数理解析研究所講究録 1424 巻 2005 年 200-213

An application of the Bethe ansatz method to certain classes of ODE *

J. Suzuki †

Department of Physics, Faculty of Science,
Shizuoka University,
836 Ohya, Shizuoka 422,
Japan

November 9, 2004

Abstract

We consider eigenvalue problems in quantum mechanics in one dimension. The strategy, developed by Sibuya, Voros, Dorey and Tateo and Bazhanov and his collaborators, has been applied to the problem with a recent tool in the study of the integrable system. The spectral problem then reduces to the study of the coupled nonlinear integral equations, which is again hard to treat analytically. We discuss an approximation scheme and its application to a problem in the PT symmetric quantum systems.

1 Introduction

The eigenvalue problem of a one-body 1D Schrödinger operator is the most fundamental subject in quantum mechanics. Still, it provides vivid materials of research and the testing ground for many ideas in the asymptotic analysis.

The exact WKB method [1, 2, 3] is unique in the sense that it provides non-perturbative information on the analytical structure of wavefunctions and spectral properties.

The mysterious link has been unveiled in [4, 5, 6], between the exact WKB method and the integrable models in the field theories, which is referred to as " the ODE/IM correspondence".

---

*Based on the talk given at "Recent Trends in Exponential Asymptotics " (28 June-2 July) RIMS, Kyoto university

†e-mail: sjsuzuk@ipc.shizuoka.ac.jp

0+1 D Quantum Mechanics: all eigenstates (values)	$\leftrightarrow$	1+1D field theory a (vacuum) state

The correspondence opens up a novel quantitative way in the analysis of the spectral problem (see, e.g., references in [7]). As a result all the information on the spectral problem and the Stokes multiplier can be obtained by solving a simple set of coupled algebraic equations, the Bethe ansatz equations: each root of the equation determines a energy level.

The price to pay is that the number of algebraic equations becomes infinitely many. This seems to be a fatal drawback at the first sight. The difficulty is partly resolved by the use of the recent technique in the integrable system, the nonlinear integral equations (NLIE). One then reduces the problem to analysis of a few NLIE. The numerical investigation through the NLIE is quite efficient and the high precision calculation is made possible for many examples.

In spite of the success in the numerical investigations, the analytical treatment of NLIE suffers from its nonlinear nature. In this report, we discuss a preliminary trial on the approximation scheme of the Bethe ansatz equations and the NLIE to develop more intuitive understanding of the resultant equations. The motivation comes from the recent finding by Dorey, Dunning and Tateo [8] on the impressive phase diagram in a PT symmetric quantum system. The wobbling phase separation curve may be obtained by solving the NLIE numerically. At the same time, it is more desirable to understand the curve by simpler equations.

2 The ODE/IM correspondence

We follow the arguments in [9, 10] to derive the Bethe ansatz equation for the spectral problem.

The first step is the analytic continuation of x, the original coordinate variable, to a complex number. Following the strategy in [2, 1], we then consider the connection problem in the complex plane.

In case that the potential term possesses a certain symmetry w.r.t. the rotation in the complex plane associated with the appropriate changes in the parameters, the problem may be simplified. A remarkable example was scrutinized in [1].

To be specific, we restrict ourselves to the eigenvalue problem,

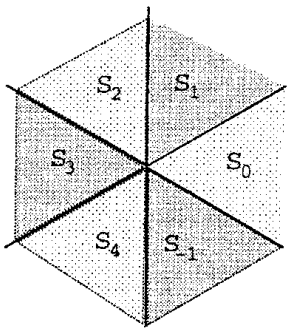
$$\mathcal{H}^{\epsilon}\Psi(x,\epsilon\alpha,E) = \Big(-\frac{d^2}{dx^2} + V(x,\epsilon\alpha)\Big)\Psi(x,\epsilon\alpha,E) = E\Psi(x,\epsilon\alpha,E)$$
$$V(x,\alpha) = x^{2M} + \alpha x^{M-1} + \frac{\ell(\ell+1)}{x^2} \tag{1}$$

Throughout this report we assume $M > 1$ and $\alpha \geq 0$. In the presence of the angular momentum term, we assume that the model is originally defined on the half line $x \in [0,\infty)$.

The potential term has an obvious symmetry,

$$V(q^{-1}x, -\alpha) = q^2 V(x, \alpha)$$

provided that $q = \mathrm{e}^{\frac{\pi}{M+1}i}$. This results the simplification of the problem. To state this further, we prepare some notations. Since the equation (1) has an irregular singular point at infinity, it is useful to introduce sectors in the complex plane. We define $\mathcal{S}_j$, a sector in the complex plane by $\mathcal{S}_j = \{x| \, |\arg x - \frac{j\pi}{M+1}| \leq \frac{\pi}{2M+2}\}$.



