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# Finite support iteration of c.c.c forcing notions and Parametrized $\diamondsuit$-principles

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**概 要**

We present several models which satisfy some $\diamondsuit$-like principles by using the $\omega_2$-stage finite support iteration of Suslin forcing notions.

## 1 Introduction

In [10] Jensen showed $V = L$ implies Suslin's Hypothesis doesn't hold. To prove this he introduced the $\diamondsuit$-principle:

$\diamondsuit$ There exists a sequence $\langle A_\alpha \subset \alpha : \alpha < \omega_1 \rangle$ such that for all $X \subset \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary.

In [9] Hrušák gave a partial solution to a question of J. Roitman who asked whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$ and answered a question of Brendle who asked whether $\mathfrak{a} = \omega_1$ in any model obtained by adding a single Laver real. To prove those he introduced the $\diamondsuit$-like principle $\diamondsuit_{\mathfrak{d}}$:

$\diamondsuit_{\mathfrak{d}}$ There exists a sequence $\langle g_\alpha : \omega \leq \alpha < \omega_1 \rangle$ such that $g_\alpha$ is a function from $\alpha$ to $\omega$ and for every $f : \omega_1 \to \omega$ there is an $\alpha \geq \omega$ with $f \restriction \alpha \leq^* g_\alpha$.

In [16] Moore, Hrušák, and Džamonja provided a broad framework of "parametrized $\diamondsuit$-principles" and they presented the following methods to construct parametrized $\diamondsuit$-principles:

**Theorem 1.1.** *Let $\mathbb{C}(\omega_1)$ and $\mathbb{B}(\omega_1)$ be the Cohen and random algebras corresponding to the product space $2^{\omega_1}$ with its usual topological and measure theoretic structures. The orders $\mathbb{C}(\omega_1)$ and $\mathbb{B}(\omega_1)$ force $\diamondsuit(non(\mathcal{M}))$ and $\diamondsuit(non(\mathcal{N}))$ respectively.*

**Theorem 1.2.** *Suppose that $\langle \mathcal{Q}_\alpha : \alpha < \omega_2 \rangle$ is a sequence of Borel partial orders such that for each $\alpha < \omega_2$ $\mathcal{Q}_\alpha$ is equivalent to $\wp(2)^+ \times \mathcal{Q}_\alpha$ as a forcing notion and let $\mathcal{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathcal{P}_{\omega_2}$ is proper and $(A, B, E)$ is a Borel invariant then $\mathcal{P}_{\omega_2}$ forces $\langle A, B, E \rangle \leq \omega_1$ iff $\mathcal{P}_{\omega_2}$ forces $\diamondsuit(A, B, E)$.*

In [15] by using $\omega_1$-stage finite support iteration of c.c.c forcing notions, several models were presented which satisfy some parametrized $\diamondsuit$-principles while others fail. The purpose of this paper is to provide several models satisfying some parametrized $\diamondsuit$-principles by using $\omega_2$-stage finite support iteration of Suslin forcing notions.

## 2 Definition and properties of Parametrized Diamonds

In [20] Vojtáš introduced a framework to describe many cardinal invariants.

**Definition 2.1.** [20][16] The triple $(A, B, E)$ is an *invariant* if

(1) $|A|, |B| \leq |\mathbb{R}|$,

(2) $E \subset A \times B$,

(3) For each $a \in A$ there exists $b \in B$ such that $(a, b) \in E$ and

(4) For each $b \in B$ there exists $a \in A$ such that $(a, b) \notin E$.

We will write $aEb$ instead of $(a, b) \in E$. If $A$ and $B$ are Borel subsets of some Polish spaces and $E$ is a Borel subset of their product, we call the triple $(A, B, E)$ Borel invariant.

Borel invariants were introduced in [3]. In the present paper we are interested only in Borel invariants.

**Definition 2.2.** Suppose $(A, B, E)$ is an invariant. Then its *evaluation* is defined by

$$\langle A, B, E \rangle = \min\{|X| : X \subset B \text{ and } \forall a \in A \exists b \in X \ (aEb)\}.$$

If $A = B$, we will write $(A, E)$ and $\langle A, E \rangle$ instead of $(A, B, E)$ and $\langle A, B, E \rangle$.

**Example 2.3.** The following Borel invariants $(\mathcal{N}, \not\supset)$, $(\mathcal{N}, \subset)$, $(\mathbb{R}, \mathcal{M}, \in)$, $(\mathcal{M}, \mathbb{R}, \not\ni)$, $(\omega^\omega, <^*)$, $(\omega^\omega, \not>^*)$ and $([\omega]^\omega, \text{is split by})$ have the evaluations $\text{add}(\mathcal{N})$, $\text{cof}(\mathcal{N})$, $\text{cov}(\mathcal{M})$, $\text{non}(\mathcal{M})$, $\mathfrak{d}$, $\mathfrak{b}$ and $\mathfrak{s}$ respectively.

**Definition 2.4.** Suppose $A$ is a Borel subset in some Polish space. Then $F : 2^{<\omega_1} \to A$ is *Borel* if for every $\alpha < \omega_1$ $F \restriction 2^\alpha$ is a Borel function.

In [7] the principle "weak diamond principle" was introduced by Devlin and Shelah. This was the starting point for the parametrized diamond principles introduced by Moore, Hrušák and Džamonja [16].

**Definition 2.5.** [16](Parametrized diamond principle)
Suppose $(A, B, E)$ is a Borel invariant. Then $\diamondsuit(A, B, E)$ is the following statement:

$\diamondsuit(A, B, E)$ For all Borel $F : 2^{<\omega_1} \to A$ there exists $g : \omega_1 \to B$ such that for every $f : \omega_1 \to 2$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha) E g(\alpha)\}$ is stationary.

The witness $g$ for a given $F$ in this statement will be called $\diamondsuit(A, B, E)$-sequence for $F$.

