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Templates and iterations Luminy 2002 lecture notes

Jörg Brendle*
The Graduate School of Science and Technology
Kobe University
Rokko–dai 1–1, Nada–ku
Kobe 657–8501, Japan
email: brendle@kurt.scitec.kobe-u.ac.jp

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Abstract

This paper is based on part of the lecture notes of a minicourse on iteration techniques presented at the Luminy workshop on set theory in 2002.[1]

The technique of iterations along templates has been developed by Saharon Shelah in 1999 to solve one of the classical problems on cardinal invariants of the continuum, namely to prove the consistency of $\mathfrak{d} < \mathfrak{a}$ [Sh] where $\mathfrak{d}$ is the dominating number and $\mathfrak{a}$ is the almost disjointness number. An alternative, axiomatic, treatment of such iterations has been developed by the present author in the survey [Br1] and used in [Br2] to show that $\mathfrak{a}$ consistently may have countable cofinality.

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[1]We apologize for not presenting the paper we talked about at the Kyōto RIMS conference. The reason is (1) the latter has not been written yet and (2) it has already been promised to another conference proceedings volume the deadline for which has past already.

We present yet another approach to iterations along templates which, unlike previous work ([Sh], [Br1], [Br2]), is based on correctness of complete embeddings. The advantage is two-fold:

(1) We obtain a concrete description of the initial steps of the iteration: they must be either standard two-step iterations or direct limits.

(2) The present framework is fairly general; in particular, the proof of complete embeddability between the initial steps of the iteration (which is often the hardest argument for nonlinear iterations) works for a large class of forcing notions, namely Suslin ccc forcing notions which preserve correctness. In [Br1], this proof had been done only for, and used the combinatorics of, Hechler forcing though it was clear that it could be done for a number of other forcing notions as well.

Correct embeddings first arose in the author's work [Br4] on shattered iterations (see also [Br3]) where they where crucial for proving complete embeddability into the amalgamated limit. The latter work shows the consistency of $\mathsf{cov}(\mathcal{N}) > \mathsf{cov}(\mathcal{M}) > \aleph_1$.

In Section 1, we give a brief outline of correctness. For a more comprehensive treatment see Section 1 of [Br3]. Section 2 presents our template framework. In the third part of the Luminy minicourse, we discussed isomorphisms of names and, putting the templates to good use, sketched Shelah's proof of the consistency of $\mathfrak{d} < \mathfrak{a}$. We omit this third part here since it does not involve any new ideas and essentially follows the exposition in [Br1].

1 Complete embeddings

Let $I = \{0 \wedge 1, 0, 1, 0 \vee 1\}$ be the four-element lattice with top element $0 \vee 1$, bottom element $0 \wedge 1$, and 0 and 1 in between. Assume we are given cBa's $\mathbb{A}_i$ for $i \in I$ and complete embeddings $e_{ij} : \mathbb{A}_i \to \mathbb{A}_j$ for $i \leq j$ from I. The latter give rise to projection mappings $h_{ji} : \mathbb{A}_j \to \mathbb{A}_i$ naturally defined by

$$h_{ji}(a_j) = \prod \{a \in \mathbb{A}_i;\ e_{ij}(a) \geq a_j\}$$

Assume without loss that $\mathbb{A}_i \subseteq \mathbb{A}_j$ for $i \leq j$ and that e_{ij} is the identity. Write $\mathbb{A}_i <\circ \mathbb{A}_j$ to denote $\mathbb{A}_i$ is completely embedded in $\mathbb{A}_j$.

Definition 1.1 *The projection $h_{0\vee 1,0}$ is* correct *(*at 1*) if $h_{0\vee 1,0}(a_1) = h_{1,0\wedge 1}(a_1)$ holds for all $a_1 \in \mathbb{A}_1$.*

Notice that $h_{0\vee 1,0}$ is correct at 1 iff for all $a_0 \in \mathbb{A}_0$ and $a_1 \in \mathbb{A}_1$, if $h_{0,0\wedge 1}(a_0) \cdot h_{1,0\wedge 1}(a_1) \neq \mathbf{0}$, then $a_0 \cdot a_1 \neq \mathbf{0}$ in $\mathbb{A}_{0\vee 1}$.

This shows that correctness is in fact a symmetric property, that is, correctness of $h_{0\vee 1,0}$ at 1 is equivalent to correctness of $h_{0\vee 1,1}$ at 0, and we shall often simply say *the diagram*

$$\langle \mathbb{A}_i \rangle = \begin{tikzcd}[column sep=small, row sep=small] & \mathbb{A}_{0\vee 1} \arrow[dr, dash] & \\ \mathbb{A}_0 \arrow[ur, dash] & & \mathbb{A}_1 \\ & \mathbb{A}_{0\wedge 1} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

is correct. Correctness implies that $\mathbb{A}_{0\wedge 1} = \mathbb{A}_0 \cap \mathbb{A}_1$, but not vice-versa (see [Br3]).