The recessive solution $\phi(x, \alpha, E)$ will be of our interest which behaves

$$\phi(x, \alpha, E) \sim x^{-\frac{M+\alpha}{2}} \mathrm{e}^{-\frac{x^{M+1}}{M+1}}, \quad \phi(x, \alpha, E) \sim -x^{\frac{M-\alpha}{2}} \mathrm{e}^{-\frac{x^{M+1}}{M+1}}$$

as $x \to \infty$ in $\mathcal{S}_0$.
We define the following function for convenience,

$$y_j^{(\epsilon)} := \frac{q^{j/2-\epsilon\alpha j/2}}{\sqrt{2i}} \phi(xq^{-j}, \epsilon\alpha, q^{2j}E).$$

for $j \in \mathbb{Z}$.

We immediately check

Lemma 1 *In the sector $\mathcal{S}_j$, the pair $(y_j^{(\epsilon_j)}, y_{j+1}^{(\epsilon_{j+1})})$ consists a basis to the solution of the eigenvalue problem with $\mathcal{H}(x, \epsilon\alpha)$ and with $\epsilon_j = \epsilon(-1)^j$. The first one $Y_j := y_j^{(\epsilon_j)}$ represents the recessive solution in the sector.*

We then introduce Stokes multipliers $\tau_j^{\pm}$ for the connection problem of the neighboring sectors. This is accomplished by the relation,

$$\tau_j^{(\epsilon_j)} Y_{j+1} = Y_j + Y_{j+2}. \tag{2}$$

It connects the two problems of which physical properties are different, the system with a simple convex potential $\mathcal{H}^+$ and the one with a double well potential $\mathcal{H}^-$.

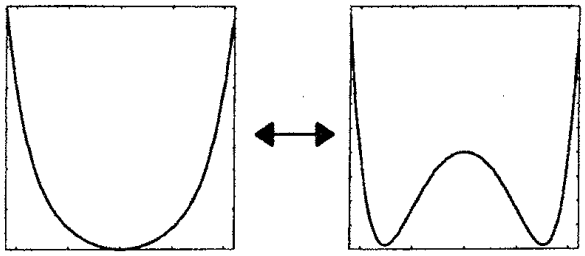


Figure 1: The double-well potential ($\mathcal{H}^-$)and the convex potential ($\mathcal{H}^+$) are connected naturally in view of the rotation in the complex plane.

There is another set of basis, characterized by the behavior near the origin. We denote by $\chi^\epsilon_\pm$ solutions to (1) which behave $\chi^\epsilon_+ \sim x^{\ell+1}$ ($\chi^\epsilon_- \sim x^{-\ell}$) respectively as $x \to 0$. The linear relation should hold

$$\phi(x, \epsilon\alpha, E) = D^\epsilon_+(E)\chi^\epsilon_+ + D^\epsilon_-(E)\chi^\epsilon_-.$$

When E is a zero of $D^\epsilon_-(E)$, the above relation claims that we have a solution to (1) which satisfies the boundary conditions $\psi \sim x^{\ell+1}$ as $x \to 0$ and $\psi \sim 0$ as $x \to \infty$. Thus $D^\epsilon_{\epsilon'}(E)$ can be identified with the spectral determinant[1]. For $M > 1$, we assume the Hadamard factorization of $D^\epsilon_{\epsilon'}(E)$,

$$D^\epsilon_{\epsilon'}(E) = \prod_j \Big(1 - \frac{E}{E^{(\epsilon)}_{\epsilon',j}}\Big)$$

The spectral determinant can be represented by Wronskian between ϕ and $\chi_\pm$. Using this and analogous one in (2) we arrive at

$$T^{-\epsilon}(E)D^\epsilon_{\epsilon'}(E) = q^{\frac{\epsilon\alpha-\epsilon'}{2}} D^{-\epsilon}_{\epsilon'}(q^2E) + q^{\frac{-\epsilon\alpha+\epsilon'}{2}} D^{-\epsilon}_{\epsilon'}(q^{-2}E) \tag{3}$$

where $T^\epsilon(E) = \tau^{(\epsilon)}_{-1}$.

