数理解析研究所講究録 1424 巻 2005 年 86-102

Deformation of the Schrödinger equation and exact asymptotic analysis.

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Abstract

Together with their associated Stokes multipliers, we analyze the dependence in the parameter $\underline{a}$ of the resurgent solutions at infinity of the Schrödinger equation $\frac{d^2}{dx^2}\Phi(x) = \frac{P_m(x,\underline{a})}{x^2}\Phi(x)$, where P_m is a monic polynomial function of order m with coefficients $\underline{a} = (a_1, \cdots, a_m)$. This provides a number of functional relations which can be used to compute the Stokes multipliers for a class of polynomials P_m.

1 Introduction

In his book [17], Sibuya provided an impressive description of the asymptotic properties when $|x| \to \infty$ of the solutions of the ordinary differential equation $-\frac{d^2\Phi}{dx^2} + P_m(x,\underline{a})\Phi = 0$, where

$$P_m(x,\underline{a}) = x^m + a_1x^{m-1} + \cdots + a_m, \quad \underline{a} = (a_1, \cdots, a_m) \in \mathbb{C}^m$$

is a complex monic polynomial function of order m. Avoiding almost systematically the use of known special functions, one of his main goal was to derive from the symmetries of the equation only informations about the Stokes multipliers, which are in essence transcendental functions of the parameter $\underline{a}$ of the equation.
This idea of Sibuya has been renewed by a number of recent works, some of them, in relation with spectral problems, having been presented during the conference (A. Voros, R. Tateo, J. Suzuki).

The present paper belongs to this trend of research. Our purpose is to analyze the ordinary differential equation

$$(\mathfrak{E}_m) \qquad \frac{d^2}{dx^2}\Phi(x) = \frac{P_m(x,\underline{a})}{x^2}\Phi(x), \quad m \in \mathbb{N}^\star,$$

from the (exact) asymptotic viewpoint, thus generalizing (part of) the work of Sibuya. We shall see what kind of information we can extract from the symmetries of equation $(\mathfrak{E}_m)$,

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focusing in particular on the Stokes multipliers and related connection matrices. Of course, when compared with the work of Sibuya [17], the main novelty comes from the existence of a regular singular point at the origin, so that a non trivial monodromy has to be taken into account.

We have to mention that our motivation for considering equation $(\mathfrak{E}_m)$ relies on problems also discussed during the conference (C. Bender, M. Znojil, R. Tateo), namely the surprising properties of the so-called PT-symmetric operators, see, e.g., [2, 6, 7, 16, 18]. A particular example of such an operator is given by $-\frac{d^2}{dx^2} - (ix)^{2m} - \alpha(ix)^{m-1} + \frac{l(l+1)}{x^2}$. Despite its non-Hermiticity, this operator has a (bounded below) real spectral set, as shown by Dorey et al [9] by ideas and tools usually used in the context of integrable models in quantum field theory. What we have in mind is to generalize this result to equation $(\mathfrak{E}_m)$ with PT-symmetry, and the present work is a first step in that direction.

The paper is organized as follows. In section 2 we localize at infinity, introducing a well-behaved, resurgent, fundamental solution. This solution can be calculated by Borel-resummation, and we present a (gentle) generalization of the known method by factorial series. We then introduce a family of fundamental solutions, and their associated Stokes-Sibuya coefficients. In section 3, we localize at the origin, introducing a convenient system of fundamental solutions (Fuchs theory). We compare in section 4 these different families of fundamental solutions and deduce, in section 5, a set of functional relations. Section 6 is devoted to functional relations in case of higher symmetries. We describe some applications in section 7.

Apart from the new results in subsections 2.2 and 6.2, most of the proofs have been omitted here, and will be published in [8]

2 Solutions of $(\mathfrak{E}_m)$ in the neighbourhood of infinity

2.1 Asymptotics and resurgence

We start with the question of the existence of solutions at infinity for equation $(\mathfrak{E}_m)$. In what follows, $\frac{\sqrt{P_m(x,\underline{a})}}{x} = x^{\frac{m}{2}-1} + \sum_{k=1}^{N} b_{\frac{m}{2}-k}(\underline{a}) x^{\frac{m}{2}-k-1} + O(x^{\frac{m}{2}-N})$ stands for the asymptotic expansion at infinity in x of $\frac{\sqrt{P_m(x,\underline{a})}}{x}$.

