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SINGULARITY BARRIERS AND BOREL PLANE ANALYTIC PROPERTIES OF 1^+ DIFFERENCE EQUATIONS

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ABSTRACT. The paper addresses generalized Borel summability of "1^+" difference equations in "critical time". We show that the Borel transform Y of a prototypical such equation is analytic and exponentially bounded for $\Re(p) < 1$ but there is no analytic continuation from 0 toward $+\infty$: the vertical line $\ell := \{p : \Re(p) = 1\}$ is a singularity barrier of Y.

There is a unique natural continuation through the barrier, based on the Borel equation dual to the difference equation, and the functions thus obtained are analytic and decaying on the other side of the barrier. In this sense, the Borel transforms are analytic and well behaved in $\mathbb{C} \setminus \ell$.

The continuation provided allows for generalized Borel summation of the formal solutions. It differs from standard "pseudocontinuation" [9]. This stresses the importance of the notion of cohesivity, a comprehensive extension of analyticity introduced and thoroughly analyzed by Écalle.

We also discuss how, in some cases, Écalle acceleration can provide a procedure of natural continuation beyond a singularity barrier.

1. INTRODUCTION

In the case of generic differential equations, generalized Borel summation of a formal power series solution, in the sense of Écalle [4], essentially consists in the following steps: (1) Borel transform with respect to a *critical time*, related to the order of exponential growth of possible solutions, (see also the note below), usual summation of the obtained series, analytic continuation along the real line or in its neighborhood, proper averaging of the analytic continuations (e.g. medianization) toward infinity, possible use of acceleration operators and Laplace transform $\mathcal{L}$.

The choice of the critical time, or of a very slight perturbation –weak acceleration– of it is crucial for Écalle summability. A slower variable (time) would hide the resurgent structure encapsulating the Stokes phenomena, and, perhaps more importantly, introduces superexponential growth preventing Laplace transformability at least in some directions. In a faster variable, convergence of the Borel transformed series would not hold.

In some functional equations and so called type 1^+ difference equations, new difficulties occur. For them, Écalle replaces analyticity with *cohesivity* [5]. This property was studied rigorously for some classes of difference equations by Immink [6]. It is the purpose of this note to show the importance of this notion: even in simple 1^+ difference equations it is shown that critical time Borel transform has barriers of singularities, preventing continuation in some half-plane. This occurs in the prototypical equation

$$y(x+1) = \frac{1}{x}y(x) + \frac{1}{x} \tag{1}$$

(example 2. of [6]). A simple proof of Borel space natural boundaries is not present in the literature, as far as the author is aware. We also show that the barrier is traversable: on the real line the associated function is well defined and Laplace transformable to a solution of the difference equation. This function is real analytic except at one point and, in fact has analytic continuation in the whole of $\mathbb{C}\setminus\ell$ with $\ell = \{p : \Re(p) = 1\}$ a singularity barrier. The present approach is adaptable to more general equations.

We expect barriers of singularities to occur quite generally in 1^+ cases, due to the fact that the pole position is periodic in the original variable, while critical time introduces a logarithmic shift in this periodicity. This leads to lacunary series in Borel plane, hence to singularity barriers.

Nonetheless, further analysis shows that, in this simple case, and likely in quite some generality, softer Borel summation methods and study of Stokes phenomena are possible, relying on the convolution equation for continuation through singularity barriers.

In spite of its simplicity, the properties in Borel plane of this equation, in the critical time, are very rich.

Note on critical time. The solution of the homogeneous equation associated to (1), $f(x) = 1/\Gamma(x)$ has large x behavior $(x/2\pi)^{1/2}e^{-x\ln x+x}$. The critical time z is then the leading asymptotic term in the exponent, $z = x\ln x$ [6]. (The origin of the terminology 1^+ is related to the exponential order slightly larger than one of f). Various slight perturbations of this variable, weak accelerations, are used and indeed are quite useful.

