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When is a Stokes Line not a Stokes Line? I. The Higher Order Stokes Phenomenon

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1 Introduction

The field of exponential asymptotics deals with the inclusion of exponentially small terms in singular perturbative expansions. Such terms would usually be neglected for reasons of numerical insignificance and a perception that their contribution to an expansion is far outweighed by the difficulty in their calculation. Work in the field of exponential asymptotics over the past two decades has shown that these leading order attitudes may not always be correct.

This is the first of three sequential articles dealing with a recent development in the field. These papers arose from a lecture course given at the meeting *Recent Trends in Exponential Asymptotics* at RIMS, Kyoto in July 2004.

It is worth recalling briefly why someone might wish to modify an expansion of the form

$$y(\mathbf{a}) \sim \sum_{n=0}^{\infty} a_n(\mathbf{a})\epsilon^n \tag{1.1}$$

to include exponentially small contributions of the form

$$y(\mathbf{a}) \sim \sum_{n=0}^{\infty} a_n(\mathbf{a})\epsilon^n + K\exp\left\{-\frac{f(\mathbf{a})}{\epsilon}\right\}\sum_{n=0}^{\infty} b_n(\mathbf{a})\epsilon^n \tag{1.2}$$

where $\mathbf{a} = \{a_1, a_2, \ldots\}$ $a_i \in C$.

- The small exponentials can be used to remove the ambiguities associated with the definition of a Poincaré expansion.

- The inclusion of small exponentials can remove the confusion over the Stokes phenomenon, whereby asymptotic expansions appear to change abruptly in form as a parameter (here $\mathbf{a}$ or ϵ) change smoothly
- Depending on the size of $f(\mathrm{a})/\epsilon$ inclusion of an exponentially small term can increase numerical accuracy of the solution.
- Analytically, inclusion of an exponentially small terms can be used to increase the range of validity of the original approximation to regions where $e^{-f(a)/\epsilon}$ may be $\mathcal{O}(1)$.
- The presence of exponentially small terms is a consequence of the theory of resurgence and provides additional information that aids rigorous work
- Terms that are initially exponentially small may actually grow to dominate as a varies. They are thus often key to understanding the stability of a system.

Several introductory articles, covering the various approaches to the subject, have appeared and, for example, a flavour of some of these approaches can be found in papers contained within Howls *et al* (2000).

We shall be concerned with a development that has a significance that spills over from the community of researchers in asymptotics to wider areas of applied mathematics. Specifically we shall discuss the existence, analysis and importance of the so-called "higher order Stokes phenomenon".

We shall discuss a variety of systems that contain finite parameters $\mathbf{a} = \{a_1, a_2, \ldots\}$ $a_i \in C$ in addition to the asymptotic variable $\epsilon \to 0$. For example the $\mathbf{a}$-parameters may be just a single spatial dimensions x or it may represent spatio-temporal coordinates (x, t). The asymptotic analysis of these systems is not only affected by the Stokes phenomenon, but also by more dramatic coalescence phenomena where underlying singularities generating the asymptotic expansions coalesce at caustics. On caustics the terms themselves in the simple asymptotic expansions become singular and more complicated uniform expansions are required. Such catastrophic effects have been extensively studied asymptotically by (for example) Chester, et al. (1957), Berry (1969), Olver (1974), Wong (1989), Berry & Howls (1993, 1994) and are also well understood.

The surprising result of recent work is that knowledge of coalescences or Stokes phenomenon alone is not always sufficient to predict the asymptotic behaviour of functions in different regions of a-space.

In the context of WKB solutions of higher order ordinary differential equations it has been known (Berk et al 1982) that when more than two possible asymptotic behaviours are present, so-called "new Stokes lines" must be introduced to fully describe the analytic continuation (Aoki *et al* 1994, 2001, 2002). These "new Stokes lines" are actually ordinary Stokes lines, that nevertheless can change their strength in the sense that the Stokes constant itself changes i) abruptly and

ii) along the Stokes line. Hence the Stokes line may even vanish at finite and perfectly regular points in a-space.

The "higher order Stokes phenomenon" is one explanation of the mechanism for this change. In the absence of knowledge of the existence of a higher order Stokes phenomenon, it possible to draw incorrect conclusions from a naive approach as to the existence of Stokes lines or coalescences as one traverses a space.

As we shall see, the higher order Stokes phenomenon is more subtle than the Stokes phenomenon. However its presence nevertheless can lead to the generation of terms elsewhere in **a**-space that can grow to dominate the asymptotics and also affect whether a caustic phenomenon can occur or not.

In this the first paper of the trilogy, we introduce the concept of the higher order Stokes phenomenon and the reason for it. In subsequent papers we deal with the practicalities of seeking out the higher order Stokes phenomenon and its influence on real-parameter situations, in particular time-evolution problems. Most of the ideas have already appeared elsewhere (Howls *et al* 2004, Chapman & Mortimer 2004). However here we present it in a different form to emphasise the breadth of application of the concept of the higher order Stokes phenomenon.

Hence in §2 of this paper we introduce the higher order Stokes phenomenon (HOSP) by means of an example involving a simple canonical integral. In section §3 we show how the higher order Stokes phenomenon is conveniently understood in terms of the remainder terms of asymptotic expansions obtained via a hyperasymptotic procedure.

