数理解析研究所講究録 1424 巻 2005 年 142-155

On the Structure of integral kernel for the Borel sum

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1 Introduction

We consider the following quasi-homogeneous linear partial differential operator with constant coefficients

$$P=P(\partial_t,\partial_x)=\prod_{j=1}^{\mu}P_j^{\ell_j},\quad P_j=\partial_t^p-\alpha_j\partial_x^q, \tag{1.1}$$

where $t,x\in\mathbb{C}$, $p,q,\mu,\ell_j\in\mathbb{N}$ with $p<q$ and $\alpha_j\in\mathbb{C}\setminus\{0\}\,(\alpha_i\neq\alpha_j\,(i\neq j))$. We put $\nu=\sum_{j=1}^{\mu}\ell_j$ and we assume that $\nu\geq 2$. Then the order of differentiation with respect to t for our operator P is $p\nu(\geq 2)$.

We consider the following Cauchy problem for a non-Kowalevski equation

$$\begin{cases} PU(t,x)=0, \\ \partial_t^k U(0,x)=0 \quad (0\leq k\leq p\nu-2), \\ \partial_t^{p\nu-1}U(0,x)=\varphi(x), \end{cases} \tag{1.2}$$

where the Cauchy data $\varphi(x)$ is assumed to be holomorphic in a neighbourhood of the origin.

The formal power series solution with respect to t of this Cauchy problem (1.2) is, in general, divergent by the assumption that $p<q$. Therefore it is natural to study the k-summability of the divergent solution (for the definitions of the terminologies, see section 2). The conditions for the k-summability of the divergent solution was obtained by Ichinobe [Ich 2] (cf. [LMS], [Miy], Theorem 4.1). Moreover, in [Ich 2], under those conditions the integral representation of the Borel sum was obtained by using the integral kernel (cf. [LMS], [Ich 1], Theorem 4.3). By the results of [Ich 2], the Borel sum is given by the summation of integrations along $q\mu$ half lines which start at the origin in the complex plane.

On the one hand, in the case of the heat equation ($(p,q,\mu)=(1,2,1)$ for our operator P) as a special case, the Borel sum is given by the integration along a line through the origin and the integral kernel is given by the heat kernel (cf. [LMS], [Ich 1,2]). Thus the integral representation of the Borel sum coincides with that of the classical solution which is obtained by the theory of Fourier integrals.

On the other hand, when $q\mu \geq 3$, the Borel sum is given by the summation of integrations along half lines in the complex plane. In this paper, we study the condition under which the integral paths of the Borel sum in the complex plane are deformed into the integration along a line. Exactly, the main interest of this paper is to give a sufficient condition under which the integral paths of the Borel sum can be deformed into the real axis.

We state the contents of the following sections. In section 2, we shall give the review of k-summability. In section 3, we shall give the decomposition formula of solutions for the Cauchy problem (1.2). We shall give the main result in section 4. In section 5, we shall give the proof of proposition.

2 Review of k-summability

We first give a short review of k-summability (cf. [Bal]).

1. Sector. For $d \in \mathbb{R}$, $\beta > 0$ and $\rho\,(0 < \rho \leq \infty)$, we define a sector $S = S(d, \beta, \rho)$ by

$$S(d,\beta,\rho) := \left\{ t \in \mathbb{C}\,;\, |d - \arg t| < \frac{\beta}{2},\, 0 < |t| < \rho \right\}, \tag{2.1}$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively.

2. Gevrey formal power series. We denote by $\mathcal{O}[[t]]$ the ring of formal power series in t-variable with coefficients in $\mathcal{O}$ which is the set of holomorphic functions in a neighbourhood of the origin. For $k > 0$, we define that $\hat{f}(t,x) = \sum_{n=0}^{\infty} f_n(x) t^n \in \mathcal{O}[[t]]_{1/k} (\subset \mathcal{O}[[t]])$, which is the ring of formal power series of Gevrey order $1/k$ in t-variable, if there exists a positive constant r such that the coefficients $f_n(x) \in \mathcal{O}(B_r)$, which denotes the set of holomorphic functions on a common closed disk $B_r := \{x \in \mathbb{C}; |x| \leq r\}$, and there exist some positive constants C and K such that for any n, we have

$$\max_{|x| \leq r} |f_n(x)| \leq C K^n \Gamma\left(1 + \frac{n}{k}\right), \tag{2.2}$$

where Γ denotes the Gamma function.

3. Gevrey asymptotic expansion. Let $k > 0$, $\hat{f}(t,x) = \sum_{n=0}^{\infty} f_n(x) t^n \in \mathcal{O}[[t]]_{1/k}$ and $f(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B_r$. Then we define that

$$f(t,x) \cong_k \hat{f}(t,x) \quad \text{in } S(d,\beta,\rho), \tag{2.3}$$

if for any closed subsector S' of $S(d,\beta,\rho)$, there exist some positive constants $r'(\le r)$, C and K such that for any N, we have

$$\max_{|x|\le r'}\left|f(t,x)-\sum_{n=0}^{N-1} f_n(x)t^n\right| \le CK^N|t|^N\Gamma\left(1+\frac{N}{k}\right), \quad t\in S'. \tag{2.4}$$

4. k-summability. For $k>0$, $\hat{f}(t,x)\in\mathcal{O}[[t]]_{1/k}$ and $d\in\mathbb{R}$, we define that $\hat{f}(t,x)$ is k-summable in d direction if there exist a sector $S(d,\beta,\rho)$ with the opening angle $\beta>\pi/k$, and a positive constant r such that there exists an analytic function $f(t,x)$ on $S(d,\beta,\rho)\times B_r$ with $f(t,x)\cong_k \hat{f}(t,x)$ in $S(d,\beta,\rho)$.