Example 1. (*amalgamation*) Let $\mathbb{A}_{0\vee 1} = \mathrm{amal}_{\mathbb{A}_{0\wedge 1}}(\mathbb{A}_0, \mathbb{A}_1)$ be the *amalgamation of* $\mathbb{A}_0$ *and* $\mathbb{A}_1$ *over* $\mathbb{A}_{0\wedge 1}$. Conditions are pairs $(a_0, a_1) \in (\mathbb{A}_0 \setminus \{\mathbf{0}\}) \times (\mathbb{A}_1 \setminus \{\mathbf{0}\})$ such that $h_{0,0\wedge 1}(a_0) \cdot h_{1,0\wedge 1}(a_1) \neq \mathbf{0}$, equipped with the coordinatewise order, $(a_0', a_1') \leq (a_0, a_1)$ if $a_0' \leq a_0$ and $a_1' \leq a_1$. $\mathbb{A}_{0\vee 1}$ is the completion of this set of conditions. Identify $\mathbb{A}_0$ with $\{(a_0, \mathbf{1});\ a_0 \in \mathbb{A}_0\}$; similarly for $\mathbb{A}_1$; thus $\mathbb{A}_0, \mathbb{A}_1 \subseteq \mathbb{A}_{0\vee 1}$, and it is well-known the embeddings are complete. The diagram is correct by definition of the amalgamation: if $h_{0,0\wedge 1}(a_0) \cdot h_{1,0\wedge 1}(a_1) \neq \mathbf{0}$, then $(a_0, a_1) \neq \mathbf{0}$ is a common extension.

The product is a special case of the amalgamation with $\mathbb{A}_{0\wedge 1} = \{\mathbf{0}, \mathbf{1}\}$. In fact, there is another way of looking at the amalgamation: forcing with $\mathrm{amal}_{\mathbb{A}_{0\wedge 1}}(\mathbb{A}_0, \mathbb{A}_1)$ is equivalent to first forcing with $\mathbb{A}_{0\wedge 1}$ and then with the product of the factors $\mathbb{A}_0 / G_{0\wedge 1}$ and $\mathbb{A}_1 / G_{0\wedge 1}$.

Example 2. (*two-step iteration*) Recall *Hechler forcing* $\mathbb{D}$ consists of pairs $(s, f) \in \omega^{<\omega} \times \omega^\omega$ such that $s \subseteq f$. The ordering is given by $(t, g) \leq (s, f)$ if $t \supseteq s$ and $g(n) \geq f(n)$ for all $n \in \omega$. Hechler forcing generically adds a real $d \in \omega^\omega$ which eventually dominates all ground model reals. Let $\mathbb{A}_0 = \mathbb{A}_1 = \mathrm{r.o.}(\mathbb{D})$, the cBa generated by $\mathbb{D}$, put $\mathbb{A}_{0\vee 1} = \mathbb{A}_0 \star \dot{\mathbb{A}}_1$, the two-step iteration (where $\dot{\mathbb{A}}_1$ is an $\mathbb{A}_0$-name for $\mathrm{r.o.}(\mathbb{D})$), and let $\mathbb{A}_{0\wedge 1} = \{\mathbf{0}, \mathbf{1}\}$ be trivial. As before identify $\mathbb{A}_0$ with $\{(a_0, \mathbf{1});\ a_0 \in \mathbb{A}_0\}$; similarly for $\mathbb{A}_1$.

Then all embeddings are complete – the only nontrivial case is $\mathbb{A}_1 <\circ\ \mathbb{A}_{0\vee 1}$: we need to argue that every maximal antichain $A \subseteq \mathbb{A}_1$ is still maximal in $\mathbb{A}_0 \star \dot{\mathbb{A}}_1$. By the above identification, this is clearly equivalent to saying that every maximal antichain $A \subseteq \mathbb{D}$ in the ground model V is still a maximal antichain of $\mathbb{D}$ in the sense of $V[G_0]$ where $G_0 \subseteq \mathbb{A}_0$ is an arbitrary generic. (Note here that $\mathbb{D}^V$ is a proper subset of $\mathbb{D}^{V[G_0]}$.) However, since $\mathbb{D}$ is Borel ccc ($\mathbb{D}$, the ordering and incompatibility are Borel), being a maximal antichain in $\mathbb{D}$ is a $\mathbf{\Pi}^1_1$-statement and therefore absolute. Correctness, then, is straightforward.

Lemma 1.2 (embeddability of direct limits [Br3]) *Let K be a directed index set. Assume $\langle \mathbb{A}_k;\ k \in K\rangle$ and $\langle \mathbb{E}_k;\ k \in K\rangle$ are systems of cBa's such that $\mathbb{A}_k <\circ \mathbb{A}_\ell$, $\mathbb{E}_k <\circ \mathbb{E}_\ell$ and $\mathbb{A}_k <\circ \mathbb{E}_k$ for any $k \leq \ell$. Assume further all diagrams*

$$\begin{tikzcd}[column sep=small, row sep=small] & \mathbb{E}_\ell \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{E}_k \arrow[dr, dash] & & \mathbb{A}_\ell \arrow[dl, dash] \\ & \mathbb{A}_k & \end{tikzcd}$$

are correct for $k \leq \ell$. Then the direct limit $\mathbb{A}$ of the $\mathbb{A}_k$'s completely embeds into the direct limit $\mathbb{E}$ of the $\mathbb{E}_k$'s.

Proof. Let $A \subseteq \bigcup_k \mathbb{A}_k$ be a maximal antichain in $\mathbb{A}$. Choose $b \in \mathbb{E}$, i.e. $b \in \mathbb{E}_k$ for some k. By maximality of A there is $a \in A$ such that $h_{\mathbb{E}_k,\mathbb{A}_k}(b) \cdot h_{\mathbb{A},\mathbb{A}_k}(a) \neq \mathbf{0}$. Find $\ell \geq k$ such that $a \in \mathbb{A}_\ell$. By correctness, $b \cdot a \neq \mathbf{0}$ in $\mathbb{E}_\ell$ as required. $\square$

Lemma 1.3 (preservation of correctness in two-step iterations with $\mathbb{D}$ [Br3]) *Assume $\langle \mathbb{A}_i\rangle$ is correct. Then so is $\langle \mathbb{E}_i\rangle$ where $\mathbb{E}_i = \mathbb{A}_i \star \mathrm{r.o.}(\dot{\mathbb{D}})$.*