2 The Higher Order Stokes Phenomenon in a Simple Integral

To illustrate the concept of a higher order Stokes phenomenon, we shall study the integral

$$I(\epsilon; a) = \int_C \exp\left\{\epsilon^{-1}\left(\tfrac{1}{4}z^4 + \tfrac{1}{2}z^2 + az\right)\right\} dz, \tag{2.1}$$

where C is a contour that starts at $V_1 = \infty \exp(-3\pi i/8)$ and ends at $V_2 = \infty \exp(\pi i/8)$ and, without loss of generality, ϵ is a small positive asymptotic parameter. The parameter a is a complex variable.

Using the definition

$$f(z; a) = -\left(\tfrac{1}{4}z^4 + \tfrac{1}{2}z^2 + az\right), \tag{2.2}$$

there are three saddlepoints, z_0, z_1, z_2 which satisfy $df/dz = 0$, that is

$$z_n^3 + z_n + a = 0, \quad n = 0, 1, 2. \tag{2.3}$$

The paths of steepest descent through the saddles z_n are the connected paths passing through z_n that satisfy

$$C_n = \{z \in C : \; k\{f(z;a) - f_n(a)\} \geq 0\}. \tag{2.4}$$

The function $I(\epsilon; a)$ is related to the Pearcey function (Berry & Howls 1991). We choose this integral to explain the higher order Stokes phenonenon because it contains the key ingredients of the higher order Stokes phenomenon:

- it contains 3 possible asymptotic contributions one from each of the saddles;
- these contributions depend on a (non-asymptotic) parameter a;

Furthermore, the integral nature allows for a better geometric explanation of the phenomenon.

We now examine the behaviour or the asymptotic expansions as a varies smoothly in the complex plane. First consider a point $a = a_1$ in figure 1. The corresponding steepest descent paths in the z-plane are shown in the same diagram in the box labeled a_1. In this case we can take $C = C_0$ as the contour of integration so that only the saddle at z_0 contributes to the small-ϵ asymptotics of $I(\epsilon; a)$.

As we cross a Stokes curve in the a-plane defined by

$$S_{i>j} = \left\{a \; : \; \epsilon^{-1}\left(f_j(a) - f_i(a)\right) > 0\right\}. \tag{2.5}$$

a Stokes phenomenon may occur and the number of asymptotic contributions may change. For example, at $a = a_2$ which is a point on such a curve, the steepest descent contours of integration is that part of C_0 that runs from V_1 to z_1 and the part of C_1 that runs from z_1 to V_2. Hence an extra, exponentially subdominant, contribution from z_1 is said to be switched on by the dominant contribution from z_0

At $a = a_3$ the steepest descent contour of integration deforms to $C = C_0 \cup C_1$. Hence saddles at z_0 and z_1 now both contribute to the small-ϵ asymptotics.

Since the definition of Stokes curves involves the values of the saddle heights it is convenient to display the corresponding values $f_j(a)$ in the Borel plane (or complex f-plane) for the three points above in figure 2. The form of the mapping from the z to f plane here generates branch-cut singularities at the images $f_j(a)$. The Borel plane thus possesses a Riemann sheet structure.

The steepest descent contours map to horizontal loop contours starting and finishing at infinity, encircling the corresponding saddlepoint-images. (Once the integral is written in an f-plane representation it is possible to deduce that it is precisely the presence of other such singularities that is responsible for the divergence of the local asymptotic expansions about the $f_j(a)$ see e.g., Olde Daalhuis 1998, Howls 1991.) The main purpose of plotting the Borel plane is that we now see from (2.5) that, in general, a Stokes phenomenon corresponds to the branch-point