2. The singularity barrier

Theorem 1. *Let $Y(p)$ be the Borel transform of y in (1) in the critical time z. Then $Y(p)$ is analytic on $\{p \neq 0 : \arg(p) \in (\pi - 2\pi, \pi + 2\pi); \Re(p) < 1\}$ and exponentially bounded as $|p| \to \infty$ in this region. The line $\ell = \{p : \Re p = 1\}$ is a singularity barrier of Y.*

Proof of the theorem. Let $\tilde{y}$ be the formal power series solution of (1). We study the analytic properties of the Borel transform $\mathcal{B}\tilde{y} := Y(p)$ of the on $\mathcal{S}_0$, the Riemann surface of the log at zero, with respect to the critical time z. In critical time the functional equation of $\mathcal{B}\tilde{y}$ (9) is unwieldy, and instead we look at the meromorphic structure of solutions on which we perform a Mittag–Leffler decomposition.

It is straightforward to check that $\tilde{y}$ is the asymptotic series for $\arg(x) \neq 0$ of the following actual solution of (1)

$$y_0(x) = \sum_{k=1}^{\infty}\prod_{j=1}^{k}\frac{1}{x-j} \tag{2}$$

The fact that $\operatorname{Res}(y_0; x = n) = e^{-1}/\Gamma(n)$ and the behavior at infinity of y_0 show that the Mittag-Leffler partial fraction decomposition of (3) is

$$y_0 = e^{-1}\sum_{k=1}^{\infty}\frac{1}{(x-k)\Gamma(k)} \tag{3}$$

(1) *Analyticity in the left half plane.* The inverse function $z \mapsto x(z)$ of $x \ln x$ is analytic on $\mathcal{S}_0 \setminus (-e^{-1}, 0)$ as it can be seen from the differential equation $\frac{dx}{dz} = (1 + \ln x)^{-1}$. Then $Y(p)$ is the analytic continuation of the function defined for p *negative* by

$$-\frac{1}{2\pi i}\int_{i\mathbb{R}-e^{-1}} e^{pz} y_0(x(z))dz = \frac{1}{2\pi i}\int_C e^{pz} y_0(x(z))dz, \quad p \in \mathbb{R}^- \tag{4}$$

where C is a contour from $\infty + i0$ around $-e^{-1}$ and to $\infty - i0$.

(2) *Identities for finding continuation in $\{z : \Re(z) < 1\}$ and exponential bounds.* For analytic continuation clockwise we start from $\arg p = \pi$ and rotate up the contour, collecting the residues:

$$Y(p) = \frac{1}{2e\pi i}\sum_{k=1}^{\infty}\frac{1}{\Gamma(k)}\int_C \frac{e^{pz}dz}{x(z)-k} = F(p) + \frac{1}{2e\pi i}\int_{C_1}\sum_{k=1}^{\infty}\frac{1}{\Gamma(k)(x(z)-k)}e^{pz}dz$$
$$\text{where } F(p) := \sum_{k=1}^{\infty}\frac{1+\ln k}{e\Gamma(k)}e^{pk\ln k} \tag{5}$$

and where for small $\phi > 0$, C_1 is the contour from $\infty e^{i\phi+i0}$ around $(-e^{-1}, 0)$ to $\infty e^{i\phi-i0}$. As $\arg p$ is decreased from to zero (and further to $-\pi$), ϕ can be increased from 0^+ to $2\pi^-$ making $\int_{C_1}$ visibly analytic in $\{p \neq 0 : \arg p \in (-\pi, \pi)\}$ and exponentially bounded as $|p| \to \infty$. We decomposed Y into a sum of a lacunary Dirichlet series and a function analytic in the right half plane.

