, Topology Proceedings, 19 (1994), 227–235.

[18] M. Scheepers. *Gaps in* ω^ω, In *Set Theory of the Reals*, volume 6 of *Israel Mathematical Conference Proceedings*, 439-561, 1993.

[19] S. Todorčević. *Partition Problems in Topology*, volume 84 of *Contemporary mathematics*, American Mathematical Society, Providence, Rhode Island, 1989.

[20] S. Todorčević and I. Farah. *Some Applications of the Method of Forcing*, Mathematical Institute, Belgrade and Yenisei, Moscow, 1995.

[21] T. Yorioka. *Forcings with the countable chain condition and the covering number of the Marczewski ideal*, Archive for Mathematical Logic, vol.42 (2003), no.7, 695–710.

[22] T. Yorioka. *The diamond principle for the uniformity of the meager ideal implies the existence of a destructible gap*, to appear in AML.

[23] T. Yorioka. *Independent families of destructible gaps*, preprint.

[24] M. Zakrzewski. *Weak product of Souslin trees can satisfy the countable chain condition*, L'Académie Polonaise des Sciences. Bullten. Série des Science Mathématiques, 29 (1981), no. 3-4, 99–102.

[25] M. Zakrzewski. *Some theorems on products of Souslin and almost Souslin trees*, Bulletin of the Polish Academy of Sciences. mathematics, 33 (1985), no. 11-12, 651–657 (1986).