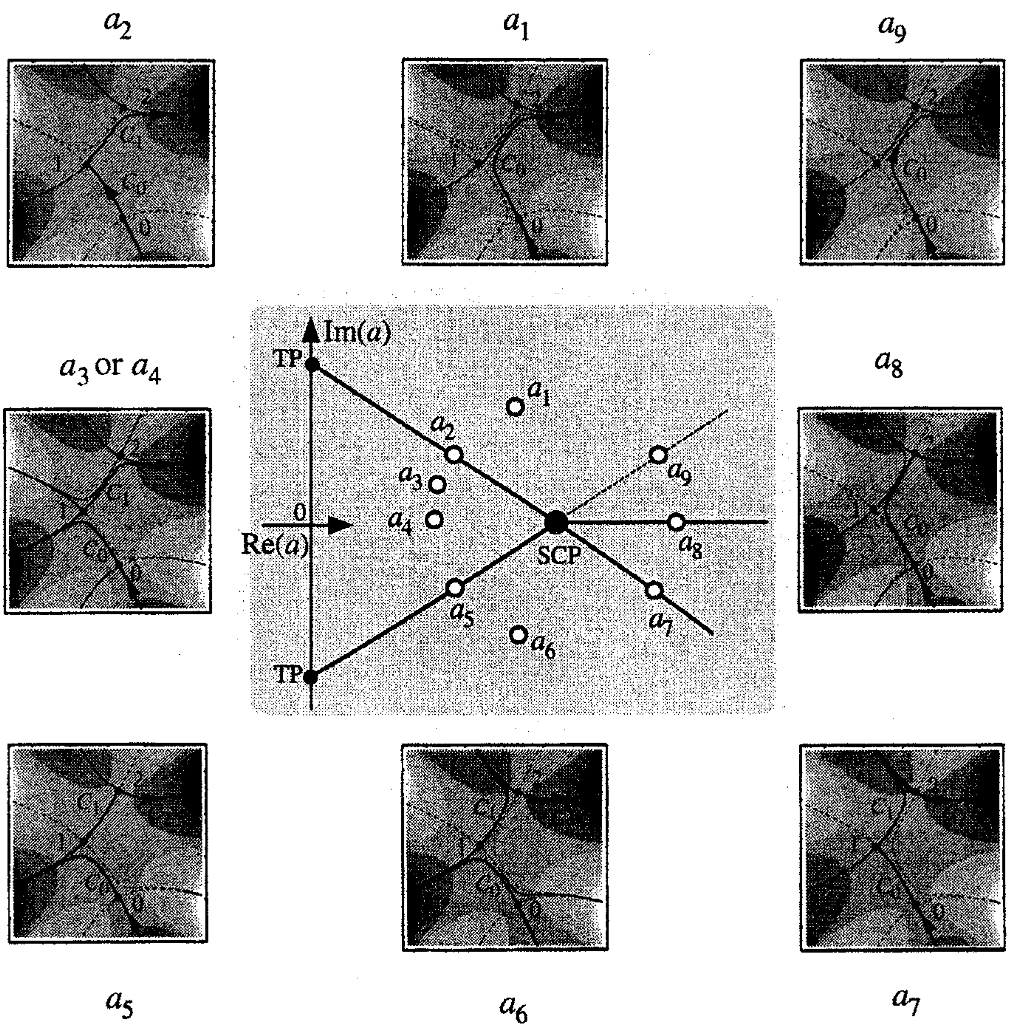

Figure 1: The Stokes curves in the a plane and the steepest descent contours of integration in the integrand z-plane passing over saddles 0, 1 and 2 for selected values a_i for integral (2.1). The dashed Stokes line passing through a_9 is active, but irrelevant to the function defined by the integral.

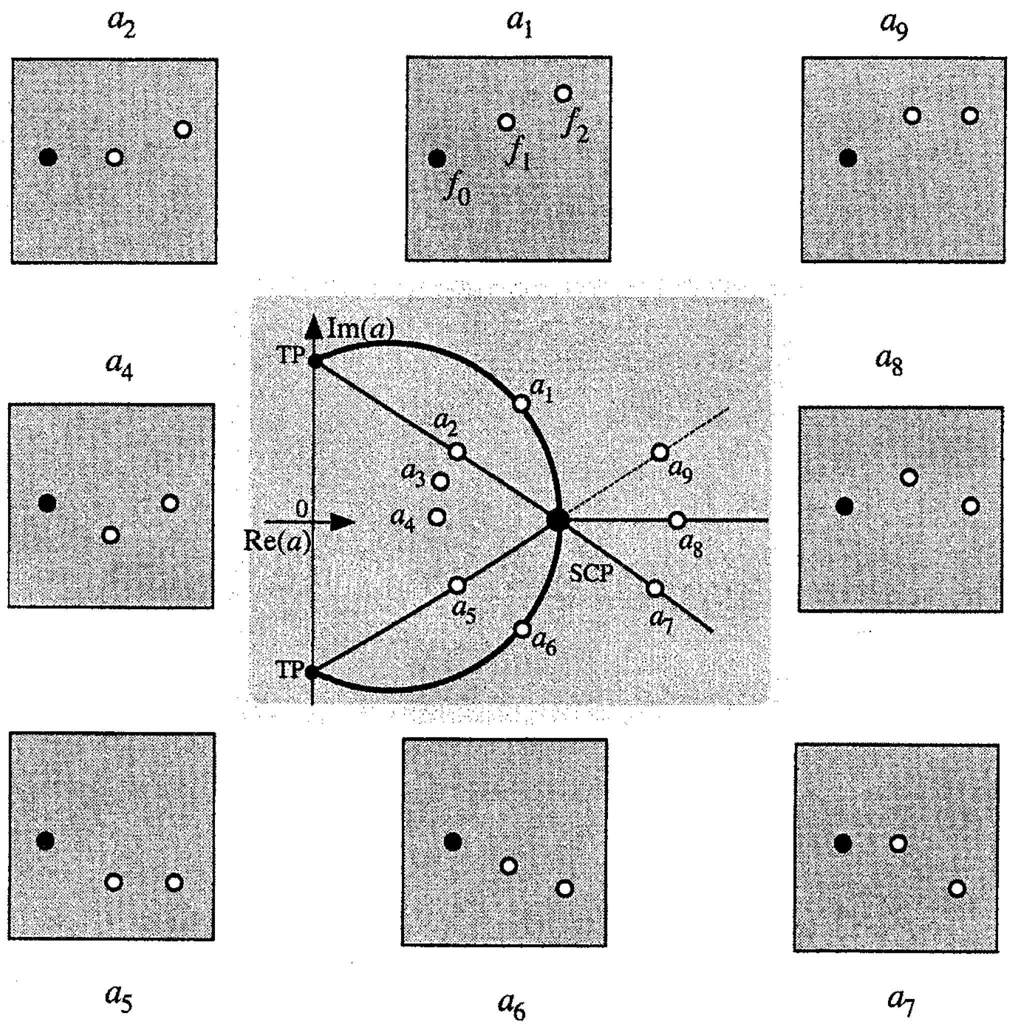

Figure 2: Sketches of the Borel planes for (2.1) at values of the a_i corresponding to those in figure 1. In each Borel plane the solid dot is the image of saddle 0. The other dots are the images of saddles 1 and 2. At a Stokes phenomenon two or more solid dots are horizontally collinear as the steepest paths map to horizontal lines. At a higher order Stokes phenomenon (a_1 and a_6) three or more are collinear in any direction. The higher order Stokes line is drawn in bold and runs between the turning points (TP) passing through the Stokes crossing point (SCP).