$\diamondsuit(A, B, E)$ and $\diamondsuit$ are related as follows:

**Proposition 2.6.** [16] Let $(A, B, E)$ be a Borel invariant. $\diamondsuit$ implies $\diamondsuit(A, B, E)$.

$\diamondsuit(A, B, E)$ and $\langle A, B, E\rangle$ are related as follows:

**Proposition 2.7.** [16] Suppose $(A, B, E)$ is a Borel invariant and $\diamondsuit(A, B, E)$ holds. Then $\langle A, B, E\rangle \le \omega_1$ holds.

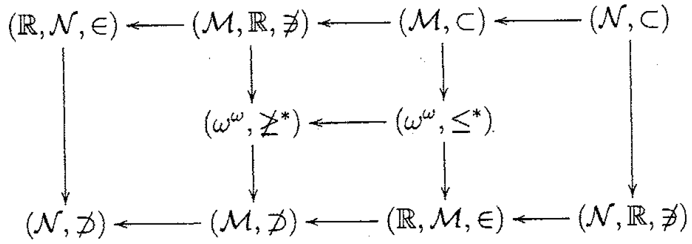
If two Borel invariants $(A_1, B_1, E_1)$,$(A_2, B_2, E_2)$ are comparable in the Borel Tukey order, then $\diamondsuit(A_1, B_1, E_1)$ and $\diamondsuit(A_2, B_2, E_2)$ are related as follows:

**Definition 2.8.** (Borel Tukey ordering [3]) Given a pair of Borel invariants $(A_1, B_1, E_1)$ and $(A_2, B_2, E_2)$, we say that $(A_1, B_1, E_1) \le_T^B (A_2, B_2, E_2)$ if there exist Borel maps $\phi : A_1 \to A_2$ and $\psi : B_2 \to B_1$ such that $(\phi(a), b) \in E_2$ implies $(a, \psi(b)) \in E_1$.

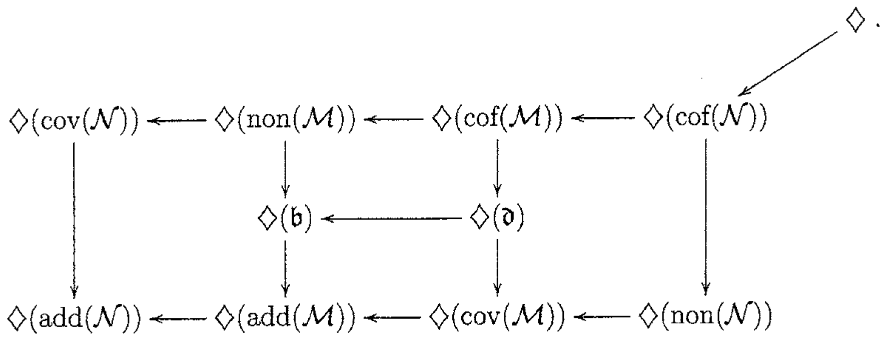
**Proposition 2.9.** [16] Let $(A_1, B_1, E_1)$ and $(A_2, B_2, E_2)$ be Borel invariants. Suppose $(A_1, B_1, E_1) \le_T^B (A_2, B_2, E_2)$ and $\diamondsuit(A_2, B_2, E_2)$ holds. Then $\diamondsuit(A_1, B_1, E_1)$ holds.

Concerning $\le_T^B$, we know the following diagram holds.

(Cichoń's diagram)



(The direction of the arrow is from larger to smaller in the Borel Tukey order). Hence the following holds:




(The direction of the arrow is the direction of the implication.)

We call this diagram "Cichoń's diagram for parametrized diamonds".

<u>Note</u> When we deal with Borel invariants in Cichoń's diagram, we will use the standard notation for their evaluations to denote the Borel invariants themselves (e.g., we will use $\diamondsuit(\mathrm{add}(\mathcal{N}))$ to denote $\diamondsuit(\mathcal{N}, \not\supset)$ ).

# 3 Construction of Parametrized Diamonds

By using $\omega_2$-stage finite support iteration of Suslin forcing notions we present several model which satisfies some parametrized $\diamondsuit$-principles.

## 3.1 Suslin forcing

Firstly we will introduce Suslin forcings and their properties.

**Definition 3.1.** *[2, p.168] A forcing notion $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ has a Suslin definition if $\mathbb{P} \subset \omega^\omega$, $\leq_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ and $\perp_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ are $\Sigma^1_1$.*

*$\mathbb{P}$ is Suslin if $\mathbb{P}$ is c.c.c and has a Suslin definition.*

**Definition 3.2.** *[2, p.168] Let $M \models ZFC^*$. A Suslin forcing $\mathbb{P}$ is in $M$ if all the parameters used in the definition of $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are in $M$.*

For convenience we will interpret Suslin forcing notion in forcing extensions using its code rather than taking the ground model forcing notion.

**Definition 3.3.** *Let $\mathbb{A}$ and $\mathbb{B}$ be forcing notions. Then $i : \mathbb{A} \to \mathbb{B}$ is a complete embedding if*

*(1)* $\forall a, a' \in \mathbb{A}(a \leq a' \to i(a) \leq i(a'))$,

*(2)* $\forall a_1, a_2 \in \mathbb{A}(a_1 \perp a_2 \leftrightarrow i(a_1) \perp i(a_2))$,

*(3)* $\forall \mathcal{A} \subset \mathbb{A}(\mathcal{A}$ *is a maximal antichain in*$\mathbb{A} \rightarrow i[\mathcal{A}]$ *is a maximal antichain in* $\mathbb{B})$.

*If there is complete embedding from* $\mathbb{A}$ *to* $\mathbb{B}$*, then we write* $\mathbb{A} \lessdot \mathbb{B}$.

Suslin forcing notion has the following good property:

**Lemma 3.4.** *Assume* $\mathbb{A} \lessdot \mathbb{B}$ *and* $\mathcal{P}$ *is a Suslin forcing notion. Then* $\mathbb{A} * \dot{\mathcal{P}} \lessdot \mathbb{B} * \dot{\mathcal{P}}$.