Proof. We may assume without loss $\mathbb{A}_{0\wedge 1} = \{\mathbf{0}, \mathbf{1}\}$. Let $e_1 = (a_1, (s_1, \dot{f}_1)) \in \mathbb{E}_1$ and fix $(s, f) \leq h_{1,0\wedge 1}(e_1)$. Without loss of generality $|s| \geq |s_1|$. Thus $s_1 \subseteq s$. Then given any $t \supseteq s$ with $t \geq f$ on the domain of t, there is $a_1' \leq a_1$ forcing $t \geq \dot{f}_1$ on the domain of t. $(*)$

Now assume $e_0 = (a_0, (s_0, \dot{f}_0)) \in \mathbb{E}_0$ is below (s, f). Thus $s_0 \supseteq s$ and $s_0 \geq f$. By $(*)$ find $a_1' \leq a_1$ forcing $s_0 \geq \dot{f}_1$. By correctness, $a_0 \cdot a_1' \neq \mathbf{0}$ and clearly $a_0 \cdot a_1' \Vdash_{\mathbb{A}_{0\vee 1}}$ "$(s_0, \dot{f}_0)$ and $(s_1, \dot{f}_1)$ are compatible." Thus $(s, f) \leq h_{0\vee 1,0}(e_1)$ as required. $\square$

Lemma 1.4 [Br4] *Assume $\langle \mathbb{A}_i\rangle$ is correct. Then so is $\langle \mathbb{E}_i\rangle$ where $\mathbb{E}_i = \mathbb{A}_i \star \dot{\mathbb{B}}$ or $\mathbb{E}_i = \mathbb{A}_i \star \dot{\mathbb{B}}_\kappa$. Here $\mathbb{B}$ is random forcing, and $\mathbb{B}_\kappa$ is the algebra for adding κ random reals.*

We actually believe this is true for a large class of forcing notions. Recall that a forcing notion $(\mathbb{P}, \leq)$ is said to be *Suslin ccc* if $\mathbb{P}$ is ccc and $\mathbb{P}$, $\leq$ as well as $\perp$ (incompatibility) are $\mathbf{\Sigma}^1_1$ sets. This implies that "$A = \{x_n;\ n \in \omega\}$ is a maximal antichain" is a $\mathbf{\Pi}^1_1$-statement as in Example 2 above. (Indeed, being a maximal antichain means that (1) $\forall n \neq m$ $(x_n \perp x_m)$ and (2) $\forall y$ $(y \in \mathbb{P}$ implies there is n such that x_n and y are compatible). (1) is $\mathbf{\Pi}^1_1$ because both $y \in \mathbb{P}$ and $y \leq z$ are $\mathbf{\Sigma}^1_1$; and (2) is $\mathbf{\Pi}^1_1$ because compatibility is $\mathbf{\Pi}^1_1$.)

Conjecture 1.5 *Let $\mathbb{P}$ be a Suslin ccc forcing and assume $\langle \mathbb{A}_i \rangle$ is correct. Then so is $\langle \mathbb{E}_i \rangle$ where $\mathbb{E}_i = \mathbb{A}_i \star \mathrm{r.o.}(\dot{\mathbb{P}})$.*

Call a Suslin ccc forcing notion *correctness-preserving* if the conclusion of the conjecture holds.

While it is obviously irrelevant for linear iterations, correctness seems to be a crucial notion when it comes to building nonlinear iterations.

2 Iterations along templates

Let $(L, \leq_L)$ be a linear order. For $x \in L$, let $L_x = \{y \in L;\ y <_L x\}$, the *initial segment* determined by x.

Definition 2.1 (Template) *A* template *is a pair $(L, \bar{\mathcal{I}} = \{\mathcal{I}_x;\ x \in L\})$ such that $(L, \leq_L)$ is a linear order, $\mathcal{I}_x \subseteq \mathcal{P}(L_x)$ for all $x \in L$, and*

(1) $[L_x]^{<\omega} \subseteq \mathcal{I}_x$,

(2) *$\mathcal{I}_x$ is closed under unions and intersections,*

(3) *$\mathcal{I}_x \subseteq \mathcal{I}_y$ for $x <_L y$,*

(4) *$\mathcal{I} := \bigcup_{x \in L} \mathcal{I}_x \cup \{L\}$ is wellfounded, as witnessed by* $\mathrm{Dp} = \mathrm{Dp}_{\mathcal{I}} : \mathcal{I} \to \mathrm{On}$.

If $A \subseteq L$ and $x \in A$, we define $\mathcal{I}_x \restriction A = \{B \cap A;\ B \in \mathcal{I}_x\}$, the trace *of $\mathcal{I}_x$ on A. Then $(A, \bar{\mathcal{I}} \restriction A = \{\mathcal{I}_x \restriction A;\ x \in A\})$ is a template as well.*

L is meant to be the index set of the iteration while $\mathcal{I}$ describes in some sense the support. (1) says that for $x < y$, the real f_y added at stage y will be generic over the real f_x added at stage x. (3) is a transitivity condition which can be construed as meaning that for $x < y$, if f_x is generic over some initial stage of the iteration, then so is f_y. Wellfoundedness (4) allows for a recursive definition of the iteration.

(2) and (3) imply that $\mathcal{I}$ is also closed under unions and intersections.