$f_j(a)$ passing through the horizontal contour of integration emanating from $f_i(a)$, to the right of $f_i(a)$. Hence at a_2, $f_1(a)$ crosses the branch-cut emanating from $f_0(a)$ this being the image of C_0.

To obtain the location of all the Stokes curves and hence deduce the asymptotic expansion in each sector of the complex a-plane it seems that all we have to do is to study the relative alignment of the $f_j(a)$ in the complex f-plane. By consideration of point a_4, it is not difficult to see that this is not sufficient.

The steepest descent contours suggests that no Stokes phenomenon occurs at a_4. However in the Borel plane $f_j(a_4)$, that $f_2(a_4)$ is actually crossing the horizontal half-line emanating from $f_0(a_4)$. Thus when viewed in the f-plane a Stokes phenomenon should be occurring.

The resolution of this paradox is that although there is a branch point at $f_2(a_4)$, when seen from $f_0(a_4)$ this branch point is not on the principal Riemann sheet. The saddle z_1 is "not adjacent" to z_0 at $a = a_4$. Hence knowledge of the Riemann sheet structure of the Borel plane is also required to establish whether a Stokes phenomenon takes place.

Continuing round in the a-plane, we see that at $a = a_5$ a Stokes phenomenon occurs between saddles z_1 and z_2, so that when $a = a_6$, $C = C_0 \cup C_1 \cup C_2$ and all three saddles contribute to the small-ϵ asymptotics. At $a = a_7$ a Stokes phenomenon switches off the contribution from z_1 so that $C = C_0 \cup C_2$ in the sector between a_7 and a_8.

To complete the a-circuit back to a_1 we now notice that the contribution from saddle z_2 must be switched off since $I(\epsilon; a)$ single-valued in a. This cannot occur at $a = a_9$, which is a typical point on the continuation of the Stokes curve $S_{1>2}$, since saddle z_1 no longer contributes to the asymptotics of $I(\epsilon; a)$. For this reason the corresponding part of that Stokes curve in the central box of figures 1 and 2 is dashed, since it is *irrelevant* for the function defined by our choice of contour in (2.1). (It would have been relevant for a different choice of valleys for the contour of integration in (2.1)). Hence another Stokes curve must be crossed somewhere on the circuit between a_7 and a_1. Consideration of the f-plane shows that an obvious choice is at $a = a_8$, that is as $f_2(a_8)$ crosses the horizontal half-line emanating from $f_0(a_8)$. This is easily confirmed by examination of the steepest descent paths in the box labeled a_8 in figure 1.

Having completed the circuit in a space we now encounter a surprise.

The Stokes curves in the central box of figures 1 and 2 all cross at a particular point in the a-plane, which we call the Stokes crossing point (SCP). In the analysis above we have shown that the part of the positive real a-axis ($S_{0>2}$) from the origin to the SCP is not an active Stokes curve. Now we see that the part of $S_{0>2}$ to the right of the SCP is an active Stokes curve. When $a \in S_{0>2}$ is on either curve in the corresponding f-plane, $f_2(a)$ is actually crossing the horizontal half-line emanating from $f_0(a)$. As mentioned above, for a to the left of the SCP the point $f_2(a)$ is not on the principal Riemann sheet seen from $f_0(a)$. For a to the right of the

SCP it is.

To explain the change in Riemann sheet structure we introduce the concept of a "higher order Stokes phenomenon" that takes place across a new curve in the complex a-plane passing through the SCP that we call a "higher order Stokes curve" (HSC). In the example above, we have chosen a_1 to lie on the HSC and from figure 2 we see that nothing of interest happens here to the steepest descent paths. However in the complex f-plane, something significant is happening.

Let a_1^+ and a_1^- be a-values slightly to the right and left of a_1 respectively. The values of $f_j(a)$ are displayed in figure 3. It is clear from this figure that at $a = a_1$, $f_2(a)$ is actually crossing the continuation of the line from $f_0(a)$ to $f_1(a)$. On the HSC $f_0(a)$, $f_1(a)$, $f_2(a)$ are collinear in the complex f-plane. When viewed from $f_0(a)$, this continuation of the line is a radial branch cut extending from $f_1(a)$ (Olde Daalhuis 1998). Thus as colinearity of all three $f_j(a)$ occurs, the Riemann sheet structure of the Borel plane in fact changes. For $a = a_1^+$ the point $f_2(a)$ is on the principal Riemann sheet as seen from $f_0(a)$, but for $a = a_1^-$ it is not. In the latter case to walk from $f_0(a_1^-)$ to $f_2(a_1^-)$ one would first have to walk around $f_1(a_1^-)$.

A higher order Stokes phenomenon is thus said to occur when at least three of the $f_j(a)$ are collinear in the f-plane. In turn, collinearity occurs at set of points in the **a**-plane that we define to be the higher order Stokes curve:

Higher Order Stokes Curve:

A higher order Stokes phenomenon takes place across a higher order Stokes curve, which is defined by the set of points:

$$\frac{f_j(a) - f_i(a)}{f_k(a) - f_j(a)} \in R. \tag{2.6}$$

We now make the following observations.