(2) *The natural boundary.* The Dirichlet series F is manifestly analytic for $\Re p < 1$. As $p \uparrow 1$ we have $F(p) \to +\infty$ and thus F is not entire. But then, by the Fabry-Wennberg-Szasz-Carlson-Landau theorem [8] pp. 18, ℓ is a singularity barrier of F and thus of Y. For a detailed analysis, see also the note below. □

Note: Description of the behavior of F at ℓ. Since all terms of the Dirichlet series are positive on the real line, it is easy to check using discrete Laplace method[1] that F increases like an iterated exponential along $\mathbb{R}^+$ toward ℓ, $F(p) \propto \exp((1-p)\exp(1/(1-p)))$. There are densely many points near ℓ where the growth is similar; it suffices to take a sequence of $k \in \mathbb{N}$, $\Re(p) = k/(1+\ln(k))$ and $(1+\ln(k))\Im(p)$ very close to an integer multiple of 2π. (A Rouché type argument shows there are also infinitely many zeros with a mean separation of order the reciprocal of the maximal order of growth, $\ln(d) \sim -(1-p)e^{1/(1-p)}$.) Rather than attempting some form of continuation through points where F is bounded, which are easy to exhibit, we prefer to soften the barrier first, by acceleration techniques.

3. General Borel summability in the direction of the barrier. Properties beyond the barrier.

Strategy of the approach. It is convenient to perform a "very weak acceleration" to smoothen the behavior of $Y(p)$ near ℓ. The natural choice of variable is $z = \ln\Gamma(k)$, but we prefer to slightly accelerate further, to $z_m(x)$ defined in Remark 1 below. We construct actual solutions of (1) starting from an incomplete Borel sum. We identify these actual solutions and show they are inverse Laplace transformable. Furthermore, they solve the associated convolution equation in Borel space. From

[1] Determining, for fixed p, the maximal term of the series and doing stationary point expansion nearby.

these points of view, we have a unique continuation on $\mathbb{R}^+$. We show that the function thus obtained is real analytic on $\mathbb{R}\setminus\{1\}$ and continuable to the whole of $\mathbb{C}\setminus\ell$.

The general solution of (1) is

$$y(x) = y_0(x) + \frac{f(x)}{\Gamma(x)} \tag{6}$$

where f is any periodic function of period one, as it can be easily seen by making a substitution of the form (6) in the equation. It can be easily checked that the following solution of (1)

$$y_1(x) = y_0 + \frac{\pi}{e}\frac{\cot \pi x}{\Gamma(x)} \tag{7}$$

is an entire function, and has the asymptotic behavior $\tilde{y}$, the formal series solution to (1) defined in the proof of the theorem.

Remark 1. *Let $m\in\mathbb{N}$ and $z_m(x) = x\ln x - x - (m+\frac{1}{2})\ln x$. For given $C>0$, there is a one-parameter family of solutions of (1) which are analytic and polynomially bounded in a region of the form $S_C = \{x : \Re(z_m(x)) \geq C\}$. They are of the form $y_c(x) = y_1(x) + c/\Gamma(x)$ for some constant c.*

Proof. The solution (7) already has the stated boundedness and analyticity properties (and in fact, it decreases at least like x^{-m} in S_C). The general solution is of the form $y_1 + f(x)/\Gamma(x)$ with f periodic, as remarked at the beginning of the section. Analyticity implies f is analytic and boundedness in the given region implies f is bounded on the line ∂S_C. By periodicity, f is polynomially bounded in the whole of $\mathbb{C}$, which means f is a polynomial, and by periodicity, a constant. □

Theorem 2 (Generalized Borel summability). *(i) There exists a one parameter family of solutions of (1) which can be written as $\mathcal{L}_{z_m}H_c := \int_0^\infty e^{-z_m p}H_c(p)dp$ where $H_c = \mathcal{B}_{z_m}\tilde{y}$ is analytic and exponentially bounded for $\Re(p) < 1$ and $H_c \in C^{m-1}(\mathbb{R}^+)$.*

(ii) H_c are real analytic on $\mathbb{R}^+\setminus\{1\}$; they extend analytically to $\mathbb{C}\setminus\ell$, and ℓ is a singularity barrier H_c and the functions are C^{m-1} on the two sides of the barrier[2]. Furthermore, for $\Re(p) > 1$, H_c decrease toward infinity in $\mathbb{C}$.