Definition and Theorem 2.2 (Iteration of correctness-preserving Suslin ccc forcing along a template) *Assume $(L, \bar{\mathcal{I}})$ is a template. Also assume $\{\mathbb{Q}_x;\ x \in L\}$ are correctness-preserving Suslin ccc forcing notions (whose definition lies in the ground model). By recursion on* $\mathrm{Dp}(A)$, $A \in \mathcal{I}$,

(a) *we define the partial order $\mathbb{P}{\restriction}A$,*

(b) *we prove that $\mathbb{P}{\restriction}D \subseteq \mathbb{P}{\restriction}A$ for $D \subseteq A$ (D is not necessarily in $\mathcal{I}$),*

(c) *we prove that $\mathbb{P}{\restriction}A$ is transitive,*

(d) *we describe how $\mathbb{P}{\restriction}A$ is obtained from $\mathbb{P}{\restriction}B$ where $B \subsetneq A$, $B \in \mathcal{I}$ (so* $\mathrm{Dp}(B) < \mathrm{Dp}(A)$*),*

(e) *we prove that $\mathbb{P}{\restriction}D <\circ \mathbb{P}{\restriction}A$ for $D \subseteq A$ (D is not necessarily in $\mathcal{I}$),*

(f) *we prove that for $D \subseteq L$ with* $\mathrm{Dp}_{\mathcal{I}{\restriction}D}(D) \leq \mathrm{Dp}(A)$, *we have $\mathbb{P}{\restriction}(A \cap D) = \mathbb{P}{\restriction}A \cap \mathbb{P}{\restriction}D$,*

(g) *we prove correctness in the sense that if $A', D \subsetneq A$, $A' \in \mathcal{I}$ and $D' = A' \cap D$, then the projection maps are correct both at A' and at D (note that D and D' do not necessarily belong to $\mathcal{I}$).*

This is, in fact, a simultaneous recursion-induction for all templates. Note that for $D \subseteq A$, we have $\mathrm{Dp}_{\mathcal{I}{\restriction}D}(D) \leq \mathrm{Dp}_{\mathcal{I}{\restriction}A}(A) = \mathrm{Dp}_{\mathcal{I}}(A)$.

Definition and Proof. (a) First consider the case $\mathrm{Dp}(A) = 0$. This is equivalent to $A = \emptyset$. Let $\mathbb{P}{\restriction}\emptyset = \{\emptyset\}$.

Now assume $\mathrm{Dp}(A) > 0$. $\mathbb{P}{\restriction}A$ consists of all finite partial functions p with domain contained in A and such that, letting $x = \max(\mathrm{dom}(p))$, there is $B \in \mathcal{I}_x{\restriction}A$ (so $B \in \mathcal{I}$ and $\mathrm{Dp}(B) < \mathrm{Dp}(A)$; in fact, $B = C \cap A$ for some $C \in \mathcal{I}_x$) such that $p{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}B$ and $p(x)$ is a $\mathbb{P}{\restriction}B$-name for a condition in $\dot{\mathbb{Q}}_x$ (where $\dot{\mathbb{Q}}_x$ is a $\mathbb{P}{\restriction}B$-name as well).

The ordering on $\mathbb{P}{\restriction}A$ is given by: $q \leq_{\mathbb{P}{\restriction}A} p$ if $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and, letting $x = \max(\mathrm{dom}(q))$, there is $B \in \mathcal{I}_x{\restriction}A$ such that $q{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}B$ and

- either $x \notin \mathrm{dom}(p)$, $p \in \mathbb{P}{\restriction}B$, and $q{\restriction}(A \cap L_x) \leq_{\mathbb{P}{\restriction}B} p$,
- or $x \in \mathrm{dom}(p)$, $p{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}B$, $q{\restriction}(A \cap L_x) \leq_{\mathbb{P}{\restriction}B} p{\restriction}(A \cap L_x)$, and $p(x)$ and $q(x)$ are $\mathbb{P}{\restriction}B$-names for conditions in $\dot{\mathbb{Q}}_x$, such that $q{\restriction}(A \cap L_x) \Vdash_{\mathbb{P}{\restriction}B} q(x) \leq_{\dot{\mathbb{Q}}_x} p(x)$.

Concerning the first alternative in the definition of the ordering note that it is straightforward that given $p \in \mathbb{P}{\restriction}A$ and $x >_L \max(\mathrm{dom}(p))$, there is $B \in \mathcal{I}_x{\restriction}A$ such that $p \in \mathbb{P}{\restriction}B$.

(b) Let $D \subseteq A$ and $p \in \mathbb{P}{\restriction}D$. Also let $x = \max(\mathrm{dom}(p))$. There is $E \in \mathcal{I}_x{\restriction}D$ such that $p{\restriction}(D \cap L_x) \in \mathbb{P}{\restriction}E$ and $p(x)$ is a $\mathbb{P}{\restriction}E$-name for a condition in $\dot{\mathbb{Q}}_x$. Let $C \in \mathcal{I}_x$ such that $E = C \cap D$. Put $B = C \cap A$. Then $E \subseteq B$ and, by induction hypothesis (b), $p{\restriction}(D \cap L_x) = p{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}B$. By induction hypothesis (e), $p(x)$ is a $\mathbb{P}{\restriction}B$-name. Thus $p \in \mathbb{P}{\restriction}A$.

(c) We use completeness (the induction hypothesis for (e)) and closure under unions.

Assume $r \leq_{\mathbb{P}{\restriction}A} q \leq_{\mathbb{P}{\restriction}A} p$. Let y and x be the maximal elements of $\mathrm{dom}(r)$ and $\mathrm{dom}(q)$, respectively. There are $B_y \in \mathcal{I}_y{\restriction}A$ and $B_x \in \mathcal{I}_x{\restriction}A$ witnessing the order relationship, i.e. $r{\restriction}(A \cap L_y), q{\restriction}(A \cap L_y) \in \mathbb{P}{\restriction}B_y$, $q{\restriction}(A \cap L_x), p{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}B_x$, and $r{\restriction}(A \cap L_y) \leq_{\mathbb{P}{\restriction}B_y} q{\restriction}(A \cap L_y)$, $q{\restriction}(A \cap L_x) \leq_{\mathbb{P}{\restriction}B_x} p{\restriction}(A \cap L_x)$. Let $B = B_y \cup B_x$. We check B witnesses $r \leq_{\mathbb{P}{\restriction}A} p$.