This can be identified with the Baxter's TQ relation in the integrable system (acting on an eigenspace). It represents the essence of the ODE/IM correspondence. We come to the main point of the argument. The lhs is vanishing at zeros of Stokes multiplier or $E = E^{(\epsilon)}_{\epsilon',j}$. The requirement that the rhs must be null at the latter leads to algebraic equations (Bethe ansatz equations),

$$A^{(\epsilon)}_{\epsilon'}(E^{(\epsilon)}_{\epsilon',j}) = 0 \tag{4}$$

where

$$a^{(\epsilon)}_{\epsilon'}(E) := q^{\epsilon\alpha-\epsilon'} \frac{D^{(-\epsilon)}_{\epsilon'}(q^2E)}{D^{(-\epsilon)}_{\epsilon'}(q^{-2}E)} \tag{5}$$

$$A^{(\epsilon)}_{\epsilon'}(E) := 1 + a^{(\epsilon)}_{\epsilon'}(E). \tag{6}$$

[1]We adopt the convention of the lower index s.t. it coincides with the parity of the wave function when $\ell = 0$

This is a remarkable consequence. Once if the energy spectra of Hamiltonian with simple convex potential ($\epsilon = 1$) is known , then all the energy spectra of Hamiltonian with the double well potential can be determined by solving (4), and vice versa. This however suffers from a flaw in practice. It is impossible to recover infinite number of energy levels. These "infinitely many input data" are however indispensable for evaluating the rhs of (4) and for locating the roots.

As an efficient method in practice, one may substitute some exact low-lying energy levels and replace higher values by those obtained from the WKB method, as proposed in [11]. As a demonstration of the simple-minded approximation, we consider the determination of the energy level of $\mathcal{H}^-$ by replacing all levels of $\mathcal{H}^+$ by the data from the WKB method for $M = 3$. The estimation by the WKB method is simple. We replace the lower levels by those from a simple harmonic oscillator and higher ones by those from the sextic potential .

As a result, we have,

$$a_{\epsilon'}^{(-)}(E) = q^{-\alpha-\epsilon'}\frac{D_{\epsilon'}^{+}(q^2E)}{D_{\epsilon'}^{+}(q^{-2}E)} \sim \frac{\Gamma(k_0+\frac{3}{2}-\frac{\epsilon'}{4}-i\frac{E}{4})\Gamma(\frac{3}{2}-\frac{\epsilon'}{4}+i\frac{E}{4})}{\Gamma(k_0+\frac{3}{2}-\frac{\epsilon'}{4}+i\frac{E}{4})\Gamma(\frac{3}{2}-\frac{\epsilon'}{4}-i\frac{E}{4})}\mathrm{e}^{i\delta(\frac{x}{E_{k_0+1}})-\frac{i\pi(\alpha+\epsilon')}{M+1}}$$

where $k_0 = \frac{1}{2}(4\alpha(\frac{b_0}{2\pi})^3 + \frac{\epsilon'}{2} - 1)$, $E_k := \frac{2\pi}{b_0}(k+\frac{1}{2})^{\frac{3}{2}}$ and

$$\delta(x) = (k_0 + \frac{3}{2} - \frac{\epsilon'}{4})\Big(\frac{1}{i}\log\frac{1+ix}{1-ix} - 3x({}_2F_1(1,\frac{1}{3},\frac{4}{3},ix) + {}_2F_1(1,\frac{1}{3},\frac{4}{3},-ix)\Big)$$

The energy levels associated to the double well potential problem, $\mathcal{H}^-$, are estimated by solving $a_{\epsilon'}^{(-)}(E) = -1$. The lowest few levels for $M = 3$ and $\alpha = 39$ and $\ell = 0$ are presented in table 1

n	$E_n^{(-)}$ direct	$E_n^{(-)}$ approx.
1	- 81.5288541540	-85.3050445336
2	- 81.5288541503	-84.2266660522
3	- 58.4951145514	-63.4034843700
4	- 58.4951124012	-62.0922740090
5	- 37.6184028692	-43.2166547767
6	- 37.6179452538	-41.5223431155

Table 1: eigenvalues $M = 3$ and $\alpha = 39, \ell = 0$

In spite of poor numerical precision, it may be still interesting to realize that energy levels of a double well Hamiltonian can be in principle estimated from those associated to a simple convex potential.

Unless one devises the way to handle "infinitely many inputs", however, the exact relation (4) is formally correct but is of no use for the exact treatment of the problem. To resolve this, we use an analogy to many-body problems in physics. The infinitely many

levels are identified with infinitely many degrees of freedom in the many-body system. A lesson from the latter theory is to divide the whole degrees of freedom into, roughly, two parts. The first part is referred to as the Fermi sea , which is stable and consists of infinitely many degrees of freedom. The second corresponds to the finitely many excitations above the sea.

We realize this separation of the degrees of freedom by applying the idea of nonlinear integral equations (NLIE), developed in the study of integrable systems[12, 13].

The figure 2 shows the pattern of zeros and singularities of $A^{(+)}(E)$ when α is less than M. (For the moment, we shall forget about the dependency on the ℓ (ϵ') .)

