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WKB ANALYSIS AND POINCARÉ'S THEOREM

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1. Introduction

In this note, we discuss the asymptotic analysis in the normal form theory of a singular vector field at the origin. By introducing a new parameter in the equation in a natural way, we will discuss the famous Poincaré's theorem for a singular vector field from the viewpoint of a WKB analysis.

Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $n \geq 2$ be the variable in $\mathbb{C}^n$. We consider a holomorphic vector field

$$\mathcal{X} = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j},$$

which is singular at the origin, i.e,

$$a_j(0) = 0, \quad j = 1, \ldots, n.$$

We set

$$X(x) = (a_1(x), \ldots, a_n(x)), \quad \frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}),$$

and write

$$\mathcal{X} = X(x) \cdot \frac{\partial}{\partial x}, \quad X(x) = \Lambda x + R(x),$$
$$R(x) = (R_1(x), \ldots, R_n(x)),$$

where Λ is an n-square constant matrix and

$$R(x) = O(|x|^2).$$

We want to linearize $\mathcal{X}$ by the change of variables,

$$x = u(y), \quad u = (u_1, \ldots, u_n), \ y = (y_1, \ldots, y_n),$$

namely,

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$$X(u(y))\frac{\partial y}{\partial x}\frac{\partial}{\partial y} = X(u(y))\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial}{\partial y} = \Lambda y\frac{\partial}{\partial y}.$$

It follows that u satisfies the equation

$$X(u(y))\left(\frac{\partial u}{\partial y}\right)^{-1} = \Lambda y,$$

that is

$$\Lambda u + R(u) = \Lambda y\frac{\partial u}{\partial y}. \tag{1.1}$$

The equation (1.1) is called a *homology equation* . In the following, we assume that Λ is semi simple. Hence, by a suitable linear change of variables, we may assume that Λ is a diagonal matrix with diagonal components given by λ_j, $(j = 1, 2, \dots, n)$.

If we denote the variable by x instead of y, then u satisfies

$$\mathcal{L}u_j = \lambda_j u_j + R_j(u_1, \dots, u_n), \quad j = 1, \dots, n, \tag{H}$$

where $\mathcal{L}$ is given by

$$\mathcal{L} = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}.$$

Remark. (Relation to a Painlevé equation). We introduce the new variable z_j by

$$e^{z_j} = x_j, \; j = 1, \dots, n.$$

We consider a plane wave solution

$$u_j = u_j(t), \quad t = z_1/(n\lambda_1) + \cdots + z_n/(n\lambda_n).$$

Then we have

$$\mathcal{L}u_j(t) = u_j'(t),$$

and we can write the equation (H) in the form

$$u_j' = \lambda_j u_j + R_j(u), \quad j = 1, \dots, n. \tag{1.2}$$

Let us consider the special case, $n = 3$ and

$$\begin{aligned} R_1(u) &= u_1^2(u_2 - u_3),\ R_2(u) = u_2^2(u_3 - u_1), \\ R_3(u) &= u_3^2(u_1 - u_2). \end{aligned}$$

By the change of unknown functions

$$u_j = \exp(U_j),$$

we obtain *a symmetric form of a homology equation*

$$\begin{aligned} U_1' &= \lambda_1 + e^{U_1}(e^{U_2} - e^{U_3}), \\ U_2' &= \lambda_2 + e^{U_2}(e^{U_3} - e^{U_1}), \\ U_3' &= \lambda_3 + e^{U_3}(e^{U_1} - e^{U_2}). \end{aligned}$$

Assuming that U_j is small, we replace e^{U_j} with $1 + U_j$ in the above equation. If we replace $1 + U_j$ with U_j, then we have *a symmetric form of a fourth Painlevé equation*

$$\begin{aligned} U_1' &= \lambda_1 + U_1(U_2 - U_3) \\ U_2' &= \lambda_2 + U_2(U_3 - U_1) \\ U_3' &= \lambda_3 + U_3(U_1 - U_2). \end{aligned}$$

Indeed, if we assume $\lambda_1 + \lambda_2 + \lambda_3 = 1$, this system of equation is equivalent to the so-called fourth Painlevé equation.