If $x < y$, (b) gives us $q \in \mathbb{P}{\restriction}B$ and $p{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}B$. Thus $p \in \mathbb{P}{\restriction}B$, and $r{\restriction}(A \cap L_y) \leq_{\mathbb{P}{\restriction}B} q \leq_{\mathbb{P}{\restriction}B} p$. So $r \leq_{\mathbb{P}{\restriction}A} p$.

If $x = y$, $r{\restriction}(A \cap L_y) \leq_{\mathbb{P}{\restriction}B} q{\restriction}(A \cap L_y) \leq_{\mathbb{P}{\restriction}B} p{\restriction}(A \cap L_y)$, so we are done if $x \notin \mathrm{dom}(p)$. So assume $x \in \mathrm{dom}(p)$. Then $r(y)$ and $q(y)$ are $\mathbb{P}{\restriction}B_y$-names and $q(x)$ and $p(x)$ are $\mathbb{P}{\restriction}B_x$-names. So they are all $\mathbb{P}{\restriction}B$-names, and $r{\restriction}(A \cap L_y) \Vdash_{\mathbb{P}{\restriction}B} r(y) \leq q(y) \leq p(y)$, as required. (Note this uses the inductive hypothesis for complete embeddability (e).)

(d) We consider several cases.

Case 1. There is $x = \max(A)$ such that $A_0 = A \cap L_x = A \setminus \{x\} \in \mathcal{I}_x{\restriction}A$. Then $\mathbb{P}{\restriction}A$ is easily seen to be the standard two-step iteration $\mathbb{P}{\restriction}A_0 \star \dot{\mathbb{Q}}_x$ where $\dot{\mathbb{Q}}_x$ is a $\mathbb{P}{\restriction}A_0$-name (indeed, if $p \in \mathbb{P}{\restriction}A$, then $p{\restriction}A_0 \in \mathbb{P}{\restriction}A_0$ and $p(x)$ is a $\mathbb{P}{\restriction}A_0$-name for a condition in $\dot{\mathbb{Q}}_x$). A fortiori, $\mathbb{P}{\restriction}A_0 <\circ\, \mathbb{P}{\restriction}A$.

Case 2. There are two subcases.

Subcase 2.(a) There is $x = \max(A)$, but $A_0 = A \cap L_x \notin \mathcal{I}_x{\restriction}A$. Let $p \in \mathbb{P}{\restriction}A$. So $p{\restriction}A_0 \in \mathbb{P}{\restriction}A_0$. There is $B_0 \in \mathcal{I}_x{\restriction}A$ (so $B_0 \subsetneq A_0$) such that $p{\restriction}A_0 \in \mathbb{P}{\restriction}B_0$ and $p(x)$ is a $\mathbb{P}{\restriction}B_0$-name. Let $B = B_0 \cup \{x\} \subsetneq A$. Then $B_0 = B \cap L_x \in \mathcal{I}_x{\restriction}A$ and thus $p \in \mathbb{P}{\restriction}B$. Since $\mathcal{I}_x{\restriction}A$ is closed under unions, the collection of $B \in \mathcal{I}$ with $B \cap L_x \in \mathcal{I}_x{\restriction}A$ is directed. Therefore $\mathbb{P}{\restriction}A$ is the direct limit of the $\mathbb{P}{\restriction}B$ where $B \subsetneq A$, $B \cap L_x \in \mathcal{I}_x{\restriction}A$. (This uses the induction hypothesis for (e).)

Subcase 2.(b) A has no maximum. Let $p \in \mathbb{P}\restriction A$ be a condition. Let $y = \max(\mathrm{dom}(p))$. Then there is $B_0 \in \mathcal{I}_y\restriction A$ such that $p\restriction(A \cap L_y) \in \mathbb{P}\restriction B_0$ and $p(y)$ is a $\mathbb{P}\restriction B_0$-name for a condition in $\dot{\mathbb{Q}}_y$ (where $\dot{\mathbb{Q}}_y$ is a $\mathbb{P}\restriction B_0$-name as well). Now $B = B_0 \cup \{y\} \in \mathcal{I}_x\restriction A$ for some $x > y$ from A. In particular $\mathrm{Dp}(B) < \mathrm{Dp}(A)$ and $p \in \mathbb{P}\restriction B$. This shows that $\mathbb{P}\restriction A$ is the direct limit of the $\mathbb{P}\restriction B$ where $B \subsetneq A$, $B \in \mathcal{I}_x\restriction A$ for some $x \in A$. (By closure under unions and because A has no maximum, the collection of such B's is directed, and we can apply again the induction hypothesis for (e).)

(e) Let $D \subseteq A$. We need to prove $\mathbb{P}\restriction D <\circ \mathbb{P}\restriction A$. We split into cases according to (d).