Theorem 2.1. *The differential equation $(\mathfrak{E}_m)$ admits a unique solution $\Phi_0(x,\underline{a})$ satisfying condition 1. :*

- *1. Φ_0 is an analytic function in x in the sector $\Sigma_0 = \{|x| > 0, \mid \arg(x) \mid < \frac{3\pi}{m}\}$ such that, in any strict sub-sector of Σ_0, Φ_0 admits an asymptotic expansion at infinity of the following form* [1]

$$T\Phi_0(x,\underline{a}) = x^{r(\underline{a})} e^{-S(x,\underline{a})} \phi_0(x,\underline{a}),$$

uniformly with respect to $\underline{a}$ in any compact set of $\mathbb{C}^m$, where:

[1] Throughout this theorem, $x^\alpha = \exp(\alpha \ln(x))$ with $\ln(x)$ real for $\arg(x) = 0$.

– *i). for odd m,*
$$\begin{cases} S(x,\underline{a}) = \dfrac{2}{m}x^{\frac{m}{2}} + \displaystyle\sum_{k=1}^{\frac{m-1}{2}} \dfrac{b_{\frac{m}{2}-k}(\underline{a})}{\frac{m}{2}-k}x^{\frac{m}{2}-k} \in \mathbb{C}[\underline{a}][x^{\frac{1}{2}}] \\ r(\underline{a}) = \dfrac{1}{2} - \dfrac{m}{4} \\ \phi_0 \in \mathbb{C}[\underline{a}][[x^{-\frac{1}{2}}]] \text{ with constant term } 1. \end{cases}$$

– *ii). for even m,*
$$\begin{cases} S(x,\underline{a}) = \dfrac{2}{m}x^{\frac{m}{2}} + \displaystyle\sum_{k=1}^{\frac{m}{2}-1} \dfrac{b_{\frac{m}{2}-k}(\underline{a})}{\frac{m}{2}-k}x^{\frac{m}{2}-k} \in \mathbb{C}[\underline{a}][x] \\ r(\underline{a}) = \dfrac{1}{2} - \dfrac{m}{4} - b_0(\underline{a}) \\ \phi_0 \in \mathbb{C}[\underline{a}][[x^{-1}]] \text{ with constant term } 1. \end{cases}$$

Moreover:

- *2. Φ_0 extends analytically in x to the universal covering of $\mathbb{C}^\star$, and is an entire function in $\underline{a}$.*
- *3. The derivative Φ_0' of Φ_0 with respect to x admits an asymptotic expansion at infinity of the form:*
$$T\left(\frac{d}{dx}\Phi_0(x,\underline{a})\right) = \frac{d}{dx}\left(T\Phi_0(x,\underline{a})\right) = x^{r(\underline{a})+\frac{m}{2}-1}e^{-S(x,\underline{a})}(-1+o(1))$$
when x tends to infinity in any strict sub-sector of Σ_0, uniformly with respect to $\underline{a}$.

This theorem can be shown with the methods developed in Sibuya's book [17], see [14, 1]. However, using the resurgent viewpoint, one can get a stronger result:

Theorem 2.2. *We use the notations of theorem 2.1. There exists a unique formal power series expansion $\psi_0 \in \mathbb{C}[\underline{a}][[S^{-1/m}]]$ for m odd , $\psi_0 \in \mathbb{C}[\underline{a}][[S^{-2/m}]]$ for m even, with constant term equal to 1, resurgent in S with regular dependence[2] on $\underline{a}$ in any given compact K, and Borel resummable, uniformly in $\underline{a} \in K$, such that Φ_0 can be described by*
$$\Phi_0(x,\underline{a}) = \left(\frac{m}{2}S\right)^{\frac{2}{m}r(\underline{a})} e^{-S}\, \mathrm{s}_\alpha\psi_0(S,\underline{a})\,|_{S=S(x,\underline{a})} \tag{1}$$
(uniformaly in $\underline{a}$ for $\underline{a}$ in any given compact set), where s_α denotes the Borel sum with respect to S, while the direction of Borel resummation α runs through $]-\pi,+\pi[$. Moreover, the minor of ψ_0 can be analytically continued to the universal covering of $\mathbb{C}\backslash\{0,-2\}$.