Remark 2. *It would not be correct at this time to conclude that, say, $\mathcal{L}^{-1}y_1$ provides Borel summation of $\tilde{y}$; we need to show that y_1 satisfies the necessary Gevrey-type estimates to identify the inverse Laplace transform with $\mathcal{B}\tilde{y}$ in the unit disk. We prefer to proceed in a more general way, not using explicit formulas, but constructing actual solutions starting with an incomplete Borel summation (and identifying them later with the explicit formulas).*

Proof of Theorem 2, (i) We redo the analysis of the proof of Theorem 1 in the variable $z = z_m$ and we get a decomposition of the form (5), where now F is replaced by

$$F_2 = \sum_{k=1}^{\infty} \frac{\ln k + \frac{m}{k}}{e\Gamma(k)} e^{p\left[k\ln k - k - (m+\frac{1}{2})\ln k\right]} \tag{8}$$

which is a Dirichlet series of the same type as F and hence has ℓ as a singularity barrier. However, F_2 is (manifestly) uniformly C^{m-1} up to ℓ and so is thus $Y(p)$.

[2]The values on the two sides cannot, obviously, be the same.

For the solutions of (1) that decrease in a sector in the right half –plane it is clear that the dominant balance is between $y(x+1)$ and $1/x$. We then rewrite the equation to prepare it for a contraction mapping argument in Borel space. By a slight abuse of notation we write $y(z)$ for $y(x(z))$ and we have

$$(x(z)-1)y(x(z)) = y(x(z)-1)+1$$
$$(x(z)-1)y(z) = y(z-g(z))+1$$

where $g(z)=\ln z - \ln\ln z + o(1)$ and then

$$(x(z)-1)y(z) = \sum_{k=0}^{\infty} y^{(k)}(z)g(z)^k/k! + 1$$

Thus, dividing by $x(z)-1$ and taking inverse Laplace transform, with $G_k(p)$ the inverse Laplace transform of $g(z)^k/(x(z)-1)/k!$, we have

$$Y(p) = \sum_{k=0}^{\infty}[(-p)^k Y] * G_k(p) + F(p) \tag{9}$$

The term G_k is (roughly) bounded by $|e^{-k(1-p)}|$, as can be seen by the saddle point method applied to the inverse Laplace transform integral. It is easy to check, using standard contraction mapping arguments (see e.g. [2]), that Y is given by a convergent ramified expansion in the open unit disk. This was to be expected from estimates of the divergence type of the formal solutions of (1). However, given the estimates on the terms of the convolution equation, the equation, as written, cannot be straightforwardly interpreted beyond $\Re(p)=1$, the threshold of convergence of the ingredient series. It is however possible to write a meaningful global equation by returning to the definition in terms of Laplace transform. We then write

$$\mathcal{L}^{-1}y(z+g(z)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz e^{pz}\int_0^\infty dq e^{-q(z+g(z))}Y(q) = \int_0^\infty H(p,q)Y(q)dq$$

where

$$H(p,q) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(p-q)z-qg(z)}dz = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(p-q)z+q(\ln\ln z+\ldots)}z^{-q}dz$$

which is well defined for $q>0$ and integrable at $q=0$; the convolution equation becomes

$$\int_0^\infty H(p,q)Y(q)dq = Y * \mathcal{L}^{-1}\Big[\frac{1}{x(z)-1}\Big] + \mathcal{L}^{-1}\Big[\frac{1}{x(z)-1}\Big] \tag{10}$$

Based on the solution on $[0,1)$ of (9) we construct solutions to (1) and their inverse Laplace transforms provide continuation of Y past $\Re(p)=1$ and implicitly solutions to (10).

We define the incomplete Borel sum

$$\hat{y} = \int_0^1 e^{-zp}Y_1(p)dp$$

Formal manipulation shows that $\hat{y}$ satisfies (1) with errors of the form[3] $o(e^{-z})$ or $o(x^m/\Gamma(x))$ in the variable x) where the estimate of the errors is uniform in the right half–plane in z, or in a region S_C w.r. to x.