Case 1. Let $D_0 = D \cap A_0 \in \mathcal{I}_x\restriction D$. By induction hypothesis (e), $\mathbb{P}\restriction D_0 <\circ \mathbb{P}\restriction A_0$. Also $\mathbb{P}\restriction A \cong \mathbb{P}\restriction A_0 \star \dot{\mathbb{Q}}_x$. If $x \notin D$, then $D = D_0$, and $\mathbb{P}\restriction D <\circ \mathbb{P}\restriction A_0 <\circ \mathbb{P}\restriction A$ follows. If $x \in D$, then $\mathbb{P}\restriction D \cong \mathbb{P}\restriction D_0 \star \dot{\mathbb{Q}}_x$ (where $\dot{\mathbb{Q}}_x$ is a $\mathbb{P}\restriction D_0$-name). Since $\mathbb{Q}_x$ is Suslin ccc, $\mathbb{P}\restriction D <\circ \mathbb{P}\restriction A$ is immediate (compare Example 2 in Section 1).

Subcase 2.(a) Assume first $D_0 = D \cap L_x \in \mathcal{I}_x\restriction D$. So there is $B_0 \in \mathcal{I}_x\restriction A$ such that $D_0 = D \cap B_0$. Put $B = B_0 \cup \{x\} \subsetneq A$. Then $D \subseteq B$ and $\mathbb{P}\restriction D <\circ \mathbb{P}\restriction B <\circ \mathbb{P}\restriction A$ by induction hypothesis (e) and subcase 2.(a) of part (d) above.

So assume $D_0 \notin \mathcal{I}_x\restriction D$. Suppose first that $x \in D$. Then (by subcase 2.(a) of part (d) applied to D instead of A) $\mathbb{P}\restriction D$ is the direct limit of the $\mathbb{P}\restriction E$ where $E \subsetneq D$, $E \cap L_x \in \mathcal{I}_x\restriction D$. Each such E is of the form $D \cap B$ where $B \cap L_x \in \mathcal{I}_x\restriction A$. Conversely, any $D \cap B$ is such an E. Using the inductive hypothesis for correctness (g), we see that all projections of such $\mathbb{P}\restriction B$ to $\mathbb{P}\restriction(D \cap B)$ are correct. By Lemma 1.2, this means, however, that the direct limit of the $\mathbb{P}\restriction E$ completely embeds into the direct limit of the $\mathbb{P}\restriction B$, as required.

Suppose finally that $x \notin D$. Then $D = D_0$ and, since $D_0 \notin \mathcal{I}_x\restriction D$, we must be in case 2 for D and, depending on whether D has a maximum or not, we are either in subcase 2.(a) or subcase 2.(b). In the first case, if $y = \max(D)$, then $\mathbb{P}\restriction D$ is the direct limit of $\mathbb{P}\restriction E$ where $E \subsetneq D, E \cap L_y \in \mathcal{I}_y\restriction D$. In the second case, $\mathbb{P}\restriction D$ is the direct limit of $\mathbb{P}\restriction E$ where $E \subsetneq D, E \in \mathcal{I}_y\restriction D$ for some $y \in D$. In either case, such E belongs to $\mathcal{I}_x\restriction D$ (though not all $E \in \mathcal{I}_x\restriction D$ are necessarily of this form). Since $\mathbb{P}\restriction E \subseteq \mathbb{P}\restriction D$ by (b) and the collection of $E \in \mathcal{I}_x\restriction D$ is directed, $\mathbb{P}\restriction D$ must in fact be the direct limit of the $\mathbb{P}\restriction E$ where $E \in \mathcal{I}_x\restriction D$. By Lemma 1.2, we see that $\mathbb{P}\restriction D <\circ \mathbb{P}\restriction A$.

Subcase 2.(b) If $D \in \mathcal{I}_x \restriction D$ for some $x \in A$, we are done because then $D \subseteq B$ for some $B \in \mathcal{I}_x \restriction A$, and $\mathbb{P}\restriction D <\circ \mathbb{P}\restriction B <\circ \mathbb{P}\restriction A$ by induction hypothesis (e) and subcase 2.(b) of part (d) above.

So assume $D \notin \mathcal{I}_x \restriction D$ for any $x \in A$. Again, we must be in case 2 for D and, as in the last paragraph of subcase 2.(a) above, we see that $\mathbb{P}\restriction D$ is the direct limit of the $\mathbb{P}\restriction E$ where $E \in \mathcal{I}_x \restriction D$ for some $x \in A$. Using again Lemma 1.2, we conclude that $\mathbb{P}\restriction D <\circ \mathbb{P}\restriction A$.

(f) $\mathbb{P}\restriction(A \cap D) \subseteq \mathbb{P}\restriction A \cap \mathbb{P}\restriction D$ is immediate from part (b). So assume $p \in \mathbb{P}\restriction A \cap \mathbb{P}\restriction D$. Let $x = \max(\mathrm{dom}(p))$. There are $B \in \mathcal{I}_x \restriction A$ and $E \in \mathcal{I}_x \restriction D$ such that $p\restriction(A \cap L_x) = p\restriction(D \cap L_x) = p\restriction L_x \in \mathbb{P}\restriction B \cap \mathbb{P}\restriction E$ and $p(x)$ is both a $\mathbb{P}\restriction B$-name and a $\mathbb{P}\restriction E$-name and thus a $\mathbb{P}\restriction B \cap \mathbb{P}\restriction E$-name. Since $\mathrm{Dp}(B) < \mathrm{Dp}(A)$ and $\mathrm{Dp}_{\mathcal{I}\restriction E}(E) = \mathrm{Dp}_{\mathcal{I}\restriction D}(E) < \mathrm{Dp}_{\mathcal{I}\restriction D}(D) \leq \mathrm{Dp}(A)$, we may apply the induction hypothesis (f) and get that $\mathbb{P}\restriction(B \cap E) = \mathbb{P}\restriction B \cap \mathbb{P}\restriction E$. Note that $B \cap E \in \mathcal{I}_x \restriction(A \cap D)$. Therefore $p \in \mathbb{P}\restriction(A \cap D)$, as required.