The proof of this theorem is detailed in [8][3]. Apart from the dependence in $\underline{a}$, this result is essentially a known theorem [10, 3, 11].

[2] in the sense of [5].

[3] In [8], for technical reasons, we used different variables of resurgence, z for m odd and $\tilde{z}$ for m even. Since the resurgence variable z (*resp.* $\tilde{z}$) reads $z = S.\Lambda(S,\underline{a})$ (*resp.* $\tilde{z} = S.\tilde{\Lambda}(S,\underline{a})$), with $\Lambda(S,\underline{a})$ (*resp.* $\tilde{\Lambda}(S,\underline{a})$) an analytic function which tends to 1 when S tends to infinity (uniformaly in $\underline{a}$, for $\underline{a}$ in a given compact set) then, from general nonsense in resurgence theory S may be chosen as a new resurgence variable and, moreover, the resurgence properties of the formal power series expansions are preserved: same singularities for their minors, same resurgence equations.

2.2 Asymptotics and summation

In theorem 2.2, the Borel sum $\mathrm{s}_\alpha\psi_0(S,\underline{a})$ can be calculated with various methods, e.g., by means of the hyperasymptotic theory. It can also be computed exactly by the resummation method by factorial series, a method which goes back to Watson, Nörlund and Nevanlinna. We describe briefly this method, computing $\mathrm{s}_0\psi_0(S,\underline{a})$ only to simplify.
By theorem 2.2, ψ_0 reads

$$\psi_0(S,\underline{a}) = 1 + \sum_{k=1}^{\infty} \frac{A_k(\underline{a})}{S^{\frac{k}{m}}} \in \mathbb{C}[\underline{a}][[S^{-\frac{1}{m}}]],$$

which we write as $\psi_0(S,\underline{a}) = \sum_{l=0}^{m-1} \frac{1}{S^{\frac{l}{m}}}\psi_0^{(l)}(S,\underline{a})$ with $\psi_0^{(l)}(S,\underline{a}) \in \mathbb{C}[\underline{a}][[S^{-1}]]$. We remark that, for $k = 0,1,\cdots,m-1$,

$$\psi_0(e^{2i\pi k}S,\underline{a}) = \sum_{l=0}^{m-1} \frac{\omega^{-lk}}{S^{\frac{l}{m}}}\psi_0^{(l)}(S,\underline{a})$$

with $\omega = e^{\frac{2i\pi}{m}}$. This reads also :

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & \omega^{-(m-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{-(m-1)} & \cdots & \omega^{-(m-1)(m-1)} \end{pmatrix} \begin{pmatrix} \psi_0^{(0)}(S,\underline{a}) \\ \dfrac{\psi_0^{(1)}(S,\underline{a})}{S^{\frac{1}{m}}} \\ \vdots \\ \dfrac{\psi_0^{(m-1)}(S,\underline{a})}{S^{\frac{m-1}{m}}} \end{pmatrix} = \begin{pmatrix} \psi_0(S,\underline{a}) \\ \psi_0(e^{2i\pi}S,\underline{a}) \\ \vdots \\ \psi_0(e^{2i\pi(m-1)}S,\underline{a}) \end{pmatrix}.$$

Since the left-hand side $m \times m$ matrix is a Vandermonde invertible matrix, each $\psi_0^{(l)}(S,\underline{a})$ can be written as a linear combination of the $\psi_0(e^{2i\pi k}S,\underline{a})$'s times $S^{\frac{l}{m}}$. By theorem 2.2, the formal series $\psi_0(e^{2i\pi k}S,a)$ are Borel-resummable in S, so that each $\psi_0^{(l)}(S,\underline{a})$ is also S-Borel-resummable. Following [13, 12], we can conclude that:

Proposition 2.3. *We consider the formal expansion* $\psi_0(S,\underline{a}) = 1 + \sum_{k=1}^{\infty} \frac{A_k(\underline{a})}{S^{\frac{k}{m}}} \in \mathbb{C}[\underline{a}][[S^{-\frac{1}{m}}]]$ *of theorem 2.2. Then, there exist* $b_n^{(l)}(a) \in C[\underline{a}]$ *and* $\tau > 0$ *such that the series expansion*

$$\sum_{l=0}^{m-1} \frac{1}{S^{\frac{l}{m}}} \left(b_0^{(l)}(\underline{a}) + \sum_{n=0}^{\infty} b_{n+1}^{(l)}(\underline{a}) \frac{\Gamma(n+1)}{S(S+1)\cdots(S+n)} \right)$$

converges absolutely for $\Re S > \tau$*, and its sum represents* $\mathrm{s}_0\psi_0(S,\underline{a})$.

The coefficients $b_n^{(l)}(\underline{a}) \in \mathbb{C}[\underline{a}]$ can be derived from the $A_k(\underline{a})$'s by the Stirling's algorithm, see [13].

We now sketch a method which can be seen as an (apparently not known) extension of the previous resummation method by factorial series.

Considering ψ_0, its minor $\widetilde{\psi}_0$ is given by

$$\widetilde{\psi}_0(\zeta,\underline{a}) = \sum_{k=1}^{\infty} A_k(\underline{a}) \frac{\zeta^{\frac{k}{m}-1}}{\Gamma(\frac{k}{m})} \in \zeta^{-1}\mathbb{C}[\underline{a}]\{\zeta^{\frac{1}{m}}\}$$

where the right-hand side series expansion converges on the universal covering of $D(0,2)\backslash\{0\}$ ($D(0,2)$ is the open disc centered at the origin, with radius 2). We now set $s = e^{-\zeta}$. This defines a biholomorphic map $s \mapsto \zeta(s)$ such $\zeta = (1-s)f(1-s)$ where f is holomorphic near 0 and $f(0) = 1$ (by Lagrange's theorem), from a neighbourhood of $s = 1$ onto a neighbourhood of $\zeta = 0$. Introducing $g_0(s,\underline{a}) = \widetilde{\psi}_0(\zeta,\underline{a})$, g_0 may be identified with its Taylor-Puiseux expansion at $s = 1$, which reads

$$g_0(s,\underline{a}) = \sum_{k=1}^{\infty} B_k(\underline{a}) \frac{(1-s)^{\frac{k}{m}-1}}{\Gamma(\frac{k}{m})}.$$

Formally, one has

$$\mathrm{s}_0\psi_0(S,\underline{a}) = 1 + \int_0^{+\infty} e^{-\zeta S}\widetilde{\psi}_0(\zeta,\underline{a})\,d\zeta = 1 + \sum_{k=1}^{\infty} \frac{B_k(a)}{\Gamma(\frac{k}{m})} \int_0^{+\infty} e^{-\zeta S}(1-e^{-\zeta})^{\frac{k}{m}-1}\,d\zeta$$

$$= 1 + \sum_{k=1}^{\infty} \frac{B_k(\underline{a})}{\Gamma(\frac{k}{m})} \int_0^1 (1-s)^{\frac{k}{m}-1} s^{S-1}\,dS,$$

so that $\mathrm{s}_0\psi_0(S,\underline{a}) = 1 + \sum_{k=1}^{\infty} \frac{B_k(\underline{a})}{\Gamma(\frac{k}{m})}\beta(\frac{k}{m},S)$ where β is the Euler beta function. We eventually get:

$$\mathrm{s}_0\psi_0(S,\underline{a}) = 1 + \sum_{k=1}^{\infty} B_k(\underline{a}) \frac{\Gamma(S)}{\Gamma(S+\frac{k}{m})}.$$

Up to this point, these transformations were formal. We have in fact the following proposition whose proof is similar to that of theorem 1.5.2.1 in Malgrange [13]:

Proposition 2.4. *We consider the expansion* $\psi_0(S,\underline{a}) = 1 + \sum_{k=1}^{\infty} \frac{A_k(\underline{a})}{S^{\frac{k}{m}}} \in \mathbb{C}[\underline{a}][[S^{-\frac{1}{m}}]]$ *of theorem 2.2. Then, there exists a* $\tau > 0$ *such that the series expansion*

$$1 + \sum_{k=1}^{\infty} B_k(\underline{a}) \frac{\Gamma(S)}{\Gamma(S+\frac{k}{m})}$$

converge absolutely for $\Re S > \tau$*, and its sum represents* $\mathrm{s}_0\psi_0(S,\underline{a})$.