**Proof.** Let $i : \mathbb{A} \rightarrow \mathbb{B}$ be a complete embedding. Then define $\hat{i} : \mathbb{A} * \dot{\mathcal{P}} \rightarrow \mathbb{A} * \dot{\mathcal{P}}$ by $\hat{i}(\langle a, \dot{p} \rangle) = \langle i(a), i_*(\dot{p}) \rangle$ where $i_*$ is the class function from $\mathbb{A}$-names to $\mathbb{B}$-names induced by $i$ (see [12, p.222]). It is enough to show following claim.

**Claim 3.4.1.** *If* $\mathcal{A} \subset \mathbb{A} * \dot{\mathcal{P}}$ *is a maximal antichain, then* $\hat{i}[\mathcal{A}]$ *is also a maximal antichain in* $\mathbb{B} * \dot{\mathcal{P}}$.

**Proof of Claim.** Let $\mathcal{A} = \{(a_\alpha, \dot{p}_\alpha) : \alpha < \kappa\}$ be a maximal antichain of $\mathbb{A} * \dot{\mathcal{P}}$. Assume there exists $(b, \dot{p}) \in \mathbb{B} * \dot{\mathcal{P}}$ such that $(b, \dot{p})$ is compatible with all $\hat{i}((a_\alpha, \dot{p}_\alpha))$. Let $G$ be $\mathbb{B}$-generic over $V$ such that $b \in G$ and let $H = i^{-1}[G]$. Look at $\{\dot{p}_\alpha[H] : i(a_\alpha) \in G\} = \mathcal{A}' \in V[H]$.

**Subclaim 3.4.1.** $V[H] \models \mathcal{A}'$ *is maximal antichain of* $\mathcal{P} = \dot{\mathcal{P}}[H]$.

<u>antichain</u>: Suppose $\alpha \neq \beta$ and $i(a_\alpha), i(a_\beta) \in G$. Since $(a_\alpha, \dot{p}_\alpha) \perp (a_\beta, \dot{p}_\beta)$, $\dot{p}_\alpha[H] \perp \dot{p}_\beta[H]$.

<u>maximality</u>: Assume to the contrary, there exists $p \in \mathcal{P}$ such that $p \perp \dot{p}_\alpha[H]$ for any $\dot{p}_\alpha[H] \in \mathcal{A}'$. Then there exists $a \in H$ such that

$$a \Vdash \forall \alpha < \kappa (a_\alpha \in \dot{H} \rightarrow \dot{p} \perp \dot{p}_\alpha).$$

Hence $(a, \dot{p}) \perp (a_\alpha, \dot{p}_\alpha)$. This is a contradiction to the maximality of $\mathcal{A}$.

Subclaim ■

Since $V[H] \models$ "$\mathcal{A}'$ is maximal antichain in $\mathcal{P}$" and "$\mathcal{A}'$ is maximal antichain of $\mathcal{P}$" is a $\Pi^1_1(\mathcal{A}', \mathcal{P}, \leq_\mathcal{P}, \perp_\mathcal{P})$-formula, $V[G] \models$ "$\mathcal{A}' = \{i_*(p_\alpha)[G] : i(a_\alpha) \in G\}$ is maximal antichain of $\mathcal{P}$" by $\Pi^1_1$-absoluteness. But this is a contradiction to the fact $V[G] \models \dot{p}[G] \perp i_*(\dot{p}_\alpha)[G]$ for $i(a_\alpha) \in G$.

Claim ■

Hence $\mathbb{A} * \dot{\mathcal{P}} \lessdot \mathbb{B} * \dot{\mathcal{P}}$.

□

**Corollary 3.5.** *Let $\langle \mathcal{Q}_\alpha : \alpha < \kappa \rangle$ be a sequence of Suslin forcing notions. Let $\mathbb{P}_\kappa$ be the finite support iteration of $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle$ where $\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha = \mathcal{Q}_\alpha^{V^{\mathbb{P}_\alpha}}$. If $\mathbb{A} \lessdot \mathbb{B}$, then $\mathbb{A} * \dot{\mathbb{P}}_\kappa \lessdot \mathbb{B} * \dot{\mathbb{P}}_\kappa$.*

**Proof.** We shall show that if $\mathcal{A}$ is a maximal antichain of $\mathbb{A} * \dot{\mathbb{P}}_\kappa$, then $\hat{\imath}[\mathcal{A}]$ is also a maximal antichain of $\mathbb{B} * \dot{\mathbb{P}}_\kappa$ where $\hat{\imath} : \mathbb{A} * \dot{\mathbb{P}}_\kappa \to \mathbb{B} * \dot{\mathbb{P}}_\kappa$ is induced by the complete embedding $i : \mathbb{A} \to \mathbb{B}$. It is enough to prove the following claim.

**Claim 3.5.1.** *Let $\mathcal{A} \subset \mathbb{A} * \dot{\mathbb{P}}_\kappa$. If for each $p \in \mathbb{A} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $q \| p$, then for each $r \in \mathbb{B} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $\hat{\imath}(q) \| r$.*

**Proof of Claim.** We shall show this by induction on $\kappa$.
The successor Step is as in Lemma 3.4.
Limit step. Let $\kappa$ be a limit ordinal and for $\alpha < \kappa$ the induction hypothesis holds. Let $\mathcal{A} \subset \mathbb{A} * \dot{\mathbb{P}}_\kappa$ such that for each $p \in \mathbb{A} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $p \| q$. Assume to the contrary there exists $p \in \mathbb{B} * \dot{\mathbb{P}}_\kappa$ such that $p \perp \hat{\imath}(q)$ for any $q \in \mathcal{A}$. Let $\alpha = \sup\{\beta < \kappa : \Vdash_{\mathbb{P}_\beta} p(\beta) \neq 1\} < \kappa$. Since for each $r \in \mathbb{A} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $r \| q$, for each $r' \in \mathbb{A} * \dot{\mathbb{P}}_\alpha$ there exists $q \in \mathcal{A}$ such that $q \restriction \alpha \| r'$. By induction hypothesis there exists $q \in \mathcal{A}$ such that $p \restriction \alpha \| \hat{\imath}_\alpha(q \restriction \alpha)$ where $\hat{\imath}_\alpha : \mathbb{A} * \dot{\mathbb{P}}_\alpha \to \mathbb{B} * \dot{\mathbb{P}}_\alpha$ is induced by $i$. By $\hat{\imath}_\alpha(q \restriction \alpha) = \hat{\imath}(q) \restriction \alpha$, $p \restriction \alpha \| \hat{\imath}(q) \restriction \alpha$. So $p \| \hat{\imath}(q)$. It is a contradiction.

