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A special value of the spectral zeta function of the non-commutative harmonic osci llators (非可換調和振動子のゼータの特殊値)

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Abstract

The non-commutative harmonic oscillator is a 2×2-system of harmonic oscillators with a non-trivial correlation. We write down explicitly the special value at $s=2$ of the spectral zeta function of the non-commutative harmonic oscillator in terms of the complete elliptic integral of the first kind, which is a special case of a hypergeometric function.

1 Introduction

The non-commutative harmonic oscillator $Q=Q(x,\partial_x)$ is defined to be the second-order ordinary differential operator

$$Q(x,\partial_x)=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}(-\frac{\partial_x^2}{2}+\frac{x^2}{2})+\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}(x\partial_x+\frac{1}{2}).$$

The first term is two harmonic oscillators, which are mutually independent, with the scaling constant $\alpha>0$ and $\beta>0$, while the second term is considered to be the correlation with a self-adjoint manner. The spectral problem is a 2×2 system of the ordinary differential equations

$$Q(x,\partial_x)u(x)=\lambda u(x)$$

with an eigenstate $u(x)=\begin{bmatrix}u_1(x)\\ u_2(x)\end{bmatrix}\in L^2(\mathbf{R})^{\oplus 2}$ and a spetrum $\lambda\in\mathbf{R}$. It is known [8] that under the natural assumption $\alpha\beta>1$ on the positivity, which is also

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assumed in this paper, the operator Q defines a positive, self-adjoint operator with a discrete spectrum

$$(0<)\lambda_1 \le \lambda_2 \le \cdots \to +\infty$$

The corrèsponding spectral zeta function is defined to be

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}.$$

An expression of the special value $\zeta_Q(2)$ is obtained in [2] in terms of a certain contour integral using the solution of a singly confluent type Heun differential equation. It would be indicated that these special values are complicated enough and highly transcendental as reflecting the transcendence of the spectra of the non-commutative harmonic oscillator.

However, in this paper, we prove the following simple expression:

$$\zeta_Q(2) = \frac{\pi^2}{4}\frac{(\alpha^{-1}+\beta^{-1})^2}{(1-\alpha^{-1}\beta^{-1})}\left(1+\left(\frac{\alpha^{-1}-\beta^{-1}}{\alpha^{-1}+\beta^{-1}}{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{1}{1-\alpha\beta}\right)\right)^2\right). \qquad (1)$$

where ${}_2F_1$ is the Gauss hypergeometric series. We also have an expression by using the complete elliptic integral of the first kind as

$$\zeta_Q(2) = \frac{\pi^2}{4}\frac{(\alpha^{-1}+\beta^{-1})^2}{(1-\alpha^{-1}\beta^{-1})}\left(1+\left(\frac{\alpha^{-1}-\beta^{-1}}{\alpha^{-1}+\beta^{-1}}\int_0^{2\pi}\frac{d\theta}{2\pi\sqrt{1+(\cos\theta)/\sqrt{1-\alpha\beta}}}\right)^2\right). \qquad (2)$$

In this sense, the special value $\zeta_Q(2)$ is written in terms of a hypergeometric series, which is much tractable and known. Note that each spectrum is related with the monodromy problem of Heun's differential equation, which is far from hypergeometric, see [5], [6]. Only the total of spectra has an extra simple form in some sense.

Here is a brief organization of the paper; In Section 2, we recall the expression of $\zeta_Q(2)$ given in [2], and derive more explicit formula of the generating function appearing in that expression. We prove in Section 3 our main results, the equations (1) and (2). The proof depends on several formulae of hypergeometric series not only for ${}_2F_1$ but also for ${}_3F_2$ such as Clausen's identity.

2 An expression of the generating function

We start from the series-expression of the special value $\zeta_Q(2)$ of the non-commutative harmonic oscillator given in [2, (4.5a)]

$$\zeta_Q(2) = Z_1(2) + \sum_{n=0}^{\infty} Z_n'(2).$$

We introduce notations. Recall that $\alpha > 0$, $\beta > 0$ with $\alpha\beta > 1$. Let us introduce the parameters $\gamma = 1/\sqrt{\alpha\beta}$ and $a = \gamma/\sqrt{1-\gamma^2} = 1/\sqrt{\alpha\beta - 1}$ as in [2, (4.1)]. Note that they satisfy $0 < \gamma < 1$ and $a > 0$.