- On a Stokes curve, at least two of the $f_j(a)$ differ by only a real number and so can be joined by a horizontal line in the Borel-plane. Stokes curves are only active when the relevant $f_j(a)$ are on the same Riemann sheet. Since the Riemann sheet structure changes as a higher order Stokes curve is crossed, the activity of a Stokes line (and therefore the Stokes constants) changes across a higher order Stokes curve.
- As one crosses a Stokes line, an series prefactored by an exponential small term appears (or disappears) in the full expansion. However as one crosses a higher order Stokes curve there is no obvious such change in the asymptotic expansion. What has changed is the existence of a term in the remainder of the truncated expansion. This will be explained below.

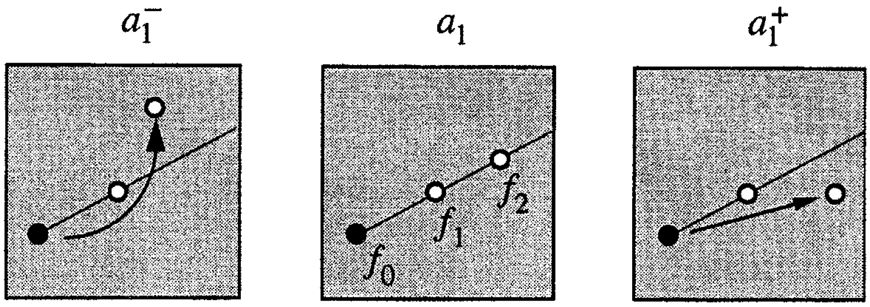

Figure 3: The higher order Stokes phenomenon in the Borel plane for values of a near to a_1. At the higher order Stokes phenomenon f_1 eclipses f_2 when viewed from f_0. The Riemann sheet structure of the Borel plane changes as f_2 passes through a radial cut from f_1. At a_1^-, f_2 is invisible from f_0 and so no Stokes phenomenon between f_0 and f_2 can take place. At a_1^+, f_2 is visible and so a Stoke phenomenon is then possible.

- Traditionally one expected that in the a-plane Stokes curves could only emanate from turning points, where two or more $f_j(a)$ coalesce, or from other singularities. However we now see that Stokes curves may start and end from other regular points in the a-plane (the SCP), where two or more other Stokes curves may cross. This effect has been observed before by Berk et al (1982) and Aoki *et al* (1994, 2001, 2002).

- It is important to note the difference between a Stokes curve being *inactive* and a Stokes curve being *irrelevant*. In the example above no Stokes curve has been drawn between $a = 0$ and the SCP. Nor has one been sketched from the SCP along the continuation of the Stokes curve $S_{1>2}$ in the direction of a_9. In the first case a Stokes phenomenon could have occurred, but it didn't because $f_2(a)$ was not on the principal Riemann sheet as viewed from $f_0(a)$ (i.e., they were not adjacent). This Stokes curve was therefore *inactive*. In the second case a Stokes phenomenon does in fact take place between the saddles at z_1 and z_2. However this particular phenomenon is irrelevant to the saddlepoint asymptotics of the specific function $I(\epsilon; a)$ as defined by our choice of valleys in (2.1): in the neighbourhood of this curve, z_1 is not contributing to the asymptotics of $I(\epsilon; a)$ anyway. If we had been interested in a different function $\tilde{I}(\epsilon; a)$ defined by (2.1) but with C running between different valleys, then this curve could have been *relevant* whereas the first Stokes curve would still be *inactive*.

- At a higher order Stokes phenomenon at least three of the $f_j(a)$ are collinear in the f-plane. At a traditional Stokes phenomenon a minimum of only two of the $f_j(a)$ are required to differ by a real number. At a higher order Stokes phenomenon there is no actual constraint on the relative positioning of the first two $f_j(a)$ in the f-plane (a straight line can join any two points in the a-plane). Hence a traditional Stokes curve and a higher-order Stokes curve (involving three $f_j(a)$) have the same codimensionality.

- As the asymptotic parameter ϵ changes in phase, the location of Stokes curves vary. However the colinearity condition that gives rise to the definition of a higher order Stokes curve is independent of ϵ. Thus the location of the higher order Stokes curve is invariant under changes of the asymptotic parameter. This is illustrated in figure 4 for $\arg\epsilon = 0$ and $\arg\epsilon = -\pi/4$.
- A higher-order Stokes curve will emanate from the same points as traditional Stokes curves. These points are turning points or singularities of the phase function f.
- The concept of a higher-order Stokes curve has been couched above in terms of asymptotic expansions arising from saddle-point integrals. However since we have expressed everything in terms of the $f_j(a)$, the ideas introduced are much more generally applicable. All that is required to apply these definitions is the ability to determine all the different types of exponential asymptotic behaviours $\exp(-kf_j(a))$ associated with an expansion, regardless of its origin. The only property peculiar to the saddlepoint integral that we have used above is the facility to determine the activity of Stokes curves from the steepest descent contours in the z-plane of the integrand. A more general way of achieving this is to compute the "Stokes multipliers" K_{ij}. If a Stokes multipler has a zero value the corresponding Stokes curve is inactive. As long as it is possible to compute the coefficients $T_r^{(j)}$ in the asymptotic expansions, the computation of Stokes multipliers is a solved problem, see Olde Daalhuis (1998), Howls (1997) for details.
- A higher-order Stokes curve requires collinearity of at least three $f_j(a)$. It is thus likely to occur in any expansion that involves more than two different asymptotic behaviours depending on a set of additional parameters $\mathbf{a}$. A higher order Stokes phenomenon could thus occur in expansions resulting from integrals involving three or more critical points (of any dimensionality), inhomogeneous second-order linear ordinary differential equations, higher order linear o.d.e.s, nonlinear o.d.e.s, and partial differential equations.