[3]Resulting from incomplete representation of $1/(x(z)-1)$.

We look for a solution of (1) in the form $\hat{y} + \delta(x)/\Gamma(x)$. Then $\delta(x)$ satisfies $\delta(x+1) = \delta(x) + R(x)$ (the 1^+ degeneracy is not present anymore) where $R(x) = o(x^m)$ with differentiable asymptotics (by Watson's lemma). A solution of this equation is $\delta(x) = P(x) - \mathcal{P}^{m+3} \sum_{k=0}^{\infty} R^{(m+3)}(x+k)$, with $\mathcal{P}$ an antiderivative and P a polynomial of degree at most $m+2$, which is manifestly analytic and polynomially bounded in regions of the form S_C, and $\hat{y}+\delta/\Gamma$ is manifestly a solution of (1), which, by construction, is also polynomially bounded in S_C.

By Remark 1, $\hat{y} + \delta/\Gamma$ is one of the solutions y_c. But y_c is inverse Laplace transformable with respect to z, and has sufficient decay to ensure the existence of $m-1$ derivatives of the transform. By Remark 1, any solution that decreases in the natural region S_C in the right half plane can be represented in this way and thus the conclusion follows. □

Corollary 3. *In $\{p : \Re(p) < 1\} \cup [1, \infty)$, there is a one parameter family of Laplace transformable solutions to (10), the functions H_c in Theorem 2 (i). They have ℓ as a barrier of singularities.*

Proof of Theorem 2 (ii). Since all Laplace transformable solutions to (10) are those provided in Remark 1, we analyze the properties of the inverse Laplace transform of these functions for $\Re(p) > 1$.

We note that, due to the fact that $y_c(z_m)$ increase at most as e^{z_m}/z_m^m, we can deform for $\Re(p) > 1$, the integral

$$\int_{c-i\infty}^{c+i\infty} e^{pz_m} y_c(z_m) dz_m \tag{11}$$

to an integral

$$\int_C e^{pz_m} y_c(z_m) dz_m \tag{12}$$

where C starts at $-\infty - i\epsilon$, avoids the origin through the right half plane and turns back to $-\infty + i\epsilon$. In view of the bound mentioned above for $y_c(z_m)$, this function is manifestly bounded and analytic for $\Re(p) > 1$, and in fact is continuous with $m-1$ derivatives up to $\Re(p) = 1$.

Cohesive continuation and pseudocontinuation. It follows from our analysis and from the fact that Écalle's cohesive continuation also provides solutions to the equation, that the results of the continuations are the same (modulo the choice of one parameter, discussed in the Appendix). This type of continuation is the natural one since it provides solutions to the associated convolution equation. It is easy to see however that this continuation is not a classical pseudocontinuation through the barrier, as it follows from the following Proposition.

Proposition 4. *The values of H_c on the two sides of ℓ are not pseudocontinuations* [9] *of each–other.*

Proof. Indeed, pseudocontinuation [9], pp. 49 requires that the analytic elements coincide almost everywhere on the two sides of the barrier. But H_c is continuous on both sides, and then the values would coincide everywhere, immediately implying analyticity through ℓ, a contradiction.

Remark 3. *The axis $\mathbb{R}^+$, which is also a Stokes line, plays a special role. No other points on the singularity barrier can be used for Borel summation, as shown in the proposition below.*

Proposition 5. *No Laplace transformable solution of (10) exists, in directions* $e^{i\phi}\mathbb{R}^+$, $\phi \in (0, \pi/2)$. *(The same conclusion holds with* $\phi \in (-\pi/2, 0)$.*)*