The term $Z_1(2)$ is given in [2, (4.5b)] and $Z'_n(2)$ are given in [2, (4.9)] as

$$Z_1(2) = \frac{(\alpha^{-1}+\beta^{-1})^2}{2(1-\gamma^2)} 3\zeta(2), \tag{3}$$

$$Z'_n(2) = (-1)^n \frac{(\alpha^{-1}-\beta^{-1})^2}{(1-\gamma^2)} \binom{2n-1}{n} \left(\frac{a}{2}\right)^{2n} J_n. \tag{4}$$

The values $\{J_n\}_{n=1,2,\cdots}$ are specified by the generating function

$$w(z) := \sum_{n=0}^{\infty} J_n z^n.$$

The function $w(z)$ is a solution of the ordinary differential equation

$$z(1-z)^2 \frac{d^2w}{dz^2} + (1-3z)(1-z)\frac{dw}{dz} + \left(z - \frac{3}{4}\right) w = 0 \tag{5}$$

which is given in [2, Theorem 4.13] and called a singly confluent Heun's differential equation. The constant term is given by $w(0) = J_0 = 3\zeta(2) = \pi^2/2$. It is easy to see that there exists a unique power-series solution of this homogeneous differential equation (5) with the initial condition $w(0) = \pi^2/2$. The final target $\zeta_Q(2)$ involving these J_n's with an infinite sum seemed to have no closed expression.

In this section, we give a simple expression of the generating function $w(z)$. We denote by $\partial_z = \partial/\partial z$.

Lemma 1 *The differential equation (5) is equivalent to*

$$4(1-z)\partial_z z \partial_z (1-z) w + w = 0. \tag{6}$$

Proof: This directly follows from Leibniz rule. QED

Lemma 2 *Let $t = z/(z-1)$ be a new independent variable, and $\eta(t) = (1-z)w(z)$ a new unknown function. Then the differential equation (6) is equivalent to*

$$t(1-t)\partial_t^2 \eta + (1-2t)\partial_t \eta - \frac{1}{4}\eta = 0. \tag{7}$$

Proof: The differential equation (6) is equivalent to

$$4(z-1)^2 \partial_z z \partial_z (z-1) w + (z-1) w = 0.$$

Note that $(z-1)(t-1) = 1$ and $\partial_t := \partial/\partial t = -(z-1)^2 \partial_z$. Then

$$4\partial_t t(t-1)\partial_t \eta + \eta = 0.$$

By Leibniz rule, this is equivalent to (7). QED

Proposition 3

$$w(z) = \frac{J_0}{1-z}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{z}{z-1}\right).$$

Proof: Since any power-series solution of (7) in t is a constant multiple of ${}_2F_1(\frac{1}{2},\frac{1}{2};1;t)$, we have the conclusion. QED

3 The special value

We introduce the auxiliary series

$$g(a) := \frac{2}{J_0}\sum_{n=0}^{\infty}(-1)^n\binom{2n-1}{n}\left(\frac{a}{2}\right)^{2n}J_n$$

so that

$$\zeta_Q(2) = \frac{(\alpha^{-1}+\beta^{-1})^2}{2(1-\gamma^2)}3\zeta(2) + \frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}3\zeta(2)g(a) \tag{8}$$

$$= \frac{\pi^2}{4}\frac{(\alpha^{-1}+\beta^{-1})^2}{(1-\alpha^{-1}\beta^{-1})}\left(1+\left(\frac{\alpha^{-1}-\beta^{-1}}{\alpha^{-1}+\beta^{-1}}\right)^2 g(a)\right) \tag{9}$$

Theorem 4

$$g(a) = {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;-a^2\right)^2.$$

Proof: We note that

$$\binom{2n-1}{n}\left(\frac{1}{2}\right)^{2n} = \frac{1}{2}\times\frac{(2n-1)!!}{(2n)!!} = \frac{1}{2\pi}\int_0^1\frac{u^n du}{\sqrt{u(1-u)}}.$$

Then, the integration by parts implies that

$$g(a) = \frac{2}{2\pi J_0}\sum_{n=0}^{\infty}(-1)^n\int_0^1\frac{u^n du}{\sqrt{u(1-u)}}a^{2n}J_n = \frac{1}{\pi J_0}\int_0^1\frac{w(-a^2u)du}{\sqrt{u(1-u)}}. \tag{10}$$

By Proposition 3, the function w is written in terms of hypergeometric series ${}_2F_1$. We substitute such an expression into the equation (10), then we obtain

$$g(a) = \frac{1}{\pi}\int_0^1\frac{1}{1+a^2u}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{a^2u}{a^2u+1}\right)\frac{du}{\sqrt{u(1-u)}}.$$