(g) Again split into cases according to (d).

Case 1. $x = \max(A)$, $A_0 = A \cap L_x \in \mathcal{I}_x \restriction A$, and $\mathbb{P}\restriction A = \mathbb{P}\restriction A_0 \star \dot{\mathbb{Q}}_x$. If $x \notin A'$, we get $\mathbb{P}\restriction A' <\circ \mathbb{P}\restriction A_0$, and correctness follows from induction hypothesis (g). Similarly if $x \notin D$. So we may assume $x \in A' \cap D = D'$. Then use induction hypothesis (g) and the assumption that the $\mathbb{Q}_x$ are correctness-preserving. (This is the only place where this assumption is needed.)

Subcase 2.(a) $x = \max(A)$, $A_0 = A \cap L_x \notin \mathcal{I}_x \restriction A$, and $\mathbb{P}\restriction A$ is the direct limit of $\mathbb{P}\restriction B$ where $B \subsetneq A$ and $B \cap L_x \in \mathcal{I}_x \restriction A$. Let $p \in \mathbb{P}\restriction A' \subseteq \mathbb{P}\restriction A$. We need to show that the projections agree: $h_{A'D'}(p) = h_{AD}(p)$.

First assume $D_0 = D \cap L_x \in \mathcal{I}_x \restriction D$. Then, by the discussion in (e) (subcase 2.(a)), $D \subseteq B$ for a B as above and, enlarging B if necessary, we may assume $p \in \mathbb{P}\restriction B$. Let $B' = A' \cap B$. By (f), we know that $p \in \mathbb{P}\restriction B'$. By $\mathrm{Dp}(B) < \mathrm{Dp}(A)$ and induction hypothesis (g), $h_{A'D'}(p) = h_{B'D'}(p) = h_{BD}(p) = h_{AD}(p)$, as required.

So assume $D_0 \notin \mathcal{I}_x \restriction D$. By the discussion in (e) (subcase 2.(a)), we know $\mathbb{P}\restriction D$ is the direct limit of $\mathbb{P}\restriction(D \cap B)$ where B is as above. Again fix such B such that $p \in \mathbb{P}\restriction B$, and let $B' = A' \cap B$. So $p \in \mathbb{P}\restriction B'$. Using $\mathrm{Dp}(A'), \mathrm{Dp}(B) < \mathrm{Dp}(A)$, the induction hypothesis (g) yields $h_{A'D'}(p) = h_{B',D' \cap B}(p) = h_{B,D \cap B}(p)$. Again by induction hypothesis (g), we have that $h_{B_0,D \cap B_0}(p) = h_{B,D \cap B}(p)$ for any $B \subseteq B_0 \subsetneq A$ with $B_0 \cap L_x \in \mathcal{I}_x \restriction A$. Since $\mathbb{P}\restriction A$ and $\mathbb{P}\restriction D$ are the direct limits of such $\mathbb{P}\restriction B_0$ and $\mathbb{P}\restriction(D \cap B_0)$, respectively, $h_{AD}(p) = h_{B,D \cap B}(p)$ follows, and we're done.

Subcase 2.(b) Depending on whether $D \in \mathcal{I}_x \restriction A$ for some $x \in A$, we repeat the previous argument, referring to subcase 2.(b) of (e). $\square$

Say a forcing notion $(\mathbb{P}, \leq)$ is *Suslin σ-linked* if it is Suslin ccc and $\mathbb{P} = \bigcup_n P_n$, all P_n are linked (any two elements of P_n have a common extension), and "$x \in P_n$" is a $\mathbf{\Sigma}^1_1$-statement. (Note that this implies "P_n is linked" is a $\mathbf{\Pi}^1_1$-statement and thus absolute. Indeed, linkedness is equivalent to $\forall x, y$ ($x, y \in P_n$ implies x and y are compatible) and compatibility is $\mathbf{\Pi}^1_1$ because incompatibility is $\mathbf{\Sigma}^1_1$.)

Lemma 2.3 *Assume $(L, \bar{\mathcal{I}})$ is a template, and the $\mathbb{Q}_x$, $x \in L$, are correctness-preserving Suslin σ-linked partial orders, $\mathbb{Q}_x = \bigcup_n Q_{x,n}$. Then for any $A \in \mathcal{I}$, $\mathbb{P} \restriction A$ is a ccc p.o.*

Proof. We argue in three steps.

Step 1. By induction on $\mathrm{Dp}(A)$, we show that given $p \in \mathbb{P}\restriction A$, there is $q \leq_{\mathbb{P}\restriction A} p$ such that for all $x \in \mathrm{dom}(q)$ there are $B \in \mathcal{I}_x \restriction A$ and $n = n_{q,x}$ such that $q \restriction (A \cap L_x) \in \mathbb{P}\restriction B$ and $q\restriction (A \cap L_x) \Vdash_{\mathbb{P}\restriction B} q(x) \in \dot{Q}_{x,n}$.

Indeed, let $p \in \mathbb{P}\restriction A$. Also let $x = \max(\mathrm{dom}(p))$. There is $B \in \mathcal{I}_x \restriction A$ such that $p \restriction (A \cap L_x) \in \mathbb{P}\restriction B$ and $p(x)$ is a $\mathbb{P}\restriction B$-name for a condition in $\dot{\mathbb{Q}}_x$. Thus we may find $r \in \mathbb{P}\restriction B$, $r \leq_{\mathbb{P}\restriction B} p\restriction (A \cap L_x)$, and n such that $r \Vdash_{\mathbb{P}\restriction B} p(x) \in \dot{Q}_{x,n}$. There is $q_0 \in \mathbb{P}\restriction B$, $q_0 \leq_{\mathbb{P}\restriction B} r$ which satisfies the induction hypothesis. Let $q \in \mathbb{P}\restriction A$ be such that $\mathrm{dom}(q) = \mathrm{dom}(q_0) \cup \{x\}$, $q\restriction (A \cap L_x) = q_0$ and $q(x) = p(x)$. Then q is as required.