Proof. Indeed, the Laplace transform y of such a solution would be analytic and decreasing in a half plane bisected by $e^{i\phi}$ and solve(1). Since $1/\Gamma(x)$ is entire and the general solution is of the form (6), by periodicity $f_1 = f - \frac{\pi}{e}\cot \pi x$ would be entire too. Taking now a ray $te^{i(\phi+\pi/2-\epsilon)}$ we see, using again periodicity, that f_1 decreases factorially in the upper half plane. Standard contour deformation shows that half of the Fourier coefficients are zero, $f_1(x) = \sum_{k\in\mathbb{N}} c_k e^{ikx}$ and that, because f is entire, c_k decrease faster than geometrically. But then $f_1(x) =: F(\exp(2\pi i x))$ with $t \mapsto F(t)$ entire. When $x \to i\infty$, $t \to 0$ and, unless $F = 0$, we have $F(t) \sim ct^n$ for some $n \in \mathbb{N}$, thus $f(x) \sim ce^{inx}$, incompatible with factorial decay. This means $f = 0$ but then (6) is not analytic on the real line[4]. □

4. Appendix: Weak acceleration, integral representation, median choice, natural crossing of the barrier

A weak acceleration is provided by the passage $x \ln x - x \mapsto x$. The $x-$ inverse Laplace transform of (1) satisfies $e^{-p}Y - \int_0^p Y(s)ds - 1 = 0$ with the solution $Y = e^{-1}\exp(p + \exp(p))$. $\mathcal{L}Y$ exists along any (combination of) paths R_n starting from the origin and ending on a ray of the form $p = \mathbb{R}^+ + (2n+1)i\pi, n \in \mathbb{Z}$. The function $f_+ = \int_{R_1} e^{-xp}e^{p+e^p-1}dp$ is manifestly entire[5]. For $x = -t; t \to \infty$ the saddle point method gives

$$f_+ \sim \sqrt{2\pi}e^{t\ln t - t + \pi i t + \frac{1}{2}\ln t - 1}$$

which identifies f_+ with $y_1 + \pi i/e/\Gamma(x)$. With obvious notations, we see that $y_1 = \frac{1}{2}(f_+ + f_-)$, reminiscing of medianization. We have also checked numerically that y_1 is approximated by least term truncation of its asymptotic series with errors $o(1/\Gamma(x))$. (The integral representation would allow for a rigorous check, but we have not done this and we state the property as a conjecture; we also conjecture that the solution constructed in Proposition 2 is y_1; this could be checked by looking at the asymptotic behavior on ∂S_C.) There is, obviously, only one solution so well approximated. It should then be considered as the natural candidate for the medianized transform in critical time and its inverse Laplace transform, defined on the whole of $\mathbb{R}^+$, and the natural continuation of the Borel transform $\mathcal{B}\tilde{y}$ past the barrier. For all these reasons it is likely, but we have not checked it rigorously, that y_1 corresponds to the medianized cohesive continuation of Écalle.

Remark 4. *The procedure described of naturally crossing a barrier does not necessarily depend on the existence of an underlying functional equation. It is sufficient to have accelerations as above that allow for Borel (over)summation along some paths, and choose as a natural actual function the one that has minimal errors in least term truncation or resort to a medianized choice. The process of continuation through the barrier can be written as the composition* $\mathcal{L}_{z_m}^{-1}\mathcal{L}_{z_1}\mathcal{B}_{z_1}\hat{\mathcal{L}}_{z_m}$ *with* $\hat{\mathcal{L}}$ *formal Laplace transform, and is expected to commute with most operations of*

[4]We should note that a procedure mimicking the proof of Theorem 2 (i) in non-horizontal directions would fail because now the remainders $R(x)$ would grow fast along the direction of evolution – parallel to $\mathbb{R}^+$.

[5]It provides, in view of the superexponential properties of the integrand, Borel oversummation.

natural origin. It is applicable to many other series including the Dirichlet series $\sum_{k=0}^{\infty} e^{(p-1)n^2}$.

Finally, it seems a plausible conjecture that in the case of nonlinear systems, infinitely many equally spaced "isolated" barriers should occur.

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