In the next section we give an explanation of how hyperasymptotic analysis is a natural way to calculate not only the required Stokes multipliers in a more general problem, but also to quantify precisely the effects of a higher-order Stokes phenomenon.

3 Explanation of Higher Order Stokes Phenomenon

The change in activity of Stokes line has been noticed or discussed by several authors Berk *et al* (1982), Aoki *et al* (1994, 2001, 2002), Chapman & Mortimer (2004). Here explain how and why a higher order Stokes phenomenon gives rise to such a fundamental change in the analytic structure of an expansion by reference to the exact remainder terms derived by hyperasymptotic procedures (Berry & Howls 1991, Olde Daalhuis & Olver 1995, Howls 1997, Olde Daalhuis 1998 Delabaere & Howls 2002). Again we will start from an integral representation involving only

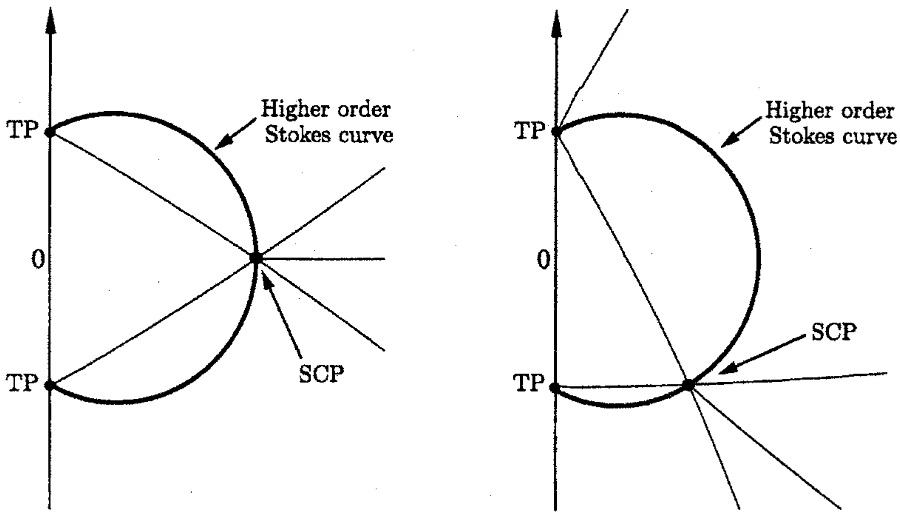

Figure 4: The Stokes geometry for $\arg \epsilon = 0$ (left) and $\arg \epsilon = -\pi/4$ (right). The thin curves are the normal Stokes curves, and the bold curves are the Higher order Stokes curves. This diagram is typical of all the examples we study in this trilogy of papers.

contributions from simple saddles. However what follows can be easily extended to any function that possesses a Borel transform.

We shall start from an integral involving the asymptotic parameter $\epsilon \to 0$ of the form

$$I^{(n)}(\epsilon; \mathbf{a}) = \int_{C_n(\theta_\epsilon; \mathbf{a})} e^{-f(z;\mathbf{a})/\epsilon} g(z; \mathbf{a})\, dz. \tag{3.1}$$

We assume that f possesses at least three saddlepoints situated at $z = z_n\,(n = 0, 1, 2)$, where $df/dz = 0$. Again, we take $f_n = f_n(\mathbf{a}) = f(z_n; \mathbf{a})$. We assume that the range of values of $\mathbf{a}$ are such that the saddles are simple so that $d^2f/dz^2 \neq 0$, however this is a technical restriction to simplify the discussion and can be removed later. The contour $C_n(\theta_\epsilon; \mathbf{a})$ is then the steepest descent path satisfying $\epsilon^{-1}\{f(z) - f_n\} > 0$ and running through (in general) a single specific saddle at z_n, between specified asymptotic valleys of $\Re\{f(z) - f_n\}$ at infinity (de Bruijn (1958) ch. 5, Copson (1965) ch. 7). The functions $f(z; \mathbf{a})$ and $g(z; \mathbf{a})$ are analytic, at least in a strip including $C_n(\theta_\epsilon; \mathbf{a})$ and in the range of $\mathbf{a}$ values considered. As $\arg\epsilon$ varies, $C_n(\theta_\epsilon; \mathbf{a})$ correspondingly deforms and for a set of discrete values of $\arg\epsilon$ it will encounter certain other saddles m. These are called adjacent saddles, generating an additional contribution to the asymptotics, prefactored by an exponentially small term: each of these births is an ordinary Stokes phenomenon. Saddles that do not connect with n as $\arg\epsilon$ varies through 2π are called non-adjacent and do not (directly) generate a Stokes phenomenon.

Without loss of generality we order the labelling of the saddles such that $\Re f_0 < \Re f_1 < \Re f_2$ for the values of $\mathbf{a}$ under discussion and consider the integral through saddle 0. We shall also choose a value of **a** such that saddle 2 is adjacent to 1, but not to 0.