Claim ■

□

Let $\langle \mathcal{R}_\alpha : \alpha < \kappa \rangle$ be a sequence of Suslin forcing notions where all parameters appear in the ground model. Let $\mathbb{P}_\kappa$ be the finite support iteration of $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle$ where $\Vdash_{P_\alpha} \dot{Q}_\alpha = \mathcal{R}_\alpha^{V^{P_\alpha}}$. Let $I \subset \kappa$. Recursively define $\mathbb{P}_I^\alpha$ by

(i) $\mathbb{P}_I^\alpha$ is given. Then $\mathbb{P}_I^{\alpha+1} = \mathbb{P}_I^\alpha * \dot{Q}'_\alpha$ where

$$\Vdash_{\mathbb{P}_I^\alpha} \dot{Q}'_\alpha = \begin{cases} R_\alpha^{V^{\mathbb{P}_I^\alpha}} & \alpha \in I \\ \{1\} & \textit{otherwise.} \end{cases}$$

(ii) Suppose $\alpha$ is a limit ordinal and $\mathbb{P}_I^\beta$ is given for $\beta < \alpha$. Define $\mathbb{P}_I^\alpha$ as the finite support iteration of $\langle \mathbb{P}_I^\beta, \dot{Q}'_\beta : \beta < \alpha \rangle$

Put $\mathbb{P}_I := \mathbb{P}_I^\kappa$.

**Lemma 3.6.** $\mathbb{P}_I \lessdot \mathbb{P}_\kappa$.
**Proof.** We shall show for $\alpha \leq \kappa$ $\mathbb{P}_I^\alpha \lessdot \mathbb{P}_\alpha$ by the induction on $\alpha \leq \kappa$.
Successor step. Suppose $\mathbb{P}_I^\alpha \lessdot \mathbb{P}_\alpha$. If $\alpha \notin I$, it is clear that $\mathbb{P}_I^{\alpha+1} \lessdot \mathbb{P}_{\alpha+1}$. If $\alpha \in I$,

then $\mathbb{P}_I^{\alpha+1} \lessdot \mathbb{P}_{\alpha+1}$ is proved as in Lemma 3.4.
Limit step. Let $\alpha$ be a limit ordinal and for $\beta < \alpha$ the induction hypothesis holds. Define $i : \mathbb{P}_I^\alpha \lessdot \mathbb{P}_\alpha$ by $i(p) = i_\beta(p)$ if $p \in \mathbb{P}_I^\beta$ for some $\beta < \alpha$ where $i_\beta : \mathbb{P}_I^\beta \to \mathbb{P}_\beta$ is the complete embedding. It is enough to prove the following claim.

**Claim 3.6.1.** *Let $\mathcal{A} \subset \mathbb{P}_I^\alpha$. If for each $p \in \mathbb{P}_I^\alpha$ there exists $q \in \mathcal{A}$ such that $q \| p$, then for each $r \in \mathbb{P}_\alpha$ there exists $q \in \mathcal{A}$ such that $i(q) \| r$.*

**Proof of Claim.** Let $\mathcal{A} \subset \mathbb{P}_I^\alpha$ such that for each $p \in \mathbb{P}_I^\alpha$ there exists $q \in \mathcal{A}$ such that $q \| p$. Let $r \in \mathbb{P}_\alpha$. Since $\mathbb{P}_\alpha$ is the finite support iteration of $\langle \mathbb{P}_\beta, \dot{Q}_\beta : \beta < \alpha \rangle$, there is $\beta < \alpha$ such that $r \in \mathbb{P}_\beta$. Since for each $p \in \mathbb{P}_I^\alpha$ there exists $q \in \mathcal{A}$ such that $q \| p$, for each $p' \in \mathbb{P}_I^\beta$ there exists $q \in \mathcal{A}$ such that $q \restriction \beta \| p'$. By induciton hypothesis there exists $q \in \mathcal{A}$ such that $i_{beta}(q \restriction \beta) = i(q) \restriction \beta \| r$. So $i(q) \| r$. Hence for each $r \in \mathbb{P}_\alpha$ there exists $q \in \mathcal{A}$ such that $i(q) \| r$.

Claim ■

Lemma □

For $\mathbb{P}_\kappa$-name $\dot{x}$ for a real, there is following property.

**Lemma 3.7.** *Let $\mathbb{P}_\kappa$ is the $\kappa$-stage finite support iteration of Suslin forcing notions. If $\dot{x}$ is $\mathbb{P}_\kappa$-name for a real. Then there exists countable $I \subset \kappa$ such that $\dot{x}$ is $\mathbb{P}_I$-name.*

## 3.2 Niceness

In this paper we will force $\diamondsuit(A, B, E)$ for Borel invariants $(A, B, E)$ which satisfy the following properties:

There exist $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ such that

(0) $E_n$ is a Borel set for $n \in \omega$,

(1) $E = \bigcap_{n \in \omega} E_n$,

(2) $E_{n+1} \subset E_n$,

(3) $U^n : A \to \wp(A)$ such that $U^n(x)$ is a Borel set

(4) $xE_ny$ implies that there exists $m \geq n$ such that $U^m(x) \subset \{z \in A : zE_ny\}$.

(5) $U^m(x) \subset \{z \in A : zE_ny\}$ is absolute with parameters $x,\ y,\ U^m$ and $E_n$.