We remark that the function $f(t,x)$ above for a k-summable $\hat{f}(t,x)$ is unique if it exists. Therefore such a function $f(t,x)$ is called the k-sum of $\hat{f}(t,x)$ in d direction. Throughout this paper, we call the k-sum the Borel sum and it is written by $f^d(t,x)$.

3 Decomposition formula of solutions

We give the following proposition which is a decomposition formula of solutions for the Cauchy problem (1.2).

Proposition 3.1 *Let $U(t,x)$ be a solution of the Cauchy problem (1.2). Then there exist ν constants c_{mn} $(1\le m\le\mu; 1\le n\le\ell_m)$ such that the following formula holds*

$$U(t,x)=\sum_{m=1}^{\mu}\sum_{n=1}^{\ell_m} c_{mn}D_t^{-p(\nu-n+1)+1}\frac{[(1/p)\delta_t]_{n-1}}{(n-1)!}D_t^{-p(n-1)}u_m(t,x), \tag{3.1}$$

where D_t^{-1} denotes the integration from 0 to t, the operator δ_t denotes the Euler operator $t\partial_t$ and $[(1/p)\delta_t]_{n-1}$ is given by

$$[(1/p)\delta_t]_{n-1} := \begin{cases} \frac{1}{p}\delta_t\left(\frac{1}{p}\delta_t-1\right)\cdots\left(\frac{1}{p}\delta_t-n+2\right), & n\ge 2,\\ 1, & n=1.\end{cases} \tag{3.2}$$

Moreover, each function $u_m(t,x)$ is a solution of the following Cauchy problem

$$\begin{cases} P_m u(t,x) = \left(\partial_t^p-\alpha_m\partial_x^q\right)u(t,x)=0,\\ u(0,x)=\varphi(x),\\ \partial_t^k u(0,x)=0 \quad (1\le k\le p-1).\end{cases} \tag{3.3}$$

We shall give the proof of Proposition 3.1 in section 5.

Remark 3.2 *Proposition* 3.1 *also holds in the case where* $p > q$. *Indeed, from Cauchy-Kowalevski theorem,* $U(t,x)$ *and all* $u_m(t,x)$*'s are analytic. Therefore the formula* (3.1) *holds in the category of analytic functions.*

Remark 3.3 *If each* $u_m(t,x)$ $(1 \le m \le \mu)$ *is the Borel sum for the Cauchy problem* (3.3), *the above* $U(t,x)$, *given by the formula* (3.1), *is the Borel sum for the Cauchy problem* (1.2) (cf. [Ich 2]).

We give some examples of Proposition 3.1.

- The case $P = \prod_{j=1}^{\mu} P_j$ (i.e. $\mu = \nu$). A solution $U(t,x)$ is given by the following expression

$$(3.4) \qquad U(t,x) = \sum_{m=1}^{\mu} c_m D_t^{-p\mu+1} u_m(t,x), \quad c_m = \frac{\alpha_m^{\mu-1}}{\prod_{1\le j\le \mu,\, j\ne m}(\alpha_m - \alpha_j)}.$$

- The case $P = P_1^{\nu}$ (i.e. $\mu = 1$). A solution $U(t,x)$ is given by the following expression

$$(3.5) \qquad U(t,x) = D_t^{-(p-1)} \frac{[(1/p)\delta_t]_{\nu-1}}{(\nu-1)!} D_t^{-p(\nu-1)} u_1(t,x).$$

- The case $p = 1$. In this case, since the operator $[(1/p)\delta_t]_{n-1} = t^{n-1}\partial_t^{n-1}$, we have

$$(3.6) \qquad U(t,x) = \sum_{m=1}^{\mu} \sum_{n=1}^{\ell_m} c_{mn} D_t^{-(\nu-n)} \frac{t^{n-1}}{(n-1)!} u_m(t,x).$$

4 Main result

From the formula (3.1), all informations of the solution $U(t,x)$ come from the one $u_m(t,x)$. Therefore, it is enough to study the property of the Borel sum for the Cauchy problem (3.3). In the following, we study (3.3) in which we replace α_m by α.

$$(4.1) \qquad \begin{cases} (\partial_t^p - \alpha \partial_x^q)\, u(t,x) = 0, \\ u(0,x) = \varphi(x), \\ \partial_t^k u(0,x) = 0 \quad (1 \le k \le p-1). \end{cases}$$

The Cauchy problem (4.1) has the following unique formal solution $\hat{u}(t,x)$

$$(4.2) \qquad \hat{u}(t,x) = \sum_{n=0}^{\infty} \alpha^n \varphi^{(qn)}(x) \frac{t^{pn}}{(pn)!}.$$

By using our terminology, we see $\hat{u}(t,x) \in \mathcal{O}[[t]]_{1/k}$, $k = p/(q-p)$.