We decompose the asymptotic expansion into the standard form of a fast varying exponential prefactor and a slowly varying algebraic part

$$I^{(0)}(\epsilon;\mathbf{a}) = \exp\left(-f_0(\mathbf{a})/\epsilon\right)\sqrt{\epsilon}\,T^{(0)}(\epsilon;\mathbf{a}), \tag{3.2}$$

(In the general case for an arbitrary Borel transform, the prefactor $\sqrt{\epsilon}$ will be replaced by an arbitrary power ϵ_0^{μ}, say.) The algebraic part is expanded as a truncated asymptotic expansion

$$T^{(0)}(\epsilon;\mathbf{a}) = \sum_{r=0}^{N_0-1} T_r^{(0)}(\mathbf{a})\epsilon^r + R_{N_0}^{(0)}(\epsilon;\mathbf{a}). \tag{3.3}$$

By assumption $T^{(0)}(\epsilon;\mathbf{a})$ is analytic in **a**. Consequently a straightforward extension of the work of Berry & Howls (1991) shows that the remainder term of this expansion may be written exactly as

$$T^{(0)}(\epsilon;\mathbf{a}) = \sum_{r=0}^{N_0-1} T_r^{(0)}(\mathbf{a})\,\epsilon^r + \frac{1}{2\pi i}\frac{K_{01}\epsilon^{N_0}}{F_{01}(\mathbf{a})^{N_0}}\int_0^\infty dv \frac{e^{-v}v^{N_0-1}}{1-v\epsilon/F_{01}(\mathbf{a})}T^{(1)}\left(\frac{F_{01}(\mathbf{a})}{v};\mathbf{a}\right) \tag{3.4}$$

Here we have defined

$$F_{nm}(\mathbf{a}) = f_m(\mathbf{a}) - f_n(\mathbf{a}) \tag{3.5}$$

is defined as the complex difference in heights between saddles n and m. The factor K_{01} is effectively the Stokes constant relating to the contribution of saddle 1 to the expansion about saddle 0. We recall that for expansions arising from integrals a Stokes constant K_{nm} is real with modulus unity if m is adjacent to n and is zero otherwise, as explained in Howls (1997). In more general cases the Stokes constant is a complex number.

The term $T^{(m)}(\ldots)$ on the right hand side of (3.5) are the slowly varying parts of integrals over the subset of the adjacent saddles analogous to the expansion $T^{(0)}$.

As ϵ varies in phase, from the definition of a Stokes phenomenon (2.5) between 0 and 1 can occur whenever $\arg(\epsilon/F_{01})$ is an integer multiple of 2π. At this phase the exact remainder integral in (3.4) encounters a pole on the real contour of integration. As the phase of ϵ advances on, the contour of integration of the remainder integral (3.4) snags on the pole. In turn this introduces a residue contribution (up to a sign):

$$\frac{\epsilon K_{01}}{F_{01}(\mathbf{a})^{N_0}}\operatorname*{Res}_{v\to F_{01}(\mathbf{a})/\epsilon}\left\{\frac{e^{-v}v^{N_0-1}}{1-v\epsilon/F_{01}(\mathbf{a})}T^{(1)}\left(\frac{F_{01}(\mathbf{a})}{v};\mathbf{a}\right)\right\} = K_{01}e^{-F_{01}(\mathbf{a})/\epsilon}T^{(1)}(\epsilon;\mathbf{a}). \tag{3.6}$$

When combined with the exponential prefactor $\exp(-f_0/\epsilon)$, (3.6) produces an exponentially small contribution $\exp(-f_1(\mathbf{a})/\epsilon)\,T^{(1)}(\epsilon;\mathbf{a})$. This is exactly the integral over the steepest contour $C_1(\theta_\epsilon)$ passing through z_1 that is acquired across the Stokes line $S_{0>1}$.

As it stands, the exact remainder term in (3.6) is implicit. In general we know no more about $T^{(1)}$ than we did $T^{(0)}$. To circumvent this problem (Berry & Howls 1991) introduced a hyperasymptotic appraoch: an expression for $T^{(1)}$, analogous to (3.4) can be written down by inspection, in terms of its own adjacent terms

$$T^{(1)}(\xi; \mathbf{a}) = \sum_{r=0}^{N_1-1} T_r^{(1)}(\mathbf{a})\,\xi^r + \frac{1}{2\pi i}\sum_{m=0,2} \frac{K_{1m}\xi^{N_1}}{F_{1m}(\mathbf{a})^{N_1}} \int_0^\infty dw \frac{e^{-w} w^{N_1-1}}{1 - w\xi/F_{1m}(\mathbf{a})} T^{(m)}\left(\frac{F_{1m}(\mathbf{a})}{w}; \mathbf{a}\right), \tag{3.7}$$

where $\xi = F_{01}(\mathbf{a})/v$. This corresponds to a re-expansion of the remainder in the locality of saddle 1. Substitution of this expression into (3.4) generates a series of terms that can be evaluated, together with a new implicit (and with suitable choice of N_0 and N_1) exponentially smaller remainder. The new expression is also valid in a larger sector, see Berry & Howls 1991, Olde Daalhuis & Olver 1995, Howls 1997 or Olde Daalhuis 1998 for further details.