**Example**

(i) For $(2^\omega, 2^\omega, \exists^\infty n\ (* \restriction I_n = *' \restriction I_n)\ )$ let $xE_ny$ if $\exists m \geq n(x \restriction I_m = y \restriction I_m)$ and $U^n(x) = [x \restriction I_n] := \{y \in 2^\omega : y \restriction I_n = x \restriction I_n\}$. Then $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfy (0)-(5).

(ii) For $(\omega^\omega, \not\geq^*)$ let $xE_ny$ if $\exists m \geq n(x(m) < y(n))$ and $U^n(x) = \bigcup_{m \leq x(n)} [\langle n, m \rangle]$. Then $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfy (0)-(5).

(iii) Let $\mathbb{LOC} = \{\phi : \phi : \omega \to [\omega]^{<\omega}$ where $|\phi(n)| \leq (n+1)^2$ for $n \in \omega\}$. If $\phi \in \mathbb{LOC}$, we call $\phi$ slalom. Then for $f \in \omega^\omega$ and $\phi \in \mathbb{LOC}$ $\phi \sqsupset f$ if $\forall^\infty n\ (f(n) \in \phi(n))$. For $(\mathbb{LOC}, \omega^\omega, \not\sqsupset)$ let $\phi E_n f$ if $\exists m \geq n(f(m) \notin \phi(m))$ and $U^n(\phi) = \bigcup_{s \subset \phi(n)} [\langle n, s \rangle]$. Then $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfy (0)-(5).

For a Borel invariant $(A, B, E)$ with $\langle U^n : n \in \omega \rangle$ and $\langle E_n : n \in \omega \rangle$ which safisfies (0)-(5), we will define the notion $(A, B, E)$-nice and show that the $\omega_2$-stage finite support iteration of some Suslin forcing notions forces parametrized $\diamondsuit$-principles.

**Definition 3.8.** *Let $(A, B, E)$ be a Borel invariant with $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfying (0)-(5). Let $\mathbb{P}$ be a forcing notion and $\mathcal{Q}$ be a Suslin forcing notion or finite support iteration of Suslin forcing notions.*

*Then $\mathcal{Q}$ is $(A, B, E)$-nice for $\mathbb{P}$ if for all $\mathcal{Q}$-names $\dot{x}$ for an element of $A$ for each $(p, \dot{q}) \in \mathbb{P} * \dot{\mathcal{Q}}$ there exists $x \in A \cap V$ such that for all $r \leq_{\mathbb{P}} p$ for all but finitely many $n$ there exists $q\prime \in \mathcal{Q}$ such that $(1, q\prime) \| (r, \dot{q})$ and $q\prime \Vdash_{\mathcal{Q}} \dot{x} \in U^n(x)$.*

There are following examples of niceness.

**Proposition 3.9.** *Suppose $I$ is countable subset of some ordinal $\kappa$. Then*

*(1)* $\mathbb{D}_I$ *is* $(2^\omega, \mathcal{N}, \in)$*-nice for* $\mathbb{D}_{\omega_1}$

*(2)* $\mathbb{B}_{\mathbb{I}}$ *is* $(\omega^\omega, \not\geq^*)$*-nice for* $\mathbb{B}_{\omega_1}$.

*(3)* $\mathbb{E}_I$ *is* $(2^\omega, \mathcal{N}, \in)$*-nice for* $\mathbb{E}_\omega$ *and* $(\omega^\omega, \not\geq^*)$*-nice for* $\mathbb{E}_{\omega_1}$.

*(4)* $(\mathbb{B} * \dot{\mathbb{D}})_I$ *is* $(\mathbb{LOC}, \omega^\omega, \not\sqsupset)$*-nice for* $(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}$.

**Proof.**
We shall show only $|I| = 1$. The General case is similar but more complicated.

(1). Let $\langle I_n : n \in \omega\rangle$ be a partition of $\omega$ such that $I_0 = \{0\}$, $I_1 = \{1,2\}, \ldots$, $I_{n+1} = \{\max(I_n)+1, \ldots, \max(I_n)+n+1\}$. For $x \in 2^\omega$ let

$$A_x = \{y \in 2^\omega : \exists^\infty n \in \omega (x \restriction I_n = y \restriction I_n)\}.$$

Then $A_x$ is null. So If $\diamondsuit(2^\omega, 2^\omega, \exists^\infty n(* \restriction I_n = *' \restriction I_n))$ holds, then $\diamondsuit(\mathrm{cov}(\mathcal{N}))$ holds. So instead of showing that $\mathbb{D}$ is $(2^\omega, \mathcal{N}, \in)$-nice for $\mathbb{D}_{\omega_1}$we shall show $\mathbb{D}$ is $(2^\omega, 2^\omega, \exists^\infty n(* \restriction I_n = *' \restriction I_n))$-nice for $\mathbb{D}_{\omega_1}$.

Let $\dot{x}$ be a $\mathbb{D}$-name such that $\Vdash_{\mathbb{D}} \dot{x} \in 2^\omega$. Let $\langle p, \dot{q}\rangle \in \mathbb{D}_{\omega_1} * \dot{\mathbb{D}}$. For $s \in \omega^{<\omega}$ define $D_s \subset \mathbb{D}$ by $p \in D_s$ if there exists $f \in \omega^\omega$ such that $p = \langle s, f\rangle$. Then $\mathbb{D} = \bigcup_{s \in \omega^{<\omega}} D_s$. Without loss of generality we can assume $p \Vdash_{\mathbb{D}_{\omega_1}} \dot{q} = \langle \check{s}, \dot{f}\rangle$ for some $s \in \omega^{<\omega}$. Then define $x_s \in 2^\omega \cap V$ so that $\forall m \in \omega \forall p \in D_s \neg p \Vdash x_s \restriction I_m \neq \dot{x} \restriction I_m$. Let $r \leq p$ and $m \in \omega$. Define $\langle r_n : n \in \omega\rangle$, $f \in \omega^\omega \cap V$ so that

(i) $r_0 \leq r$, $r_{n+1} \leq r_n$ and

(ii) $r_n$ decides $\dot{f}(n)$ and $r_n \Vdash \dot{f}(n) = f(n)$.