4.1 Known results

We shall give the results of the k-summability of the formal solution (4.2) and the integral representation of the Borel sum by using the integral kernel.

First, the result of the k-summability is stated as follows, which was proved by Miyake [Miy].

Theorem 4.1 *Let $d \in \mathbb{R}$ and $\hat{u}(t,x)$ be the formal solution of the Cauchy problem* (4.1). *Then the following conditions are equivalent:*

(i) *$\hat{u}(t,x)$ is k-summable in d direction.*

(ii) *The Cauchy data $\varphi(x)$ can be continued analytically in*

$$\Omega_{\alpha}^{(p,q)}(d,\varepsilon) := \bigcup_{m=0}^{q-1} S\left(\frac{dp + \arg\alpha + 2\pi m}{q}, \varepsilon, \infty\right) \tag{4.3}$$

and has the growth condition of exponential order at most $q/(q-p)$ there, which means that there exist positive constants C and δ such that

$$|\varphi(x)| \leq C \exp\left(\delta |x|^{q/(q-p)}\right), \quad x \in \Omega_{\alpha}^{(p,q)}(d,\varepsilon). \tag{4.4}$$

Before stating the result of the integral representation of the Borel sum, we need some preparations.

We use the following abbreviations.

$$\boldsymbol{p} = (1,2,\ldots,p), \quad \boldsymbol{q} = (1,2,\ldots,q), \quad \boldsymbol{q}/q = (1/q, 2/q, \ldots, q/q),$$

$$\widehat{\boldsymbol{q}}_j = (1,2,\ldots,j-1,j+1,\ldots,q) \in \mathbb{N}^{q-1}, \quad \boldsymbol{q}+c = (1+c, 2+c, \ldots, q+c) \quad (c \in \mathbb{C}),$$

$$\Gamma(\boldsymbol{q}/q + c) = \prod_{j=1}^{q} \Gamma(j/q + c).$$

We give the definition of Meijer G-function (cf. [MS, p.2], [Luk, p.144]).

Let $0 \leq n \leq p$, $0 \leq m \leq q$. For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p) \in \mathbb{C}^p$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q) \in \mathbb{C}^q$ with $\beta_\ell - \gamma_j \notin \mathbb{N}$ ($\ell = 1,2,\ldots,n; j = 1,2,\ldots,m$), we define

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{matrix}\right) = \frac{1}{2\pi i} \int_I \frac{\prod_{j=1}^{m} \Gamma(\gamma_j + \tau) \prod_{\ell=1}^{n} \Gamma(1 - \beta_\ell - \tau)}{\prod_{j=m+1}^{q} \Gamma(1 - \gamma_j - \tau) \prod_{\ell=n+1}^{p} \Gamma(\beta_\ell + \tau)} z^{-\tau} d\tau, \tag{4.5}$$

where $z^{-\tau} = \exp\{-\tau(\log|z| + i \arg z)\}$ and the path of integration I runs from $\kappa - i\infty$ to $\kappa + i\infty$ for any fixed $\kappa \in \mathbb{R}$ in such a manner that, if $|\tau|$ is sufficiently large, then $\tau \in I$ lies on the line $\mathrm{Re}\,\tau = \kappa$, all poles of $\Gamma(\gamma_j + \tau)$, $\{-\gamma_j - k\,;\, k \geq 0, j = 1,2,\ldots,m\}$, lie to the left of the path and all poles of $\Gamma(1-\beta_\ell-\tau)$, $\{1-\beta_\ell+k\,;\,k\geq 0, \ell=1,2,\ldots,n\}$, lie to the right of the path.

The integral converges absolutely on any compact set in the sector $S(0,\sigma\pi,\infty)$ if $\sigma=2(m+n)-(p+q)>0$. If $|\arg z|=\sigma\pi$, $\sigma\geq 0$, the integral converges absolutely when $p=q$ if Re $\Xi<-1$ where

$$\Xi=\sum_{j=1}^{q}\gamma_j-\sum_{\ell=1}^{p}\beta_\ell, \tag{4.6}$$

and when $p\neq q$, if with $\tau=\kappa+i\eta$, κ and η real, κ is chosen so that for $\eta\to\pm\infty$

$$(p-q)\kappa>\text{Re }\Xi+1+\frac{1}{2}(p-q). \tag{4.7}$$

Next, the result of the integral representation of the Borel sum is stated as follows, which was proved by myself (cf. [Ich 1,2]).

Theorem 4.2 *Under the condition* (ii) *in Theorem* 4.1 *for the Cauchy data* $\varphi(x)$*, the Borel sum* $u^d(t,x)$ *is given by*

$$\begin{aligned} u^d(t,x) &= \sum\nolimits_{m=0}^{q-1}\int_0^{\infty((dp+\arg\alpha+2\pi m)/q)}\varphi(x+\zeta)\ E_\alpha^{(p,q)}(t,\zeta\omega_q^{-m})\omega_q^{-m}d\zeta \\ &= \int_0^{\infty((dp+\arg\alpha)/q)}\sum\nolimits_{m=0}^{q-1}\varphi(x+\zeta\omega_q^m)\ E_\alpha^{(p,q)}(t,\zeta)d\zeta, \end{aligned} \tag{4.8}$$

where the integration $\int_0^{\infty(\theta)}$ *is taken from* 0 *to* ∞ *along the half line of argument* θ*,* $\omega_q=\exp(2\pi i/q)$ *and the kernel function* $E_\alpha^{(p,q)}(t,\zeta)$ *is given by the following expression*

$$E_\alpha^{(p,q)}(t,\zeta)=\frac{C_{pq}}{\zeta}\times G_{p,q}^{q,0}\left(\frac{p^p}{q^q}\frac{1}{\alpha}\frac{\zeta^q}{t^p}\,\middle|\,\begin{matrix}\boldsymbol{p}/p\\ \boldsymbol{q}/q\end{matrix}\right),\quad C_{pq}=\frac{\Gamma(\boldsymbol{p}/p)}{\Gamma(\boldsymbol{q}/q)}. \tag{4.9}$$