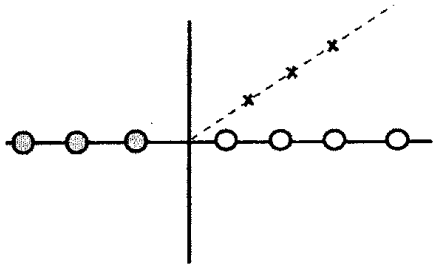


Figure 2: The zeros (circle) and singularities (cross) of $A^{(+)}(E)$.

The zeros on the positive real axis coincide with those of $D^{(+)}(E)$ while those on the negative real axis agree with zeros of Stokes multipliers $T^{(-)}(E)$. The singularities are images of $q^2E_j^{(-)}$ s.t. $E_j^{(-)} \geq 0$. We remark that these zeros and poles are order 1.

We start by rewriting $\ln a^{(-)}$ as as

$$\ln a^{(-)}(E) = \sum_j \ln\Big(\frac{1-\frac{q^2E}{E_j^{(-)}}}{1-\frac{q^{-2}E}{E_j^{(-)}}}\Big).$$

Then the summation in the rhs can be rewritten in terms of the integral over the contour C_1 depicted in fig. 3

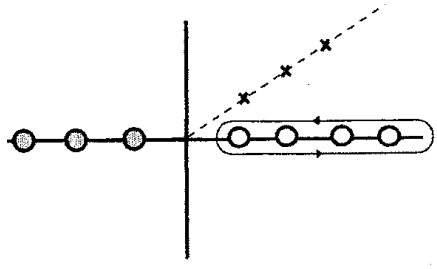


Figure 3: The integration contour in eq. (7).

$$\ln a^{(-)}(E) = \frac{1}{2\pi i}\int_{C_1} \ln\Big(\frac{1-\frac{q^2E}{E'}}{1-\frac{q^{-2}E}{E'}}\Big)\frac{d}{dE'}\ln A^{(+)}(E'). \tag{7}$$

This equation reminds us of the notation of "resurgence": the information of the capital $A = 1 + a$ around the branch cut line determines a (if even the upper signature is neglected.)

In case of the equation for $\ln a^{(+)}(E)$, an extra treatment is necessary. Again, we like to rewrite the summation by a contour integral over $\frac{d}{dE'} \ln A^{(-)}(E')$. Contrary to the previous case, some relevant zeros of $A^{(-)}(E)$ distribute also on the negative real axis. Thus one must be careful in the choice of the contour $\mathcal{C}_2$ (fig. 4 left) such that only zeros related to $E_j^{(-)}$ must be encircled, not those related to zeros of Stokes multipliers.

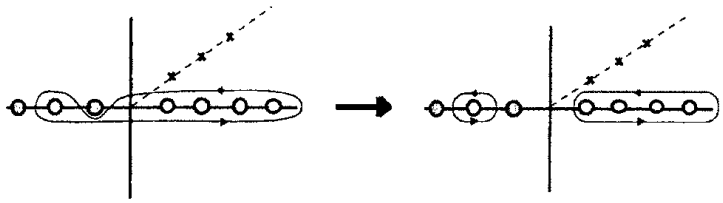


Figure 4: The integration contour needs the deformation.

The Cauchy theorem tells that the contour $\mathcal{C}_2$ can be break into pieces: $\mathcal{C}_1$ which tends to infinity and small portions encircling $E_j^{(-)}$ on the negative real axis. (fig. 4 right)

The number of the latter circles may be finitely many for finite values of α. Our rough idea is to identify the contributions from the contour integral over $\mathcal{C}_1$ as the one from the "Fermi sea" and those from small circles as contributions from the excitations.

We shall further comment on these additional zeros which can be divided into two families. The first is already mentioned zeros of $A^{(-)}$ on the negative real axis of E, which we referred to as "particles". The second species is the zeros of $A^{(+)}$ on the positive real axis of E which are not zeros of $D^{(+)}$, which will be referred to as "holes".

When $\ell = 0(-1)$, there exist simple patterns of particles and holes

- At $\alpha = (2n-1)(M+1) \mp 1$, there exists zero at the origin for $A_{\pm}^{(-)}(E)$.
- For $(2n-1)(M+1) - 1 < \alpha < (2n-1)(M+1) + 1$, there are $n(n-1)$ particles for $A_{+}^{(-)}(E)(A_{-}^{(-)}(E))$. For $(2n-1)(M+1)+1 < \alpha < (2n+1)(M+1) - 1$, n particles for both $A_{\pm}^{(-)}(E)$.
- For $(2n-1)(M+1) - 1 < \alpha < (2n-1)(M+1) + 1$, there exists a hole of $A_{\pm}^{(+)}(E)$ For other ranges of α, the holes do not exist.

When $\ell \neq 0, -1$, the numbers of holes can be more than one.

We need to further rewrite the equation for the separation of the contributions precisely, which is the topic in the next section.