There are now two contributions to the remainder term of the original expansion (3.4) since 1 is adjacent to both 0 and 2 by assumption. The first is a "backscatter" contribution from the Borel singularities corresponding to the saddles from/to 0 via 1. This contribution to the remainder of the original expansion about from saddle 0 contains no further potential singularities at this first level of re-expansion and so is not important for the argument here. However the contribution from saddle 2, does contain a further possible singularity, as we now outline.

We focus now on the exponentially smaller unevaluated remainder from the term involving saddle 2. After substitution of (3.6) into (3.4) we discover it to be of the form:

$$\begin{aligned} R^{(012)}(\xi; \mathbf{a}) =& \frac{1}{(2\pi i)^2} \frac{K_{01}K_{12}\epsilon^{N_0}}{F_{01}(\mathbf{a})^{N_0-N_1} F_{12}(\mathbf{a})^{N_1}} \\ &\times \int_0^\infty dv \frac{e^{-v} v^{N_0-N_1-1}}{1 - v\epsilon F_{01}(\mathbf{a})} \int_0^\infty dw \frac{e^{-w} w^{N_1-1}}{1 - wF_{01}(\mathbf{a})/vF_{12}(\mathbf{a})} T^{(2)}\left(\frac{F_{12}(\mathbf{a})}{w}; \mathbf{a}\right). \end{aligned} \tag{3.8}$$

$R^{(012)}$ clearly has a pole when the Stokes phenomenon takes place where $F_{01}(\mathbf{a})/\epsilon > 0$. It must have this pole as it is a re-expansion of the original remainder in (3.4) and so must contain the Stokes phenomenon that was present in that exact remainder. However, if $\mathbf{a}$ is now varied (independently from ϵ), another potential pole in $R^{(012)}$ can occur when

$$F_{01}(\mathbf{a})/F_{12}(\mathbf{a}) > 0. \tag{3.9}$$

Note that this pole condition is identical to the colinearity condition (2.6). The occurrence of the pole is therefore synonymous with a higher order Stokes phenomenon.

The residue from this pole is (up to a sign)

$$\operatorname*{Res}_{w \to vF_{12}/F_{01}} R^{(012)} = \frac{K_{01}K_{12}k^{N_0}}{2\pi i\, F_{02}(\mathbf{a})^{N_0}} \int_0^\infty dv \frac{e^{-v} v^{N_0-1}}{1 - \frac{v\epsilon}{F_{02}(\mathbf{a})}} T^{(2)}\left(\frac{F_{02}(\mathbf{a})}{v}\right), \tag{3.10}$$

We now observe that as a higher order Stokes curve is crossed, a pole arising in the higher order hyperasymptotic remainder term switches on a new contribution to the remainder term.

Since K_{01} and K_{12} are ± 1, a comparison of the form of (3.10) with (3.4) and (3.7) shows that this new contribution to be precisely the contribution to the remainder term of the expansion about 0 that one would expect if saddle 2 is adjacent to 0.

Note that now if the parameters $\mathbf{a}$ (or ϵ) are further varied such that $F_{02}(\mathbf{a})/\epsilon$ becomes real and positive then integration in the new remainder term (3.10) will encounter a pole and a Stokes phenomenon will now take place between 0 and 2. Prior to the higher order Stokes phenomenon it could not.

Equally, we note that if 0 and 2 now coalesce so that $F_{02}(\mathbf{a}) = 0$ then the form of the expansion (*cf* (3.10)) breaks down and a caustic takes place. Prior to the higher order Stokes phenomenon, although there may have been values of $\mathbf{a}$ such that $F_{02}(\mathbf{a}) = 0$, the corresponding Borel singularities were actually vertically above each other on different Riemann sheets.

The hyperasymptotic approach has thus automatically carried out the accounting of all contributions involved during the higher order Stokes phenomenon correctly, concisely and exactly. The changes activity of Stokes lines can be traced to the changes in the presence of terms in the exact remainder contributions. In turn this can thought of as abrupt changes in the Stokes constants themselves that prefactor these remainder terms.

Finally, note that in a full hyperasymptotic theory, expressions for $T^{(j)}$ analogous to (3.7) can be successively substituted into each implicit remainder term, generating a tree-like expansion of self-similiar multiple-integral contributions called hyperterminants (which can be straightforwardly evaluated by the methods of Olde Daalhuis (1998b). Each such hyperterminant has a denominator in the integrand of the form $1 - wF_{ij}(\mathbf{a})/vF_{jk}(\mathbf{a})$ and so a higher order Stokes phenomenon will occur in any branch of the hyperasymptotic expansion-tree whenever the appropriate condition $F_{ij}(\mathbf{a})/F_{jk}(\mathbf{a}) > 0$ is satisfied.

Consequently if there are more than 3 singularities the $\mathbf{a}$ plane will be interweaved with a warp and weft of increasingly higher order Stokes curves.

4 Conclusion

In this paper we have introduced the higher order Stokes phenomenon in terms of systems that have natural integral representations. We have used a hyperasymptotic approach to quantify the changes in activity of Stokes lines to changes in the Riemann sheet structure of the corresponding Borel plane. Other explanations of the change in activity of Stokes lines certainly exist and can be found in Aoki *et al* (1994, 2001, 2002), Chapman & Mortimer (2004).

In the next paper of the trilogy we discuss two examples in which we examine how to quantify the higher order Stokes phenomenon in the absence of convenient integral representations.

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