4.2 Deformation of integral paths of the Borel sum

Now, we give a sufficient condition for the deformation of the integral paths of the Borel sum in 0 direction into the real axis. For simplicity of the statement of our main result, we put

$$\begin{cases} \alpha=1\ (\arg\alpha=0) & \text{when}\ \ q\equiv 2,3\ \ (\text{mod}\,4)\\ \alpha=-1\ (\arg\alpha=\pi) & \text{when}\ \ q\equiv 0,1\ \ (\text{mod}\,4). \end{cases} \tag{4.10}$$

Then our result is stated as follows.

Theorem 4.3 *Under the additional conditions for the Cauchy data* $\varphi(x)$ *which are stated below, the integral paths of the Borel sum* (4.8) *in* 0 *direction can be deformed into the* *real axis as the following manner. We divide q rays of integrations in the integral representation (4.8) into two groups R_+ and R_-. Here R_+ (resp. R_-) denotes the group of the rays which are in the right (resp. left) half plane of the complex plane. Then all the integrations along the rays in R_+ (resp. R_-) can be changed into the integration on the positive (resp. negative) real axis.*

- *The case $p = 1$.*

(I) *When q is even, the Cauchy data $\varphi(x)$ can be continued analytically in two sectors $\Delta_{\text{even}\, q} = S(0, \pi - 2\pi/q, \infty) \cup S(\pi, \pi - 2\pi/q, \infty)$ with the same growth condition as in the k-summability in Theorem 4.1.*

(II) *When q is odd, the Cauchy data $\varphi(x)$ can be continued analytically in two sectors $\Delta_{\text{odd}\, q} = S(0, \pi - 3\pi/q, \infty) \cup S(\pi, \pi - \pi/q, \infty)$ with the same growth condition as in the k-summability in Theorem 4.1. We define $\Delta_{\text{odd}\, 3} = S(\pi, 2\pi/3, \infty)$ as an exceptional case. Further, we assume that there exists a positive constant δ such that, in the region $S(\pi, \delta, \infty)$, $\varphi(x)$ has the following decreasing condition of polynomial order*

$$|\varphi(x)| \leq \frac{C}{|x|^{q/2(q-1)+\lambda}}, \quad x \in S(\pi, \delta, \infty), \tag{4.11}$$

for some positive constants λ and C.

- *The case $p = 2$ and q is even.*

The Cauchy data $\varphi(x)$ can be continued analytically in two sectors $\Delta_{\text{even}\, q} = S(0, \pi - 2\pi/q, \infty) \cup S(\pi, \pi - 2\pi/q, \infty)$ with the same growth condition as in the k-summability in Theorem 4.1. Further, we assume that there exists a positive constant δ such that, in the region $S(0, \delta, \infty) \cup S(\pi, \delta, \infty)$, $\varphi(x)$ has the following decreasing condition of polynomial order

$$|\varphi(x)| \leq \frac{C}{|x|^{q/2(q-2)+\lambda}}, \quad x \in S(0, \delta, \infty) \cup S(\pi, \delta, \infty), \tag{4.12}$$

for some positive constants λ and C.

4.3 Proof of Theorem 4.3

Before giving the proof of Theorem 4.3, we prepare some properties of Meijer G-function.

We remark that G-function in the integral kernel of the Borel sum is given by the following expression

$$G^{q,0}_{p,q}\left(z \,\middle|\, \begin{matrix} \boldsymbol{p}/p \\ \boldsymbol{q}/q \end{matrix}\right) = \frac{1}{2\pi i} \int_I \frac{\Gamma(\boldsymbol{q}/q + \tau)}{\Gamma(\boldsymbol{p}/p + \tau)} z^{-\tau} d\tau, \quad z = \frac{p^p}{q^q} \frac{1}{\alpha} \frac{\zeta^q}{t^p}, \tag{4.13}$$

where we choose the path I such that if $|\tau|$ is sufficiently large, then $\tau \in I$ lies on the line $\mathrm{Re}\,\tau = \kappa < -1/(q-p)$ and it is possible to take such a path. (see the definition (4.5) of Meijer G-function.) Then G-function is well-defined in

$$|\arg z| \le \frac{q-p}{2}\pi. \tag{4.14}$$

In this region, G-function has the following estimates as $|z| \to \infty$ (cf. [Luk, p. 179])

$$\left| G_{p,q}^{q,0}\left(z \,\middle|\, \begin{matrix} \boldsymbol{p}/p \\ \boldsymbol{q}/q \end{matrix} \right) \right| \le \begin{cases} C_\varepsilon \exp\left(-\sigma_\varepsilon |z|^{1/(q-p)}\right), & |\arg z| \le \frac{q-p}{2}\pi - \varepsilon, \\ C|z|^{1/2(q-p)}, & \frac{q-p}{2}\pi - \varepsilon \le |\arg z| \le \frac{q-p}{2}\pi. \end{cases} \tag{4.15}$$