3 Nonlinear Integral Equations

The numerical investigation indicates that the magnitude of $a_{\epsilon'}^{(\epsilon)}$ is small just below the real axis for $\alpha \leq M$. To avoid the complexity due to the logarithmic branches of $\log A_{\epsilon'}^{(\epsilon)}$, it may be thus better to use $a_{\epsilon'}^{(\epsilon)}$ just below the real axis and to use instead $\overline{a_{\epsilon'}^{(\epsilon)}} := \frac{1}{a_{\epsilon'}^{(\epsilon)}}$ slightly above the real axis.

The following parameterization will be convenient for our purpose, $E = \frac{1}{\nu^2}\exp(\frac{\theta}{\mu})$, $\mu = \frac{M+1}{2M}$, $\frac{1}{\nu} = (2M+2)^{\frac{1}{2\mu}}\Gamma(\frac{1}{2\mu})$. The locations of particles and holes also need parameterizations, $E = -\frac{\exp(\frac{\theta^{(p)}}{\mu})}{\nu^2}$, $E = \frac{\exp(\frac{\theta^{(h)}}{\mu})}{\nu^2}$.

With these preparations, the nonlinear integral equations are readily derived in the form,

$$\begin{aligned}\ln a_{\epsilon'}^{(\epsilon)}(\theta) &= D_{\epsilon'}^{(\epsilon)}(\theta) \\ &+ \int_{\mathcal{C}_-} K_{\epsilon,\epsilon}(\theta-\theta')\ln A_{\epsilon'}^{(\epsilon)}(\theta')d\theta' - \int_{\mathcal{C}_+} K_{\epsilon,\epsilon}(\theta-\theta')\ln \overline{A}_{\epsilon'}^{(\epsilon)}(\theta')d\theta' \\ &+ \int_{\mathcal{C}_-} K_{\epsilon,-\epsilon}(\theta-\theta')\ln A_{\epsilon'}^{(-\epsilon)}(\theta')d\theta' - \int_{\mathcal{C}_+} K_{\epsilon,-\epsilon}(\theta-\theta')\ln \overline{A}_{\epsilon'}^{(-\epsilon)}(\theta')d\theta' \qquad (8)\end{aligned}$$

where $\mathcal{C}_+(\mathcal{C}_-)$ specify the contours slightly above (below) the real axis of θ. The validity of the equation is restricted to $-\frac{\pi}{M} < \Im\theta \leq 0$. This is the NLIE we seek for.

The integral kernels read

$$\begin{aligned}K_{+,+}(\theta) = K_{-,-}(\theta) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw\theta}\frac{\sinh^2\frac{\pi(M-1)w}{2M}}{\sinh\pi w\sinh\frac{\pi w}{M}}dw \\ K_{-,+}(\theta) = K_{+,-}(\theta) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw\theta}\frac{\sinh\frac{\pi(M+1)w}{2M}\sinh\frac{\pi(M-1)w}{2M}}{\sinh\pi w\sinh\frac{\pi w}{M}}dw\end{aligned}$$

We choose the principle branch of log : $-\pi \leq \log x \leq \pi$ for real x.

The drive term $D_{\epsilon'}^{(\epsilon)}$ consists of two parts. The first part is determined by the naive WKB method which takes account of the asymptotic behavior $E \to +\infty$. It is thus solely determined by the leading term x^{2M} in the potential and it does not include the existence of the double well at all. This contribution can be regarded as the "bulk" contribution. The second represents the contributions from the "particles" and "holes". The constants, determined by the asymptotic condition as $\theta \to -\infty$ should be also added. By summing up it reads,

$$D_{\epsilon'}^{(\epsilon)}(\theta) = -\frac{i}{2}b_0\nu^{-2\mu}e^{\theta} + \mathcal{C}_{\epsilon'}^{(\epsilon)} + i\Big(\sum_j \epsilon'_{h,j}f^{(\epsilon)}(\theta-\theta_j^{(h)}) - \sum_j g^{(\epsilon)}(\theta-\theta_j^{(p)})\Big)$$

where $b_0 = \frac{1}{\mu M}B(\frac{1}{2},\frac{1}{2M})$.

The functions representing the contributions form "particles" and "holes" are given by odd primitives of kernel functions with shifts in arguments.

