数理解析研究所講究録 1419 巻 2005 年 68-70

Reflection groups of geodesic spaces and Coxeter groups

宇都宮大学教育学部

保坂 哲也 (Tetsuya Hosaka)

The purpose of this note is to introduce a result of my recent paper [8] about *cocompact discrete reflection groups of geodesic spaces.*

A metric space (X,d) is called a *geodesic space* if for each $x,y\in X$, there exists an isometry $\xi:[0,d(x,y)]\to X$ such that $\xi(0)=x$ and $\xi(d(x,y))=y$ (such ξ is called a *geodesic*). We say that an isometry r of a geodesic space X is a *reflection* of X, if

(1) r^2 is the identity of X,

(2) $X\setminus F_r$ has strictly two convex components X_r^+ and X_r^-, and

(3) $\operatorname{Int} F_r=\emptyset$,

where F_r is the fixed-point set of r which is called the *wall* of r. An isometry group Γ of a geodesic space X is called a *reflection group*, if some set of reflections of X generates Γ. Let Γ be a reflection group of a geodesic space X and let R be the set of all reflections of X in Γ. We note that R generates Γ by definition. Now we suppose that the action of Γ on X is proper, that is, $\{\gamma\in\Gamma\,|\,\gamma x\in B(x,N)\}$ is finite for each $x\in X$ and $N>0$ (cf. [2, p.131]). Then the set $\{F_r\,|\,r\in R\}$ is locally finite. Let C be a component of $X\setminus\bigcup_{r\in R}F_r$, which is called a *chamber*. Here we can show that $\Gamma C=X\setminus\bigcup_{r\in R}F_r$. Then $\Gamma\overline{C}=X$ and for each $\gamma\in\Gamma$, either $C\cap\gamma C=\emptyset$ or $C=\gamma C$. We say that Γ is a *cocompact discrete reflection group* of X, if $\overline{C}$ is compact and $\{\gamma\in\Gamma\,|\,C=\gamma C\}=\{1\}$.

Definition. A group Γ is called a *cocompact discrete reflection group* of a geodesic space X, if

(1) Γ is a reflection group of X,
(2) the action of Γ on X is proper,
(3) for a chamber C, $\overline{C}$ is compact, and
(4) $\{\gamma \in \Gamma \mid C = \gamma C\} = \{1\}$.

For example, every Coxeter group is a cocompact discrete reflection group of some geodesic space.

A *Coxeter group* is a group W having a presentation

$$\langle\, S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S\,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for each $s,t \in S$,
(2) $m(s,s) = 1$ for each $s \in S$, and
(3) $m(s,t) \geq 2$ for each $s,t \in S$ such that $s \neq t$.

The pair (W, S) is called a *Coxeter system.* H.S.M. Coxeter showed that a group Γ is a finite reflection group of some Euclidean space if and only if Γ is a finite Coxeter group. Every Coxeter system (W, S) induces the Davis-Moussong complex $\Sigma(W, S)$ which is a CAT(0) space ([6], [7], [10]). Then the Coxeter group W is a cocompact discrete reflection group of the CAT(0) space $\Sigma(W, S)$.

Here we obtained the following theorem in [8].

Theorem. *A group Γ is a cocompact discrete reflection group of some geodesic space if and only if Γ is a Coxeter group.*

Let Γ be a cocompact discrete reflection group of a geodesic space X, let C be a chamber and let S be a *minimal* subset of R such that $C = \bigcap_{s \in S} X_s^+$ (i.e. $C \neq \bigcap_{s \in S \setminus \{s_0\}} X_s^+$ for each $s_0 \in S$). Then we can show that $\langle S \rangle C = X \setminus \bigcup_{r \in R} F_r = \Gamma C$. Since $\{\gamma \in \Gamma \mid C = \gamma C\} = \{1\}$, S generates Γ. In [8], we have proved that the pair (Γ, S) is a Coxeter system.

References

[1] N. Bourbaki, *Groupes et Algebrès de Lie*, Chapters IV-VI, Masson, Paris, 1981.

[2] M.R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[3] K.S. Brown, *Buildings*, Springer-Verlag, 1980.

[4] H.S.M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. **35** (1934), 588–621.

[5] ______, *The complete enumeration of finite groups of the form* $R_i^2 = (R_iR_j)^{k_{ij}} = 1$, J. London Math. Soc. **10** (1935), 21–25.

[6] M.W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. **117** (1983), 293–324.

[7] ______, *Nonpositive curvature and reflection groups*, in Handbook of geometric topology (Edited by R. J. Daverman and R. B. Sher), pp. 373–422, North-Holland, Amsterdam, 2002.

[8] T. Hosaka, *Reflection groups of geodesic spaces and Coxeter groups*, preprint.

[9] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[10] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D. thesis, The Ohio State University, 1988.

Department of Mathematics, Utsunomiya University, Utsunomiya, 321-8505, Japan

E-mail address: hosaka@cc.utsunomiya-u.ac.jp