From these estimates, the integral kernel of the Borel sum has the following estimates as $|\zeta| \to \infty$ for a fixed $t > 0$

$$\begin{aligned} |E_\alpha^{(p,q)}(t, \zeta\omega_q^{-m})| &= \left| \frac{C_{p,q}}{\zeta} G_{p,q}^{q,0}\left(\frac{c_{pq}}{\alpha} \frac{(\zeta\omega_q^{-m})^q}{t^p} \,\middle|\, \begin{matrix} \boldsymbol{p}/p \\ \boldsymbol{q}/q \end{matrix} \right) \right| \\ &\le \begin{cases} C_\varepsilon \exp\left(-\sigma_\varepsilon |\zeta|^{q/(q-p)}/t^{p/(q-p)}\right), & |q\arg(\zeta\omega_q^{-m}) - \arg\alpha| \le \frac{q-p}{2}\pi - \varepsilon, \\ C|\zeta|^{q/2(q-p)-1}/t^{p/2(q-p)}, & \frac{q-p}{2}\pi - \varepsilon \le |q\arg(\zeta\omega_q^{-m}) - \arg\alpha| \le \frac{q-p}{2}\pi, \end{cases} \end{aligned} \tag{4.16}$$

where $C_{p,q} = \Gamma(\boldsymbol{p}/p)/\Gamma(\boldsymbol{q}/q)$ and $c_{pq} = p^p/q^q$.

G-function can be evaluated as a sum of residues as follows.

$$G_{p,q}^{q,0}\left(z \,\middle|\, \begin{matrix} \boldsymbol{p}/p \\ \boldsymbol{q}/q \end{matrix} \right) = \sum_{j=1}^{q} \frac{\Gamma(\widehat{\boldsymbol{q}}_j/q - j/q)}{\Gamma(\boldsymbol{p}/p - j/q)} z^{j/q} {}_pF_{q-1}\left(\begin{matrix} 1 + j/q - \boldsymbol{p}/p \\ 1 + j/q - \widehat{\boldsymbol{q}}_j/q \end{matrix} ; (-1)^{q-p} z \right). \tag{4.17}$$

Here ${}_pF_{q-1}$ denotes the generalized hypergeometric series which is defined by

$${}_pF_{q-1}\left(\begin{matrix} \beta_1, \beta_2, \ldots, \beta_p \\ \gamma_1, \gamma_2, \ldots, \gamma_{q-1} \end{matrix} ; z \right) = \sum_{n \ge 0} \frac{(\beta_1)_n (\beta_2)_n \cdots (\beta_p)_n}{(\gamma_1)_n (\gamma_2)_n \cdots (\gamma_{q-1})_n} \frac{z^n}{n!}, \tag{4.18}$$

where $(c)_n = \Gamma(c+n)/\Gamma(c)$ $(c \in \mathbb{C})$.

From (4.17) and $z =$ (constant) $\times \zeta^q/t^p$, we notice that G-function in the integral kernel and itself $E_\alpha^{(p,q)}(t, \zeta)$ are entire functions and single-valued with respect to ζ for a fixed t.

Proof of Theorem 4.3. We only give the proof in the case where $p = 2$ and $q = 4n$ $(n \ge 1)$, because the proofs in the other cases are given in the similar way.

In this case, we note that $\alpha = -1$ $(\arg\alpha = \pi)$ and the Borel sum $u^0(t,x)$ is given by the following expression

$$u^0(t,x) = \sum_{m=0}^{q-1} \int_0^{\infty((2\pi m + \pi)/q)} \varphi(x+\zeta)\, E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m}) \omega_q^{-m} d\zeta. \tag{4.19}$$

Now, we fix $t > 0$. It is enough to prove the following formula when the Cauchy data $\varphi(x)$ satisfies the conditions (II) in Theorem 4.3

$$\int_0^{\infty((2\pi m+\pi)/q)} \varphi(x+\zeta)E_{-1}^{(2,q)}(t,\zeta\omega_q^{-m})\omega_q^{-m}d\zeta = \begin{cases} \displaystyle\int_0^{+\infty} \varphi(x+\zeta)E_{-1}^{(2,q)}(t,\zeta\omega_q^{-m})\omega_q^{-m}d\zeta & \text{if the ray is in } R_+, \\ \displaystyle\int_0^{-\infty} \varphi(x+\zeta)E_{-1}^{(2,q)}(t,\zeta\omega_q^{-m})\omega_q^{-m}d\zeta & \text{if the ray is in } R_-. \end{cases} \tag{4.20}$$

In the following, we prove the expression (4.20). In the case where $q = 4n$, the rays of integrations with $m = 0, 1, \ldots, n-1, 3n, \ldots, q-1$ (resp. $m = n, \ldots, 3n-1$) in the expression (4.19) belong to R_+ (resp. R_-).