Their explicit forms for $M = 3$ read

$$f^{(+)}(\theta) = g^{(-)}(\theta) = \frac{1}{i}\log -\frac{\sinh(\theta/2 + i\pi/3)}{\sinh(\theta/2 - i\pi/3)} - \frac{\pi}{3}$$
$$g^{(+)}(\theta) = f^{(-)}(\theta) = \frac{1}{i}\log -\frac{\sinh(\theta/2 + i\pi/6)}{\sinh(\theta/2 - i\pi/6)} - \frac{2\pi}{3}$$

Note we adopt the convention $f, g \to 0$ as $\theta \to -\infty$. The parameter $\epsilon'_{h,j}$ takes ± 1 individually for general ℓ and equals to ϵ' for $\ell = 0$. The constants C is given by

$$C^{(+)}_{\epsilon'} = -C^{(-)}_{-\epsilon'} = -\frac{(M-1)\pi}{4M}[\alpha + \epsilon'] + \frac{(M+1)\pi}{4M}[\alpha - \epsilon']$$
$$[x] = \frac{M+1}{\pi}\log(\exp(\frac{\pi i x}{M+1})).$$

In the absence of particles and holes, the NLIE (8) and analogous equation for $\bar{a}$ provide an autonomous set of equations which determine the energy spectra as demonstrated in [9]

The presence of particles requires an additional equation which encodes the locations of θ_p. This can be obtained by the analytic continuation of (8) with $\epsilon = -1$ from $\Im\theta \sim 0^-$ to $\Im\theta = -\mu\pi$. We denote the resultant function $b_{\epsilon'}(\theta) := a^{(-)}_{\epsilon'}(\theta - \mu\pi)$.

Hereafter we restrict our argument to $M = 3, \ell = 0$. Assume that we solve NLIE and obtain functions a and b as functions of θ^2. Then the following "quantization condition" determines the locations of particles, holes and energy levels.

$$\begin{aligned}
\ln b_{\epsilon'}(\theta^{(p)}_j) &= (-2(j - n_p) + 1)\pi i, \qquad j = 1, 2, \cdots, n_p \\
\ln a^{(+)}_{\epsilon'}(\theta^{(+)}_j) &= (-2(j - \epsilon' n_h) + 1)\pi i \qquad j = 1, 2, \cdots \\
\ln a^{(-)}_{\epsilon'}(\theta^{(-)}_j) &= (-2j + 1)\pi i \qquad j = 1, 2, \cdots \\
\ln a^{(+)}_{\epsilon'}(\theta^{(h)}_1) &= \epsilon'\pi i
\end{aligned}$$

Here we assume the ordering, $\theta^{(p)}_1 > \theta^{(p)}_2 > \cdots > \theta^{(p)}_{n_p}$ and $\theta^{(\pm)}_{i+1} > \theta^{(\pm)}_i$ for $i \geq 1$.

As a benchmark, we first consider a case where the model enjoys the quasi-exactly solvable property: one only has to locate roots of the Bender-Dunne polynomial [14] (of finite order) to find some of exact spectra. We take $|\alpha| = 23$ and consider the positive parity case . See Table 2. The leftmost column represents the spectra evaluated from a numerical routine in order to check its reliability. The first 12 decimal positions are kept and they show reasonable agreement of three results within numerical accuracy.

The energy levels with negative parity are no longer possessing the QES property, nevertheless their estimation through NLIE seems to be in excellent agreement with the result from the numerical routine .

[2]We still have a numerical instability for $(2n-1)(M+1) - 1 \leq \alpha < (2n-1)(M+1) + 1$ for $n \in \mathbb{Z}_{\geq 0}$. The rest of ranges, however, the NLIE converges quickly.

n	$E_n^{(-)}$ direct	$E_n^{(-)}$ NLIE	Exact
1	-33.2269336257	-33.2269336226	-33.2269336257
3	-16.7077849943	-16.7077849929	-16.7077849944
5	-4.10131715550	-4.10131715511	-4.10131715548
7	4.10131715552	4.10131715512	4.10131715548
9	16.7077849968	16.7077849929	16.7077849944

Table 2: eigenvalues with positive parity: $|\alpha| = 23$

n	$E_n^{(-)}$ direct	$E_n^{(-)}$ NLIE
2	-33.2268566236	- 33.2268566205
4	-16.6880935159	- 16.6880935144
6	-3.08280029197	-3.08280029165
8	9.54245531639	9.54245531549
10	24.6001535110	24.6001534881

Table 3: eigenvalues with negative parity: $|\alpha| = 23$

For larger values of α, the numerical package sometime can be unstable. We therefore check the result again the instanton calculation.

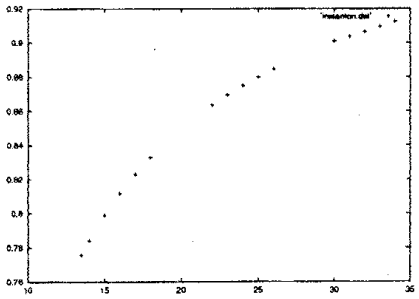


Figure 5: The estimation on level splitting. The horizontal axis denotes $|\alpha|$ while the vertical axis corresponds to $\triangle$(NLIE)/$\triangle$(Instanton) .