Let $q' \leq \langle s, f\rangle$ such that $q' \Vdash_{\mathbb{D}} \dot{x} \in [x_s \restriction I_m] = U^m(x_s)$.

**Claim 3.9.1.** $\langle 1, q'\rangle \| \langle r, \langle s, \dot{f}\rangle\rangle$.

**Proof of Claim.** Let $q' = \langle t, g\rangle$. Then $r_{|t|} \Vdash \dot{f} \restriction |t| = f \restriction |t|$. So $\langle r_{|t|}, \langle t, \dot{f}\rangle\rangle \leq \langle r, \langle s, \dot{f}\rangle\rangle$. Hence $\langle 1, q'\rangle \| \langle r, \langle s, \dot{f}\rangle\rangle$.

Claim ■ (1)□

(2). Let $\dot{x}$ be a $\mathbb{B}$-name such that $\Vdash_{\mathbb{B}} \dot{x} \in \omega^\omega$. Define $x \in \omega^\omega \cap V$ so that $\mu([\dot{x}(n) \leq x(n)]) \geq 1 - \frac{1}{2^{n+1}}$. Let $(p, \dot{q}) \in \mathbb{B}_{\omega_1} * \dot{\mathbb{B}}$. Without loss of generality we can assume $p \Vdash_{\mathbb{B}_{\omega_1}} \mu(\dot{q}) \geq \frac{1}{2^n}$. Then for any $r \leq p$ and $m \geq n$ $(r, \dot{q}) \| (1, [\dot{x}(m) \leq x(m)])$ and $[\dot{x}(m) \leq x(m)] \Vdash_{\mathbb{B}} \dot{x} \in \bigcup_{i \leq x(m)} [\langle m, i\rangle] = U^m(x)$.

□

(3). $(2^\omega, \mathcal{N}, \in)$-niceness is shown as (1).
$(\omega^\omega, \not\geq^*)$-niceness: For $s \in \omega^{<\omega}$ and $k \in \omega$ let $E_{s,k} = \{p \in \mathbb{E} : p = \langle s, F\rangle$ and $|F| = k\}$. Then $\mathbb{E} = \bigcup_{s \in \omega^\omega, k \in \omega} E_{s,k}$. Let $\dot{x}$ be $\mathbb{E}$-name such that $\Vdash_{\mathbb{E}} \dot{x} \in \omega^\omega$. Let $\langle p, \dot{q}\rangle \in \mathbb{E}_{\omega_1} * \dot{\mathbb{E}}$. Without loss of generality we can assume $p \Vdash_{\mathbb{E}_{\omega_1}} \dot{q} \in E_{s,k}$. Then define $x_{s,k} \in \omega^\omega \cap V$ by

$$x_{s,k}(i) = \min\{j : \forall p \in E_{s,k} \neg(p \Vdash \dot{x} > j)\}.$$

For $j < k$ let $\dot{f}_j$ be a $\mathbb{E}_{\omega_1}$-name such that $p \Vdash_{\mathbb{E}_{\omega_1}}$ "$\dot{q} = \langle s, \dot{F}\rangle$ and $\dot{F} = \{\dot{f}_j : j < k\}$".

Let $r \leq p$ and $m \in \omega$. Then define $\langle r_n : n \in \omega\rangle$ and $\{f_i : i < k\} \in \omega^\omega \cap V$ so that

(i) $r_0 \leq r$ ,$r_{n+1} \leq r_n$ and

(ii) $r_m$ decides $\dot{f}_j \restriction m$ for $j < k$ and $r_m \Vdash_{\mathbb{E}_{\omega_1}} f_j \restriction m = \dot{f}_j \restriction m$ for $j < m$.

Let $F = \{f_j; j < k\}$ and $q' \leq \langle s, F \rangle$ such that $q' \Vdash_{\mathbb{E}} \dot{x}(m) < x_{s,k}(m)$. Then $q' \Vdash_{\mathbb{E}} \dot{x}(m) \in \bigcup_{i < x_{s,k}(m)} [\langle m, i \rangle] = U^m(x_{s,k})$.

**Claim 3.9.2.** $(r, \dot{q}) \| (1, q')$.

**Proof of Claim.** Let $q' = \langle t, G \rangle$. Since $r_{|t|} \Vdash_{\mathbb{E}_{\omega_1}} \dot{f}_j \restriction |t| = f_j \restriction |t|$ for $j < k$. $r_{|t|} \Vdash_{\mathbb{E}_{\omega_1}} q' \| \dot{q}$. So $(1, q') \| (r, \dot{q})$.

Claim■ (3) □

(4) By [11] we can assume $\mathbb{A} := (\mathbb{B} * \dot{\mathbb{D}})_I$ is Boolean Algebra with strictly positive finitely additive measure $\mu$. Let $\dot{\phi}$ is $\mathbb{A}$-name such that $\Vdash_{\mathbb{A}} \dot{\phi} \in \mathbb{LOC}$. For each $n \in \omega$ define $k_n \in \omega$ so that $\mu([k_n \in \dot{\phi}(n)]) < \frac{1}{n}$. Then define $\phi \in \mathbb{LOC} \cap V$ by $\phi(n) = \{k_n\}$. Let $\langle p, \dot{q} \rangle \in (\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}$. Without loss of generality we can assume $p \Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}} \mu(\dot{q}) > \frac{1}{k}$. Let $r \leq q$. Since $\mu([k_n \notin \dot{\phi}(n)]) \geq 1 - \frac{1}{k}$ for $n \geq k$, $r \Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}}$ "$\mu(\dot{q} \cap [k_n \notin \dot{\phi}]) \geq 0$" for $n \geq k$. Since $[\dot{\phi} \in U^n(\phi)] = [k_n \notin \dot{\phi}(n)]$, $r \Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}} \mu(\dot{q} \cap [\dot{\phi} \in U^n(\phi)]) > 0$. Hence $(r, \dot{q}) \| (1, [\dot{\phi} \in U^n(\phi)])$.