Because we have $U_1 + U_2 + U_3 = t$, we obtain a system of equations for U_1 and U_2. Then we eliminate U_1: we finally obtain the second order equation for U_2. Indeed, if we set

$$u = \sqrt{2}U_2, \quad s = -\sqrt{2}t,$$

then we have

$$u'' = \frac{1}{2u}(u')^2 + \left(\frac{3}{2}u^3 + 4su^2 + 2(s^2 + (1 - 2\lambda_1 - \lambda_2))u - \frac{2\lambda_2^2}{u}\right).$$

2. WKB solution of a homology equation

Introduction of a parameter

The natural way of introducing a large parameter in the symmetric form of a Painlevé equation is the following

$$\begin{aligned} U_1' &= \eta(\lambda_1 + U_1(U_2 - U_3)), \\ U_2' &= \eta(\lambda_2 + U_2(U_3 - U_1)), \\ U_3' &= \eta(\lambda_3 + U_3(U_1 - U_2)). \end{aligned}$$

This system of equations is essentially equivalent to the fourth Painlevé equation with a large parameter due to Aoki-Kawai- Takei, where a parameter was introduced via monodromy preserving deformation of a Schrödinger equation. (cf. [10]). Indeed, the equation introduced by them was

$$u'' = \frac{1}{2u}(u')^2\frac{2}{u} + \eta^2\left(\frac{3}{2}u^3 + 4su^2 + 2(s^2 + \beta_1)u - \frac{8\beta_0}{u}\right).$$

In view of the presentation in the preceeding section, we have the following relations

$$\beta_0 = \frac{\eta^2\lambda_2^2 - 1}{4\eta^2}, \quad \beta_1 = \frac{1 - 2\eta\lambda_1 - \eta\lambda_2}{4\eta}.$$

Hence we introduce a parameter in the homology equation in the following way

$$\mathcal{L}u_j = \eta\left(\lambda_j u_j + R_j(u)\right), \quad j = 1, \dots, n. \tag{2.1}$$

Remark. If we make the change of variables $e^{z_j} = x_j$ in the homology equation (H), then we obtain

$$\mathcal{L}u = \Lambda u + R(u_1, \dots, u_n), \ \mathcal{L} = \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial z_j}. \tag{2.2}$$

By making the change of variables $z_j = \eta y_j$, we obtain

$$\sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial z_j} = \eta^{-1} \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial y_j}.$$

If we write e^{y_j} as x_j, then we obtain (2.1). We note that a parameter η is also introduced in the homology equation through the following blowing up transformation

$$x_j = y_j^{\eta}, \quad j = 1, \dots, n.$$

A WKB solution (0 - instanton solution)
For the sake of simplicity we set $u(x) = x + v(x)$ in the original homology equation and we introduce a parameter η by the above argument. The resultant equation is

$$\mathcal{L}v_j = \eta\left(\lambda_j v_j(x) + R_j(x + v(x))\right), \quad j = 1, \dots, n. \tag{2.3}$$

Definition. *A WKB solution* (0 - instanton solution) $v(x, \eta)$ of (2.3) is a formal power series solution of the form

$$v(x, \eta) = \sum_{\nu=0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1} v_1(x) + \cdots, \tag{2.4}$$

where the series is a formal power series in η with coefficients $v_\nu(x)$ being holomorphic vector functions in x in some open set independent of ν.

By substituting the expansion (2.4) into (2.3), we obtain

$$\mathcal{L}v_j = \sum_{\nu=0}^{\infty} \mathcal{L}v_\nu^j(x)\eta^{-\nu}, \tag{2.5}$$

$$\begin{aligned} R_j(x+v) &= R_j(x+v_0+v_1\eta^{-1}+v_2\eta^{-2}+\cdots) \\ &= R_j(x+v_0)+\eta^{-1}\sum_{k=1}^{n}\left(\frac{\partial R_j}{\partial z_k}\right)(x+v_0)v_1^k+O(\eta^{-2}). \end{aligned} \tag{2.6}$$

By comparing the coefficients of η, $\eta^0 = 1$ we obtain

$$\lambda_j v_0^j(x) + R_j(x_1+v_0^1,\dots,x_n+v_0^n) = 0. \tag{2.7}$$

$$\mathcal{L}v_0^j = \lambda_j v_1^j + \sum_{k=1}^{n}\left(\frac{\partial R_j}{\partial z_k}\right)(x+v_0)v_1^k. \tag{2.8}$$

In order to determine $v_\nu(x)$ $(\nu \geq 2)$ we compare the coefficients of $\eta^{-\nu}$. Then we obtain

$$\mathcal{L}v_{\nu-1}^j = \lambda_j v_\nu^j + \sum_{k=1}^{n}\left(\frac{\partial R_j}{\partial z_k}\right)(x+v_0)v_\nu^k \tag{2.9}$$

$$+\quad \text{(terms consisting of } v_k^j,\ k \leq \nu-1 \text{ and } j=1,\dots,n). \tag{2.10}$$

In order to determine v_ν from the above recurrence relations we first make a definition.

Definition. The point x such that

$$\det(\Lambda + (\partial R/\partial z)(x+v_0)) = 0$$

is called a *turning point* of the equation (2.3).

Assumption. *We assume*

$$\det\Lambda \neq 0.$$

If $\det\Lambda \neq 0$, then the origin $x=0$ is not a turning point of (2.3) for any holomorphic $v_0(x) = O(|x|^2)$.