The quantum tunneling brings about fine splitting of energy spectra between $\epsilon' = \pm$. This is due to nonzero value of $\hbar$. In the present problem, the Symanzik scaling shows that $1/\alpha$ plays the role of the Planck constant. Thus the quantum effect is enhanced for larger values of α. We compare the simplest one-instanton result with numerical data for intermediate values of α. See fig. 5. Two results seem to coincide as $\alpha \to \infty$.

We have a comment on the splitting. Consider E plane again, instead of θ. See fig 6. When α is small, all zeros of $A_{\epsilon'}^{(-)}(E)$ on the negative real axis comes from those of Stokes multiplier whose locations do not depend on the parity. Therefore locations of zeros of $A_{+}^{(-)}(E)$ and $A_{-}^{(-)}(E)$ are completely " bounded". This binding force exists on the negative real axis for larger values of α. With increase of α, a zero of $A_{+}^{(-)}(E)$ comes

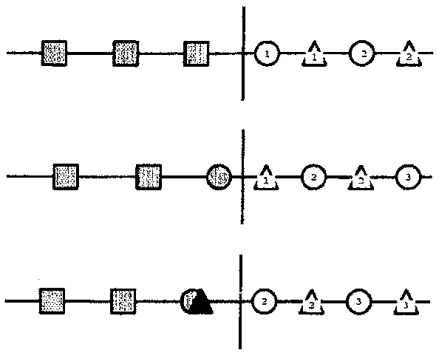


Figure 6: The formation of pairs. The square boxes on the negative real axis represent zeros of Stokes multipliers. The circles (triangles) represent energy levels with positive (negative) parity. Just distinguish them we assign number labels. The case $\alpha < 3$ (top), $3 < \alpha < 5$ (middle), $\alpha = 5^+$ (bottom).

onto the negative E axis at $\alpha = 3$ and it act as a single particle until $\alpha = 5$. For α slightly bigger than 5, a zero of $A_-^{(-)}(E)$ invades into the negative E region and the "binding force" immediately glues them together. The binding is perfect for zeros of Stokes multiplier, while it is incomplete for particles . The incompleteness results in the slight splitting of the levels. This interpretation of the quantum tunneling sounds quite intuitive.

4 Higher Bethe ansatz approximation and its application

The numerical data in the preceding section supports the validity of the NLIE approach. The equations however seem to be involved and it is desirable to understand them in a simpler manner. For this purpose, we discuss an approximation scheme in this section.

The most simple-minded approximation is to drop convolution integrals from NLIE. Although this procedure seems to be too crude, it leads to the famous higher Bethe ansatz equation which plays a crucial role in many problems of the low dimensional quantum many body system. We refer to [15, 16] for instance. It is known that the approximation becomes exact in the thermodynamic limit.

In our framework, the meaning of "the thermodynamic limit" is obscure. We nevertheless employ this scheme.

We focus on the equation which determines $\theta_j^{(p)}$, equivalently, negative eigenvalues. In the present approximation it reads explicitly,

$$m_b \exp(\theta_j^{(p)}) + \sum_k \frac{1}{i} \log\Big(-\frac{\sinh(\frac{\theta_j^{(p)} - \theta_k^{(p)}}{2} + i\frac{\pi}{3})}{\sinh(\frac{\theta_j^{(p)} - \theta_k^{(p)}}{2} - i\frac{\pi}{3})} \Big) = (-2j+1)\pi + \phi \qquad (9)$$

where

$$\phi = \frac{7\pi n_p}{3} - \mathcal{C}^b_{\epsilon'}, \qquad m_b = \frac{b_0}{2}\nu^{-2\mu}$$

Unfortunately, this approximation can not yield the tunneling splitting. This simply comes from the fact that $\mathcal{C}^b_{\epsilon'}$ does not depend on ϵ'.

It still provides rather accurate estimation to the eigenvalue spectra in spite of its facility.

Direct	HBAE	WKB (Harmonic)
-208.891449382762105858	-208.86468154847168	-208.86206203621745
-208.891449382762105867		
-176.842948216853944772	-176.81243541001473	-176.81201222397732
-176.842948216853956226		
-146.283750686771029419	-146.24823501971161	-146.25070765647126
-146.283750686777096947		
-117.306263295979684855	-117.2637488536079	-117.2701931189165
-117.306263297797877911		

Table 4: some negative eigenvalues $\alpha = 70$

Note that the rightmost WKB approximation takes account of the double well structure from the beginning while it treats the existence of x^6 term only secondarily . On the other hand, the main (bulk)term in our equation comes from the x^6 term, while the double well structure is only reflected in the number of particles n_p. Interestingly, they nevertheless draw almost the same conclusions.

We can improve the above result by use of the "dilogarithmic trick" known in the evaluation of scaling dimensions in CFT but will be omitted here.