From the estimates (4.16), we see that each integral kernel $E_{-1}^{(2,q)}(t,\zeta\omega_q^{-m})$ with $m = 0, 1, \ldots, n-2, 3n+1, \ldots, q-1$ (resp. $m = n+1, \ldots, 3n-2$) in (4.19) has the exponential decreasing estimate of order $q/(q-2)$ as $|\zeta| \to \infty$ in each sector

$$-\frac{q-4m-4}{2q}\pi + \frac{\varepsilon}{q} \le \arg\zeta \le \frac{q+4m}{2q}\pi - \frac{\varepsilon}{q} \tag{4.21}$$

which contains the positive (resp. negative) real axis.

We have to remark that the cases where $m = n-1, n, 3n-1$ and $3n$ are exceptional, because the integral kernels $E_{-1}^{(2,q)}(t,\zeta\omega_q^{-m})$ with $m = n-1$ and $m = 3n$ (resp. $m = n$ and $m = 3n-1$) in (4.19) do not have the exponential decreasing estimate as $\zeta \to +\infty$ (resp. $\zeta \to -\infty$) on the positive (resp. negative) real axis. Indeed, from the estimates (4.16), the integral kernels $E_{-1}^{(2,q)}(t,\zeta\omega_q^{-m})$ with $m = n-1$ and $m = 3n$ have the following estimates for some $\varepsilon > 0$

$$\left|E_{-1}^{(2,q)}(t,\zeta\omega_q^{-n+1})\right| \le \begin{cases} C_\varepsilon \exp\left(-\sigma_\varepsilon|\zeta|^{q/(q-2)}/t^{1/(q-2)}\right), & \varepsilon/q \le \arg\zeta \le \pi - 2\pi/q - \varepsilon/q, \\ C|\zeta|^{q/2(q-2)-1}/t^{1/2(q-2)}, & 0 \le \arg\zeta \le \varepsilon/q, \end{cases} \tag{4.22}$$

$$\left|E_{-1}^{(2,q)}(t,\zeta\omega_q^{-3n})\right| \le \begin{cases} C_\varepsilon \exp\left(-\sigma_\varepsilon|\zeta|^{q/(q-2)}/t^{1/(q-2)}\right), & \pi + 2\pi/q + \varepsilon/q \le \arg\zeta \le 2\pi - \varepsilon/q, \\ C|\zeta|^{q/2(q-2)-1}/t^{1/2(q-2)}, & 2\pi - \varepsilon/q \le \arg\zeta \le 2\pi, \end{cases} \tag{4.23}$$

and the integral kernels $E_{-1}^{(2,q)}(t,\zeta\omega_q^{-m})$ with $m = n$ and $m = 3n-1$ have the following estimates for some $\varepsilon > 0$

$$\left|E_{-1}^{(2,q)}(t,\zeta\omega_q^{-n})\right| \tag{4.24}$$

$$\leq \begin{cases} C_\varepsilon \exp\left(-\sigma_\varepsilon |\zeta|^{q/(q-2)}/t^{1/(q-2)}\right), & 2\pi/q + \varepsilon/q \leq \arg\zeta \leq \pi - \varepsilon/q, \\ C|\zeta|^{q/2(q-2)-1}/t^{1/2(q-2)}, & \pi - \varepsilon/q \leq \arg\zeta \leq \pi, \end{cases}$$

$$(4.25) \quad \left|E_{-1}^{(2,q)}(t, \zeta\omega_q^{-3n+1})\right| \leq \begin{cases} C_\varepsilon \exp\left(-\sigma_\varepsilon |\zeta|^{q/(q-2)}/t^{1/(q-2)}\right), & \pi + \varepsilon/q \leq \arg\zeta \leq 2\pi - 2\pi/q - \varepsilon/q, \\ C|\zeta|^{q/2(q-2)-1}/t^{1/2(q-2)}, & \pi \leq \arg\zeta \leq \pi + \varepsilon/q. \end{cases}$$

Therefore if the Cauchy data $\varphi(x)$ is analytic in $\Delta_{\text{odd}\, q}$ and has the growth condition of exponential order at most $q/(q-2)$ there, then for a small fixed $\varepsilon > 0$ we have

$$(4.26) \quad \int_0^{\infty((2\pi m+\pi)/q)} \varphi(x+\zeta) E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m})\omega_q^{-m} d\zeta$$

$$= \begin{cases} \int_0^{\infty(0+\varepsilon/q)} \varphi(x+\zeta) E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m})\omega_q^{-m} d\zeta & \text{if the ray is in the first quadrant,} \\ \int_0^{\infty(2\pi-\varepsilon/q)} \varphi(x+\zeta) E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m})\omega_q^{-m} d\zeta & \text{if the ray is in the fourth quadrant,} \\ \int_0^{\infty(\pi-\varepsilon/q)} \varphi(x+\zeta) E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m})\omega_q^{-m} d\zeta & \text{if the ray is in the second quadrant,} \\ \int_0^{\infty(\pi+\varepsilon/q)} \varphi(x+\zeta) E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m})\omega_q^{-m} d\zeta & \text{if the ray is in third quadrant.} \end{cases}$$

Here since the integral kernels are single-valued with respect to ζ for a fixed t, we may change $2\pi - \varepsilon/q$ to $0 - \varepsilon/q$ for argument of the second expression in the right hand side of (4.26).