Step 2. Assume $p, q \in \mathbb{P}\restriction A$ are as above; i.e. $n_{p,x}$ ($n_{q,x}$, respectively) exists for all $x \in \mathrm{dom}(p)$ ($x \in \mathrm{dom}(q)$, resp.). Also suppose that $n_{p,x} = n_{q,x}$ for all $x \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. We claim that p and q are compatible.

This is proved by building a common extension r by recursion on $\mathrm{dom}(p) \cup \mathrm{dom}(q)$. For $x = \min(\mathrm{dom}(p) \cup \mathrm{dom}(q))$, $r_x \in \mathbb{P}\restriction (A \cap L_x)$ is the trivial condition. Assume $r_x \in \mathbb{P}\restriction (A \cap L_x)$ has been produced for $x \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$. Let y be the successor of x in $\mathrm{dom}(p) \cup \mathrm{dom}(q)$, or let $y = \infty$ if x is the maximum of $\mathrm{dom}(p) \cup \mathrm{dom}(q)$. If $x \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$, let $r_y \in \mathbb{P}\restriction (A \cap L_y)$ such that $\mathrm{dom}(r_y) = \mathrm{dom}(r_x) \cup \{x\}$, $r_y \restriction (A \cap L_x) = r_x$ and $r_y(x) = p(x)$. If $x \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$ define r_y analogously. If $x \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$, find $r_y \restriction (A \cap L_x) \leq r_x$ and $r_y(x)$ such that $r_y \restriction (A \cap L_x) \Vdash r_y(x) \leq p(x), q(x)$. This is possible because $n_{p,x} = n_{q,x}$. Set $r = r_\infty$.

It is immediate from the construction that $r \leq p, q$.

Step 3. Ccc-ness now follows by a straightforward delta system argument.

$\square$

Lemma 2.4 *Let $(L,\bar{\mathcal{I}})$ be a template. Also assume the $\mathbb{Q}_x$ are as in the previous lemma.*

(i) *If $p \in \mathbb{P}{\restriction}L$, then there is $C \subseteq L$ countable such that $p \in \mathbb{P}{\restriction}C$;*

(ii) *If $\dot{f}$ is a $\mathbb{P}{\restriction}L$–name for a real, then there is $C \subseteq L$ countable such that $\dot{f}$ is a $\mathbb{P}{\restriction}C$–name.*

Note that C does not necessarily belong to $\mathcal{I}$.

Proof. We make a simultaneous induction on $\mathrm{Dp}(A)$, $A \in \mathcal{I}$.

(i) Assume $p \in \mathbb{P}{\restriction}A$. Let $x = \max(\mathrm{dom}(p))$. There is $B \in \mathcal{I}_x{\restriction}A$ such that $p{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}B$ and $p(x)$ is a $\mathbb{P}{\restriction}B$-name. By induction hypothesis (i), there is $C_0 \subseteq B$ countable such that $p{\restriction}(A \cap L_x) \in \mathbb{P}{\restriction}C_0$. By induction hypothesis (ii), since $p(x)$ is a name for a real, there is $C_1 \subseteq B$ such that $p(x)$ is a $\mathbb{P}{\restriction}C_1$-name. Let $C = C_0 \cup C_1 \cup \{x\}$. Then C is countable and $p \in \mathbb{P}{\restriction}C$.

(ii) Assume $\dot{f}$ is a $\mathbb{P}{\restriction}A$-name. For $n \in \omega$, let $\{p_{n,i};\ i \in \omega\}$ be maximal antichains of conditions deciding $\dot{f}(n)$. By part (i), there are countable $C_{n,i} \subseteq A$ such that $p_{n,i} \in \mathbb{P}{\restriction}C_{n,i}$. Let $C = \bigcup_{n,i} C_{n,i}$. Then $\dot{f}$ is a $\mathbb{P}{\restriction}C$-name. □

This immediately entails that the whole iteration is the direct limit of its countable fragments.

Corollary 2.5 *Let $(L,\bar{\mathcal{I}})$ be a template. Also assume the $\mathbb{Q}_x$ are as in the previous lemma. Then $\mathbb{P}{\restriction}L$ is the direct limit of the $\mathbb{P}{\restriction}A$ where $A \subseteq L$ is countable.*

Classical Example. Let $L = \mu$. Also let $\mathcal{I} = \{L_\alpha;\ \alpha \leq \mu\} = \mu + 1$, the collection of initial segments. Put $\mathcal{I}_\alpha = \mathcal{I} \cap \mathcal{P}(L_\alpha) = \{L_\beta;\ \beta \leq \alpha\}$. Let $\mathbb{Q}_x$, $x \in L$, be Hechler forcing $\mathbb{D}$. This is a Suslin σ-linked (even σ-centered) forcing notion. $\mathbb{P}{\restriction}L$ is easily seen to be nothing else than the usual finite support iteration of $\mathbb{D}$ of length μ, $\mathbb{D}_\mu$. So we also see that $\mathbb{D}_\mu$ is the direct limit of the $\mathbb{D}{\restriction}A$ where $A \subseteq \mu$ is countable, a fact which, of course, has been known before.

Main Example. The (much more complicated) template used for the consistency proof of $\mathfrak{d} < \mathfrak{a}$. See [Br1] for details.

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