The crucial observation is that the equation (9) contains only locations of particles. This is a great simplification from the original problem where all energy levels are taken into account. We shall apply this simple approximation to the analysis of the impressive phase separation curve found by Dorey, Dunning and Tateo[8].

Hamiltonians in PT Quantum Mechanics are invariant under the simultaneous P, T transformations, $P : x \to -x$, $T : i \to -i$.

We consider a specific Hamiltonian

$$\mathcal{H}_{\rm PT} = -\frac{d}{dx^2} - (ix)^{2M} - \epsilon\alpha(ix)^{M-1} + \frac{\ell(\ell+1)}{x^2} \tag{10}$$

which is similar to our original Hamiltonian. The quantization contour is however different from the real axis but the curve depicted in fig 7.

The argument in [8] clarifies the spectral relation between two similar Hamiltonians: it is shown that the zeros of Stokes multipliers associated to the original Hamiltonian are identified with minus of the eigenvalues of a PT symmetric Hamiltonian $\mathcal{H}_{\rm PT}$.

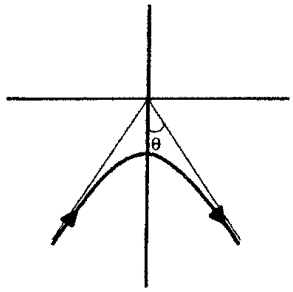


Figure 7: The quantization contour for $\mathcal{H}_{\mathcal{PT}}$. The angle θ is equal to $\frac{\pi}{M+1}$. The wave-function possesses L^2 property on the curve.

In spite of being non-Hermitian, the energy spectra is real for a large area in the parameter space. When $\epsilon\alpha$ gets positive and larger, two of energy levels collide with change in parameters, and they constitute a complex conjugate pair. This phenomena has been identified with the spontaneous breakdown of the PT symmetry. The phase separation curve is evaluated in [8] and it results impressive wobbling pattern with cusp like singularities. The patterns of collision are different for different portions of the curve. We divide the region $\ell \in [0, 3]$ into 4 parts and locate the curve of collisions using the higher Bethe ansatz equations. Skipping the detail, we present the final result of our estimation in fig. 8. The phase separation curve seems to be qualitatively well recovered

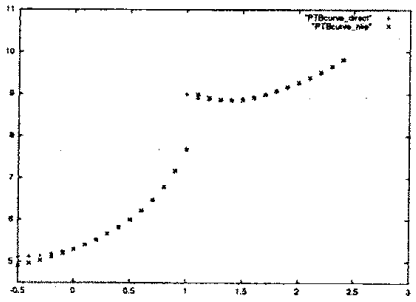


Figure 8: The curve of the PT symmetry breaking, direct v.s. approximated NLIE. The vertical (horizontal) axis presents value of $\alpha(\ell)$.

even within this simple approximation.

Acknowledgements

The author would like to thank the organizers of "Recent Trends in Exponential Asymptotics " (RIMS, Kyoto) for kind invitations. He thanks C. Bender, P. Dorey, Y. Takei, R. Tateo, M. Znojil and other participants for comments and discussions. This work has been supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports and Technology of Japan, no. 14540376.

References

[1] A. Voros, Ann. Inst. H. Poincare **A39** (1983) 211-338.

[2] Y. Sibuya,*'Global theory of a second-order linear ordinary differential equation with polynomial coefficient'*, (Amsterdam North-Holland 1975)

[3] T. Kawai and Y. Takei ,*' Algebraic analysis of the singular perturbations'*, (Japanese) (Iwanami 1997)

[4] P. Dorey and R. Tateo, J. Phys. **A32**, (1999) L419.

[5] J. Suzuki, J. Phys. **A32**, (1999) L183.

[6] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, J. Stat. Phys. **102**, (2001) 567.

[7] P. Dorey, J. Suzuki and R. Tateo, J. Phys. **A 37** (2004) 2047-2061.

[8] P. Dorey, C. Dunning and R. Tateo, J.Phys. **A34** (2001) L391.

[9] J. Suzuki, J. Stat. Phys. **102** (2001) 1029.

[10] P. Dorey, C. Dunning and R. Tateo, J.Phys. **A34** (2001) 5679-5704

[11] A. Voros, J. Phys. **A32** (1999) 1301-1311.

[12] A. Klümper , M.T. Batchelor and P.A. Pearce J. Phys. **A 24** (1991) 3111.

[13] C. Destri and H.J. de Vega, Phys. Rev. Lett. **69** (1992) 2313

[14] C.Bender ang G. Dunne, J. Math. Phys. **37** (1996) 6

[15] C. Destri and J. H. Lowenstein, Nucl.Phys.**B205** (1982) 369.

[16] O. Babelon, H.J. de Vega and C.M. Viallet, Nucl.Phys.**B220** (1983) 13.