By the implicit function theorem, we can uniquely determine a holomorphic function $v_0(x)$ from (2.7) such that $v_0(x) = O(|x|^2)$. Next, every v_ν are calculated algebraically via differentiation and division. Consequently, we have

Proposition *Assume that* $\det\Lambda \neq 0$. *Then every coefficients of a WKB solution is holomorphic in a common neighborhood of the origin independent of* ν.

Definition (Resonance condition). We say that η is *resonant*, if

$$\sum_{i=1}^{n} \lambda_i \alpha_i - \eta \lambda_j = 0,$$

for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| \geq 2$ and $j, 1 \leq j \leq n$. If η is not resonant we say that η is *nonresonant.*

Definition (Poincaré condition) *We say that a homolgy equation satisfies a Poincaré condition, if the convex hull of $\lambda_j, j = 1, \ldots, n$ in the complex plane does not contain the origin.*

If a Poincaré condition is not verified, then we assume the following condition

$$\lambda_j \in \mathbb{R}, \quad j = 1, \ldots, n.$$

If a Poincaré condition is not verified, then there are two important cases, namely, a Diophantine case and Liouville case. In the former case, either a Siegel condition or a Bruno type Diophantine condition is verified among λ_j, $j = 1, \ldots, n$. If no such conditions are satisfied, then we say that we are in a Liouville case.

We note that, if a Poincaré condition is verified, then the number of resonance is finite, while in a Siegel case, the number of resonance is, in general, infinite. Moreover the resonance may be a dense subset of a real line.

3. A WKB SOLUTION IN A SECTOR

For the direction ξ, $(0 \leq \xi < 2\pi)$ and the opening $\theta > 0$ the sector $S_{\xi,\theta}$ is defined by

$$S_{\xi,\theta} = \left\{ \eta \in \mathbb{C}; |\arg \eta - \xi| < \frac{\theta}{2} \right\}.$$

Then we have

Theorem *There exist a direction ξ, an opening $\theta > 0$, a neighborhood U of the origin $x = 0$ and $V(x, \eta)$ such that $V(x, \eta)$ is holomorphic in $(x, \eta) \in U \times S_{\xi,\theta}$ and satisfies (2.3). $V(x, \eta)$ is an asymptotic expansion of the WKB solution $v(x, \eta)$ in $U \times S_{\xi,\theta}$ when $\eta \to \infty$. Namely, for every $N \geq 1$ and $R > 0$, there exists $C > 0$ such that*

$$\left| V(x, \eta) - \sum_{\nu=0}^{N} \eta^{-\nu} v_\nu(x) \right| \leq C |\eta|^{-N-1},$$

$$\forall (x, \eta) \in U \times S_{\xi,\theta}, \ |\eta| \geq R.$$

In addition, if a Poincaré condition is verified, then we can take $\xi = \pi$.

Remark. In the above theorem we do not assume that the Poincaré condition is satisfied and the Liouville case is also admitted. The difference of the behavior of WKB solutions in the Poincaré and the Liouville case appears if we consider the analytic continuation of WKB solutions to the right half plane.

Sketch of the proof. First we show that there exist $\theta > 0$, ξ and $c_0 > 0$ such that if $\eta \in S_{\xi,\theta}$, then

$$\left|\eta^{-1}\langle\lambda,\alpha\rangle - \lambda_j\right| \geq c_0, \quad \forall\alpha \in \mathbb{Z}_+^n,\ |\alpha| \geq 2, \forall\eta \in S_{\xi,\theta}.$$

Then we construct $V(x,\eta) = \sum_\alpha V_\alpha(\eta)x^\alpha$ as a formal power series solution of a homology equation, where x is in some neighborhood of the origin and $\eta \in S_{\xi,\theta}$ for some ξ and θ. Indeed, we solve the homology equation with a parameter η by the method of sucessive approximations, because the equation is semilinear.

If we expand the coefficients $V_\alpha(\eta)$ in the power of η formally, and if we change the order of summation we obtain a WKB solution. Precise estimate of the derivatives of $V_\alpha(\eta)$ gives the desired results.

4. Analytic continuation of a WKB solution to the right half plane

In the following we assume that a Poincaré condition is satisfied. Without loss of generality, we may assume

$$\mathrm{Re}\,\lambda_j > 0, \quad j = 1,\ldots,n.$$

It follows that we can choose a realization of the WKB solution such that it is holomorphic in x and η when x is in some neighborhood of the origin and η in a sufficiently small sector containing a negative real axis. We want to make an analytic continuation to the right half plane. We can easily see that the solution has a singularity on the set of resonances of η. We have

Theorem (Poincaré case) *If a Poincaré condition is verified, then the above WKB solution is analytically continued to the right half plane as a single valued function except for resonance points. If $\eta = 1$ is not resonant, then the analytic continuation of the WKB solution to $\eta = 1$ coincides with a classical Poincaré solution of a homology equation.*

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