We introduce a new variable $v = (1+a^2)u/(1+a^2u)$. Then

$$g(a) = \frac{1}{\pi}\int_0^1 {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{a^2v}{1+a^2}\right)\frac{dv}{\sqrt{v(1-v)(1+a^2)}}.$$

Now we use the formula (2.2.2) of [1]

$$ {}_3F_2(a_1,a_2,a_3;b_1,b_2;x) = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2-a_3)} \int_0^1 t^{a_3-1}(1-t)^{b_2-a_3-1} {}_2F_1(a_1,a_2;b_1;xt)dt. $$

This shows

$$ g(a) = \frac{1}{\sqrt{1+a^2}} {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;\frac{a^2}{1+a^2}\right). $$

By Clausen's identity (in e.g., Exercise 13 of Chapter 2 in [1])

$$ {}_2F_1\left(a,b;a+b+\frac{1}{2};x\right)^2 = {}_3F_2\left(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};x\right), $$

we obtain

$$ g(a) = \frac{1}{\sqrt{1+a^2}} {}_2F_1\left(\frac{1}{4},\frac{1}{4};1;\frac{a^2}{1+a^2}\right)^2. $$

Moreover by Pfaff formula, Theorem 2.2.5 of [1]

$$ {}_2F_1(a,b;c;x) = (1-x)^{-a} {}_2F_1(a,c-b;c;x/(x-1)), $$

we obtain

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;-a^2\right) = (1+a^2)^{-1/4} {}_2F_1\left(\frac{1}{4},\frac{1}{4};1;\frac{a^2}{a^2+1}\right). $$

This shows

$$ g(a) = {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;-a^2\right)^2. $$

QED

Remark 5 *In the earlier version of the paper, it was suggested to make use of the hypergeometric series* ${}_3F_2$ *with this special parameter* $(1/2,1/2,1/2;1,1)$ *by the multi-variable hypergeometric function of type* $(3,6)$*, especially by its restriction on the stratum called* X_{1b} *in [4]. However, we can avoid to use a multi-variable hypergeometric function in the present version as is seen above.*

Theorem 4 with the help of the equation (9) shows the equation (1). The equation (2) is shown as follows. By Theorem 3.13 of [1]

$$ {}_2F_1(a,b;2a;x) = \left(1-\frac{x}{2}\right)^{-b} {}_2F_1\left(\frac{b}{2},\frac{b+1}{2};a+\frac{1}{2};\left(\frac{x}{2-x}\right)^2\right), $$

we have

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2ia}{ia+1}\right) = (1+ia)^{1/2} {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;-a^2\right). $$

Let us recall the definition of the elliptic integral of the first kind;

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right).$$

Then we have

$${}_2F_1\left(\frac{1}{4},\frac{3}{4};1;-a^2\right) = \frac{2}{\pi}(1+ia)^{-1/2}K\left(\frac{2ia}{ia+1}\right) = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1+ia\cos 2\theta}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{1+ia\cos\theta}},$$

and the equation (2).

References

[1] G. E. Andrews, R. Askey and R. Roy, Special functions. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.

[2] T. Ichinoze and M.Wakayama, Special values of the spectral zeta function of the non-commutative harmonic oscillator and confluent Heun equations, to appear in Kyushu Journal of Math., preprint 04-311 avaliable at `http://rene.ma.utexas.edu/mp_arc/`.

[3] T. Ichinoze and M.Wakayama, Zeta functions for the spectrum of the non-commutative harmonic oscillators, to appear in Comm. Math. Phys., preprint 04-312 at [2].

[4] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4-parameter family of $K3$ surfaces and the hypergeometric function of type $(3,6)$, Internat. J. Math. 3 (1992), no. 1, 164 pp.

[5] H. Ochiai, Non-commutative harmonic oscillators and Fuchsian ordinary differential operators, Comm. Math. Phys. **217** (2001), 357–373.

[6] H. Ochiai, Non-commutative harmonic oscillators and the connection problem of the Heun differential equation , preprint, 2004, 6 pages.

[7] A. Parmeggiani and M. Wakayama, Oscillator representations and systems of ordinary differential equations, Proc. Nat. Acad. Sci. U.S.A. **98**(2001), 26–30.

[8] A. Parmeggiani and M. Wakayama, Non-commutative harmonic oscillators. I, II. Forum Math. **14** (2002), 539–604, 669–690.

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