Further, if the Cauchy data $\varphi(x)$ has the polynomial decreasing condition (4.12) in the sector $S(0, \delta, \infty) \cup S(\pi, \delta, \infty)$ with $\delta > \varepsilon/q$, then the absolute integrability on the real axis do hold for all integrals in the right hand side of (4.26), and we obtain the formula (4.20).

Finally, in the case where $q = 4n$ $(n \geq 1)$, we have the following formula

$$(4.27) \quad u^0(t,x) = \int_0^{+\infty} \varphi(x+\zeta) \left\{ \sum_{m=0}^{n-1} + \sum_{m=3n}^{q-1} \right\} E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m})\omega_q^{-m} d\zeta + \int_{-\infty}^{0} \varphi(x+\zeta) \left\{ -\sum_{m=n}^{3n-1} E_{-1}^{(2,q)}(t, \zeta\omega_q^{-m})\omega_q^{-m} \right\} d\zeta, \quad t > 0.$$

This completes the proof of Theorem 4.3. □

Remark 4.4 *In* [Ich 3], *we proved that the integral representation of the Borel sum with integral path along the real axis, which is obtained in such a manner as in Theorem 4.3,* *just coincides with that of the classical solution when $(p,q) = (1,3)$ and $(1,4)$. In a forthcoming paper* [Ich 4], *we shall prove that the same results hold when $p = 1$ and q is arbitrary. As we shall show in the below, the same results also hold when $p = 2$ and q is even.*

We put $q = 2\tilde{q}$. The classical solution of the Cauchy problem (4.1) *is given by*

$$u_c(t,x) = \int_{-\infty}^{+\infty} \varphi(x+y) K_\alpha^{\tilde{q}}(t,y)dy, \quad t>0,\ x\in\mathbb{R}, \tag{4.28}$$

where the integral kernel $K_\alpha^{\tilde{q}}(t,y)$ is given by

$$K_\alpha^{\tilde{q}}(t,y) = \frac{1}{2}\frac{1}{(\tilde{q}t)^{1/\tilde{q}}}\left[\widetilde{K_+}\left(\frac{y}{(\tilde{q}t)^{1/\tilde{q}}}\right) + \widetilde{K_-}\left(\frac{y}{(\tilde{q}t)^{1/\tilde{q}}}\right)\right], \tag{4.29}$$

with

$$\widetilde{K_\pm}(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \exp\left(zs \pm \alpha^{1/2}\frac{(-s)^{\tilde{q}}}{\tilde{q}}\right) ds, \quad z\in\mathbb{C}. \tag{4.30}$$

Here we assume that the Cauchy data $\varphi(x)$ belongs to Schwartz' rapidly decreasing functions in x variable.

We remark that the function $\widetilde{K_\pm}(y/(\tilde{q}t)^{1/\tilde{q}})/(\tilde{q}t)^{1/\tilde{q}}$ is the fundamental solution of the equation $(\partial_t \mp \alpha^{1/2}\partial_y^{\tilde{q}})u(t,y) = 0$.

Now, in the case where $q = 4n$, we can prove the following formula for $t > 0$

$$K_\alpha^{\tilde{q}}(t,y) = \begin{cases} \left\{\sum_{m=0}^{n-1} + \sum_{m=3n}^{q-1}\right\} E_{-1}^{(2,q)}(t, y\omega_q^{-m})\omega_q^{-m}, & y>0, \\ -\sum_{m=n}^{3n-1} E_{-1}^{(2,q)}(t, y\omega_q^{-m})\omega_q^{-m}, & y<0. \end{cases} \tag{4.31}$$

Indeed, by using the multiplication formula of the Gamma function

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2), \tag{4.32}$$

with $z, z+1/2 \neq 0, -1, -2, \ldots$, we have

$$C_{2,q}G_{2,q}^{q,0}\left(\frac{2^2}{q^q}\frac{1}{\alpha}\frac{y^q}{t^2}\,\middle|\,\begin{matrix}\mathbf{2}/2\\ \mathbf{q}/q\end{matrix}\right) = \frac{1}{2}C_{1,\tilde{q}}G_{1,\tilde{q}}^{\tilde{q},0}\left(\frac{1}{\tilde{q}^{\tilde{q}}}\frac{1}{\alpha^{1/2}}\frac{y^{\tilde{q}}}{t}\,\middle|\,\begin{matrix}1\\ \tilde{\mathbf{q}}/\tilde{q}\end{matrix}\right). \tag{4.33}$$

where $C_{2,q}$ and $C_{1,\tilde{q}}$ are constants which are given by (4.9). *Therefore, it can be reduced to the problem of the case where $p = 1$ (cf.* [Ich 4]). *We omit the details, but they will be published elsewhere.*

5 Proof of Proposition 3.1

We give the proof of Proposition 3.1.

We recall that

$$P = \prod_{j=1}^{\mu} P_j^{\ell_j}, \quad P_j = \partial_t^p - \alpha_j \partial_x^q. \tag{5.1}$$

First of all, we can choose ν constants $c_{mn}\,(1 \le m \le \mu; 1 \le n \le \ell_m)$ such that the following identity for the operator holds

$$\partial_t^{p(\nu-1)} = \sum_{m=1}^{\mu}\sum_{n=1}^{\ell_{mu}} c_{mn}\partial_t^{p(n-1)} \prod_{j=1, j\ne m}^{\mu} P_j^{\ell_j} \cdot P_m^{\ell_m - n}. \tag{5.2}$$

Indeed, it is enough to compare coefficients of ∂_t^{pj} in the both hand sides (cf. Remark 5.1 below).