□

If $Q$ is $(A, B, E)$-nice for $\mathbb{P}$, then elements of $A \cap V^{\mathbb{Q}}$ have a following property.

**Theorem 3.10.** *[Minami] Let $(A, B, E)$ be a Borel invariant with $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ which satisfy (0)-(5). Let $\mathbb{P}$ be a forcing notion such that there exists $\mathbb{P}$-name $\dot{r}$ for an element of $B$ such that $\Vdash_{\mathbb{P}}$ "$xE\dot{r}$" for $x \in A \cap V$ and let $Q$ be a Suslin forcing notion or the finite support iteration of Suslin forcing notions. If $Q$ is $(A, B, E)$-nice for $\mathbb{P}$ and $\dot{x}$ is a $Q$-name for an element of $A \cap V^{\mathbb{P}}$, then $\Vdash_{\mathbb{P} * \dot{Q}} \dot{x} E \dot{r}$.*

**Proof.** Suppose $Q$ is $(A, B, E)$-nice for $\mathbb{P}$. Let $\dot{r}$ be a $\mathbb{P}$-name for an element of $B \cap V^{\mathbb{P}}$ such that $\Vdash xE\dot{r}$ for $x \in A \cap V$. Let $\dot{x}$ be a $Q$-name for an element of $A \cap V^{\mathbb{Q}}$. It suffices to show for each $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ there exists $(r, \dot{s}) \leq (p, \dot{q})$ such that

$$(r, \dot{s}) \Vdash \dot{x} E \dot{r}.$$

Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$. Since $\mathbb{P}$ is $(A, B, E)$-nice for $Q$, there exists $x \in A \cap V$ such that

$$\forall r \leq_{\mathbb{P}} p \forall^\infty n \exists q\prime \in Q(\ (1, q\prime) \| (r, \dot{q}) \text{ and } q\prime \Vdash_Q \dot{x} \in U^n(x)).$$

Let $r \leq p$ and $n \in \omega$ such that $r \Vdash_{\mathbb{P}}$ "$xE_n\dot{r}$" and if $m \geq n$, there exists $q' \in \mathcal{Q}(\ (1, q\prime) \| (r, \dot{q})$ and $q\prime \Vdash_{\mathcal{Q}} \dot{x} \in U^m(x))$. Since $r \Vdash xE_n\dot{r}$, there exists $m \geq n$ such that

$$r \Vdash U^m(x) \subset \{z \in A : zE_n\dot{r}\}.$$

Pick $q' \in \mathcal{Q}$ such that $(1, q') \| (r, \dot{q})$ and $q' \Vdash_{\mathcal{Q}} \dot{x} \in U^m(x)$. Let $(p', \dot{q}^*) \leq (1, q'), (r, \dot{q})$. Then

$$(p', \dot{q}^*) \Vdash \dot{x} \in U^m(x) \subset \{z \in A : zE_n\dot{r}\}.$$

Hence $(p', \dot{q}^*) \Vdash \dot{x}E_n\dot{r}$. Therefore $\Vdash \dot{x}E\dot{r}$.

□

**Theorem 3.11.** *Let $(A, B, E)$ be a Borel invariant with $\langle E_n, n \in \omega\rangle$ and $\langle U^n : n \in \omega\rangle$ satisfying (0)-(5). Let $\mathbb{P}_{\omega_2}$ be a $\omega_2$-stage finite support iteration of Suslin forcing notion and*

*(1) for all $\beta < \omega_2$ there exists a $\mathbb{P}_{\beta+\omega_1}$-name $\dot{r}$ for an element of $A$ such that $\Vdash_{\mathbb{P}_{\beta+\omega_1}}$ "$xE\dot{r}$" for $x \in A \cap V^{\mathbb{P}_\beta}$.*

*(2) for all $\beta < \omega_2$ for all $I$ countable subset of $\omega_2 \setminus (\beta + \omega_1)$ $V^{\mathbb{P}_\beta} \models$ "$\mathbb{P}_I$ is $(A, B, E) -$ nice for $\mathbb{P}_{[\beta,\beta+\omega_1)}$".*

*Then $\mathbb{P}_{\omega_2} \models \diamondsuit(A, B, E)$.*

**Proof.** Let $\dot{F}$ be a $\mathbb{P}_{\omega_2}$-name for a Borel function. Since $\mathbb{P}_{\omega_2}$ has c.c.c and $\mathbb{P}_{\omega_2}$ is the finite support iteration of $\langle \mathbb{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\rangle$ without loss of generality we can assume $F$ is in ground model. By (1) let $\dot{r}_\alpha$ be a $\mathbb{P}_{\omega_1}$-name such that $\Vdash_{\mathbb{P}_{\omega_1}} xE\dot{r}_\alpha$ for $x \in A \cap V^{\mathbb{P}_\alpha}$ for $\alpha < \omega_1$. We shall show $\Vdash_{\mathbb{P}_{\omega_2}}$ "$\langle r_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit(A, B, E)$-sequence for $F$".

**Claim 3.11.1.** *Let $\dot{f}$ be a $\mathbb{P}_{\omega_2}$-name such that $\Vdash_{\mathbb{P}_{\omega_2}} \dot{f} : \omega_1 \to 2$. Then*

$$\{\alpha \in \omega_1 : \dot{f} \restriction \alpha \text{ is } \mathbb{P}_I\text{-name where } I \cap \omega_1 \subset \alpha \text{ and } I \text{ is countable}\}$$

*contains a club.*

■

Let $\dot{x} = F(\dot{f} \restriction \alpha)$ such that $\dot{x}$ is a $\mathbb{P}_I$-name, $I$ is countable and $I \cap \omega_1 \subset \alpha$. In $V^{\mathbb{P}_\alpha}$ we can assume $\dot{r}_\alpha$ is $\mathbb{P}_{[\alpha,\omega_1)}$-name and $\dot{x}$ is $\mathbb{P}_{I\cap[\omega_1,\omega_2)}$-name. Hence to show $\Vdash_{\mathbb{P}_{\omega_2}}$ "$\langle r_\alpha : \alpha < \omega_1\rangle$ is $\diamondsuit(A, B, E)$-sequence for $F$", it suffices to show that $\Vdash_{\mathbb{P}_{\omega_1} * \mathbb{P}_I}$ "$\dot{x}E\dot{r}_\alpha$" where $\dot{x}$ is $\mathbb{P}_I$-name for an element of $A \cap V^{\mathbb{P}_I}$.