Let $U(t,x)$ be a formal solution of the Cauchy problem (1.2). We operate $U(t,x)$ to this identity (5.2) and we put

$$U[m,n](t,x) := \prod_{j=1, j\ne m}^{\mu} P_j^{\ell_j} \cdot P_m^{\ell_m - n} U(t,x). \tag{5.3}$$

Then we see that $U[m,n](t,x)$ satisfies the following Cauchy problem

$$\begin{cases} P_m^n U[m,n](t,x) = 0, \\ \partial_t^k U[m,n](0,x) = 0 \quad (0 \le k \le pn-2), \\ \partial_t^{pn-1} U[m,n](0,x) = \varphi(x). \end{cases} \tag{5.4}$$

Because $P_m^n U[m,n](t,x)$ is equal to $PU(t,x)$, and $U(t,x)$ is the formal solution of the Cauchy problem (1.2).

Moreover, we consider the Cauchy problem (3.3)

$$\begin{cases} P_m u(t,x) = 0, \\ u(0,x) = \varphi(x), \\ \partial_t^k u(0,x) = 0 \quad (1 \le k \le p-1). \end{cases} \tag{5.5}$$

Let $U[m,n](t,x)$ and $u_m(t,x)$ be formal solutions of the Cauchy problem (5.4) and (5.5), respectively. Then each formal solution is given by the following series

$$U[m,n](t,x) = \sum_{j\ge 0} \alpha_m^j \frac{(j+1)_{n-1}}{(n-1)!}\varphi^{(qj)}(x)\frac{t^{pj+pn-1}}{(pj+pn-1)!}, \tag{5.6}$$

$$u_m(t,x) = \sum_{j\ge 0} \alpha_m^j \varphi^{(qj)}(x)\frac{t^{pj}}{(pj)!}, \tag{5.7}$$

where $(j+1)_{n-1} = \Gamma(j+n)/\Gamma(j+1)$. The relationship between these formal solutions is given by the following formula

$$U[m,n](t,x) = D_t^{-(p-1)} \frac{[(1/p)\delta_t]_{n-1}}{(n-1)!} D_t^{-p(n-1)} u_m(t,x). \tag{5.8}$$

Therefore by substituting (5.8) to (5.2) in which we operate $U(t,x)$, and by operating integral operator $D_t^{-p(\nu-1)}$ in the both hand sides, we have

$$U(t,x) = D_t^{-p(\nu-1)} \sum_{m=1}^{\mu} \sum_{n=1}^{\ell_{mu}} c_{mn} \partial_t^{p(n-1)} \cdot D_t^{-(p-1)} \frac{[(1/p)\delta_t]_{n-1}}{(n-1)!} D_t^{-p(n-1)} u_m(t,x). \tag{5.9}$$

By calculating integral and differential operators, we have the desired result (3.1). □

Remark 5.1 *Constants c_{mn} in (5.2) are given in the following way.*

We put

$$f_{m,n} = f_{m,n}[\alpha_1, \ldots, \alpha_\mu] := \prod_{j=1}^{\mu} \alpha_j^{\ell_j} / \alpha_m^n,$$

$$\partial_\alpha := \sum_{j=1}^{\mu} \partial_{\alpha_j},$$

$$\Delta_k := \frac{\partial_\alpha^k}{k!} = \sum_{\substack{0 \le k_1, k_2, \ldots, k_\mu \le k \\ k_1 + k_2 + \cdots + k_\mu = k}} \frac{1}{k_1! k_2! \cdots k_\mu!} \partial_{\alpha_1}^{k_1} \partial_{\alpha_2}^{k_2} \cdots \partial_{\alpha_\mu}^{k_\mu}.$$

Then c_{mn} are determined as a unique solution of the following system of linear equations

$$\mathcal{A}\vec{c} = \vec{e}, \tag{5.10}$$

where $\mathcal{A}$ denotes a $\nu \times \nu$ matrix which is given by

$$\mathcal{A} = \left(\begin{array}{ccccccccccc} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & & 1 & \cdots & 1 \\ \Delta_{\nu-2} f_{1,1} & \Delta_{\nu-3} f_{1,2} & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \Delta_0 f_{1\ell_1} & \vdots & & \Delta_0 f_{2\ell_2} & & \vdots & & \Delta_0 f_{\mu\ell_\mu} \\ \vdots & \vdots & \iddots & 0 & \vdots & \iddots & & & \vdots & \iddots & \\ \Delta_1 f_{1,1} & \Delta_0 f_{1,2} & \iddots & \vdots & \vdots & & O & \cdots & \vdots & & O \\ \Delta_0 f_{1,1} & 0 & \cdots & 0 & \Delta_0 f_{2,1} & & & & \Delta_0 f_{\mu 1} & & \end{array} \right)$$

$\vec{c} = {}^t(c_{11}, c_{12}, \ldots, c_{1\ell_1}, c_{21}, \ldots, c_{2\ell_2}, \ldots, c_{\mu 1}, \ldots, c_{\mu\ell_\mu})$ and $\vec{e} = {}^t(1, 0, \ldots, 0)$ are ν-column vectors.

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