By (2) $\mathbb{P}_I$ is $(A, B, E)$-nice for $\mathbb{P}_{\omega_1}$. By Theorem 3.10 $\Vdash \dot{x}E\dot{r}_\alpha$. Hence $\langle \dot{r}_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit(A, B, E)$-sequence for $F$.

□

**Remark 3.11.2.** *Same argument holds for $\mathbb{P}_\kappa$ if $cf(\kappa) \geq \omega_2$.*

**Corollary 3.12.** *Each of the following are relatively consistent with ZFC:*

*(i)* $\mathfrak{c} = add(\mathcal{M}) = \omega_2 + \diamondsuit(cov(\mathcal{N}))$*(see Diagram 1).*

*(ii)* $\mathfrak{c} = cov(\mathcal{N}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b})$*(see Diagram 2).*

*(iii)* $\mathfrak{c} = non(\mathcal{M}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b}) + \diamondsuit(cov(\mathcal{N}))$*(see Diagram 3).*

*(iv)* $\mathfrak{c} = cov(\mathcal{N}) = add(\mathcal{M}) = \omega_2 + \diamondsuit(add(\mathcal{N}))$*(see Diagram 4).*

**Proof.** (i) Suppose $V \models$ CH. By Theorem 3.11 and Proposition 3.9 (1) $V^{\mathbb{D}_{\omega_2}} \models \diamondsuit(\text{cov}(\mathcal{N}))$. Since $\mathbb{D}_{\omega_2}$ adds $\omega_2$-many dominating reals and Cohen reals, $V^{\mathbb{D}_{\omega_2}} \models \mathfrak{c} = \mathfrak{b} = \text{cov}(\mathcal{M}) = \omega_2$. Since $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\}$ (see [19], [14]),

$$V^{\mathbb{D}_{\omega_2}} \models \diamondsuit(\text{cov}(\mathcal{N})) + \mathfrak{c} = \text{add}(\mathcal{M}) = \omega_2.$$

Cichoń's's diagram for parametrized diamond looks as follows where a $\omega_2$ means the corresponding evaluation of Borel invariant is $\omega_2$ while parametrized diamonds principle for the others hold.

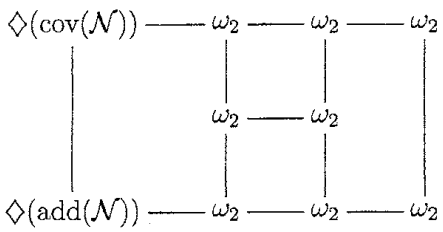



Diagram 1.

(ii) Suppose $V \models$ CH. By Theorem 3.11 and Proposition 3.9 (2) $V^{\mathbb{B}_{\omega_2}} \models \diamondsuit(\mathfrak{b})$. Since $\mathbb{B}_{\omega_2}$ adds $\omega_2$ many Cohen and random reals, $V^{\mathbb{B}_{\omega_2}} \models \mathfrak{c} = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{B}_{\omega_2}} \models \diamondsuit(\mathfrak{b}) + \mathfrak{c} = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2.$$

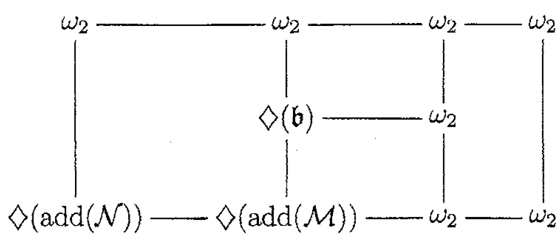



Diagram 2.

(iii) Suppose $V \models$ CH. By Theorem 3.11 and Proposition 3.9 (3) $V^{\mathbb{E}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \diamondsuit(\mathfrak{b})$. Since $\mathbb{E}_{\omega_2}$ adds $\omega_2$ many Cohen and almost different reals, $\mathfrak{c} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{E}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \diamondsuit(\mathrm{cov}(\mathcal{M})) + \mathfrak{c} = \mathrm{non}(\mathcal{M}) + \mathrm{cov}(\mathcal{M}).$$

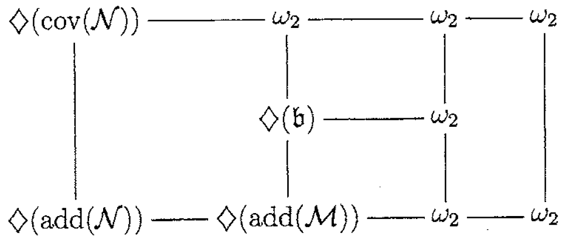



Diagram 3.

(iv) Suppose $V \models$ CH. By Theorem 3.11 and Proposition 3.9 (4) $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}} \models \diamondsuit(\mathrm{add}(\mathcal{N}))$. Since $(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}$ adds $\omega_2$ many random, Cohen and dominating reals, $\mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\} = \omega_2$. Hence

$$V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}} \models \diamondsuit(\mathrm{add}(\mathcal{N})) + \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \omega_2.$$

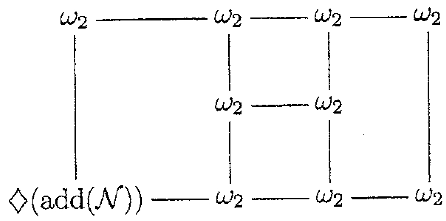



Diagram 4

□

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