To derive the coefficients $B_k(\underline{a})$ from the $A_k(\underline{a})$, all we have to do is to compute the decomposition given by proposition 2.4 for $\frac{1}{S^r}$, $r > 0$. For $\Re S > 0$, we have

$$\frac{1}{S^r} = \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-uS} u^{r-1} du$$

so that, with $u = -\ln(s)$, $\dfrac{1}{S^r} = \dfrac{1}{\Gamma(r)} \int_0^1 s^{S-1} \left(-\ln(s)\right)^{r-1} ds$. For $s \in]0,1[$, one can write $-\dfrac{\ln(s)}{1-s} = \sum_{j=0}^{\infty} \dfrac{j!}{j+1} \dfrac{(1-s)^j}{j!}$. Then,

$$\left(-\frac{\ln(s)}{1-s}\right)^{r-1} = \sum_{j=0}^{\infty} c_{r,j} \frac{(1-s)^j}{j!}$$

with

$$c_{r,0} = 1, \quad c_{r,j} = \sum_{1 \le p \le j} \frac{\Gamma(r)}{\Gamma(r-p)} B_{j,p}\left(\frac{1!}{2}, \frac{2!}{3}, \cdots, \frac{l!}{l+1}, \cdots\right), \; j \ge 1,$$

where the $B_{j,p}$'s are the exponential partial Bell polynomial functions ([4]). By exchanging $\sum$ and $\int$ (this can be justified, cf. [15]), one deduces that

$$\frac{1}{S^r} = \sum_{j=0}^{\infty} \frac{c_{r,j}}{\Gamma(r) j!} \int_0^1 s^{S-1} (1-s)^{r+j-1} ds = \sum_{j=0}^{\infty} \frac{c_{r,j}}{\Gamma(r) j!} \beta(r+j, S)$$

$$\frac{1}{S^r} = \sum_{j=0}^{\infty} \frac{c_{r,j}}{\Gamma(r) j!} \frac{\Gamma(r+j)\Gamma(S)}{\Gamma(r+j+S)}.$$

We eventually get:

$$\frac{1}{S^r} = \frac{\Gamma(S)}{\Gamma(r+S)} + \sum_{j=1}^{\infty} d_{r,j} \frac{\Gamma(S)}{\Gamma(r+j+S)}$$

with

$$d_{r,j} = \left(\sum_{1 \le p \le j} \frac{B_{j,p}(\frac{1!}{2}, \frac{2!}{3}, \cdots, \frac{l!}{l+1}, \cdots)}{\Gamma(r-p)} \right) \frac{\Gamma(r+j)}{j!}.$$

Lemma 2.5. *In proposition 2.4, we have*

$$B_k(\underline{a}) = A_k(\underline{a}) + \sum_{\substack{j,l \ge 1 \\ l+jm=k}} d_{\frac{l}{m},j} . A_l(\underline{a}).$$

To end this subsection, we note that it would be interesting to compare the method of summation based on propositions 2.4 and 2.3 with the hyperasymptotics, which is based on the knowledge of the resurgent structure. In particular, we believe that this could provide explicit error bound (which seems to be not known) when truncating the factorial series expansions, for concrete computations. We are presently engaged in this study.

2.3 Fundamental systems and Stokes-Sibuya coefficients

In this subsection, using the quasi-homogeneity property of equation $(\mathfrak{E}_m)$, we derive from Φ_0 (given by theorem 2.1) a family of fundamental systems of solutions of $(\mathfrak{E}_m)$, and introduce the Stokes-Sibuya coefficients which govern the Stokes phenomena. For that purpose, it is useful to introduce the following notations:

Notation 2.6. We set $\omega = e^{+\frac{2i\pi}{m}}$. For all $\lambda \in \mathbb{C}$ and all $\underline{a} = (a_1, \cdots, a_m) \in \mathbb{C}^m$, we note

$$\lambda . \underline{a} := (\lambda a_1, \cdots, \lambda^m a_m).$$

Using the fact that equation $(\mathfrak{E}_m)$ is invariant under the transformation $(x,a) \mapsto (\omega x, \omega.\underline{a})$, one easily gets the following result:

Lemma 2.7. *For all $k \in \mathbb{Z}$, we define $\Phi_k(x,\underline{a}) = \Phi_0(\omega^k x, \omega^k.\underline{a})$. Then, Φ_k is a solution of $(\mathfrak{E}_m)$, and is entire in a. Its asymptotic expansion when x tends to infinity in the sector $\Sigma_k = \{|x| > 0, \mid arg(x) + k.arg(\omega) \mid < \frac{3\pi}{m}\}$, uniformly in $\underline{a}$ in any compact set of $\mathbb{C}^m$, is given by :*

$$T\Phi_k(x,\underline{a}) = T\Phi_0(\omega^k x, \omega^k.\underline{a})$$

where $T\Phi_0$ is the asymptotic expansion of Φ_0 in Σ_0 described in theorem 2.1.

We remark that for all $k \in \mathbb{Z}$, the solution Φ_k is a "subdominant function" (in the sense of [17], p. 19) in the sector $\Lambda_k = \{\mid arg(x) + k.arg(\omega) \mid < \frac{\pi}{m}\}$. Since the sectors Λ_{k-1}, Λ_k and Λ_{k+1} are included in Σ_k, the previous lemma 2.7 allows to calculate the Wronskian of (Φ_k, Φ_{k+1}) (cf [8]):

$$W(\Phi_k, \Phi_{k+1}) = 2(-1)^k \omega^{k(1-\frac{m}{2})+r(\omega^{k+1}.a)}$$

where r is given by theorem 2.1. This means that for all $k \in \mathbb{Z}$, $\{\Phi_k, \Phi_{k+1}\}$ constitutes a fundamental system of solutions of $(\mathfrak{E}_m)$. Since $(\mathfrak{E}_m)$ is a second order linear differential equation, we deduce the existence of functions $C_k(\underline{a})$, $\widetilde{C}_k(\underline{a})$ such that :

$$\forall k \in \mathbb{Z}, \Phi_{k-1} = C_k(\underline{a})\Phi_k + \widetilde{C}_k(\underline{a})\Phi_{k+1}. \tag{2}$$

Definition 2.8. The functions $C_k(\underline{a})$ and $\widetilde{C}_k(\underline{a})$ are called the *Stokes-Sibuya coefficients* of Φ_{k-1} associated respectively with Φ_k and Φ_{k+1}. The matrices $\mathfrak{S}_k(\underline{a}) := \begin{pmatrix} C_k(\underline{a}) & \widetilde{C}_k(\underline{a}) \\ 1 & 0 \end{pmatrix}$ are called the *Stokes-Sibuya connection matrices.*

The analytic properties of the Stokes-Sibuya coefficients are described by the following theorem, proved in [8]:

Theorem 2.9. *For all $k \in \mathbb{Z}$ we note $\Phi_k(x,\underline{a}) = \Phi_0(\omega^k x, \omega^k.\underline{a})$, where Φ_0 is the solution of $(\mathfrak{E}_m)$ defined by theorem 2.1. Then, for all $k \in \mathbb{Z}$, $\Phi_k(x,\underline{a})$ is analytic in x on the universal covering of $\mathbb{C}^*$ and entire in $\underline{a}$; moreover, the system $\{\Phi_k, \Phi_{k+1}\}$ constitutes a fundamental system of solutions of $(\mathfrak{E}_m)$.*

- *We have $\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x,\underline{a}) = \mathfrak{S}_k(\underline{a}) \begin{pmatrix} \Phi_k \\ \Phi_{k+1} \end{pmatrix}(x,\underline{a})$, where the Stokes-Sibuya connection matrix $\mathfrak{S}_k(\underline{a})$ is invertible, and entire in $\underline{a}$. Also, for all $k \in \mathbb{Z}$,*

$$\mathfrak{S}_k(\underline{a}) = \mathfrak{S}_{k-1}(\omega.\underline{a}), \quad \mathfrak{S}_k(\underline{a}) = \mathfrak{S}_0(\omega^k.\underline{a}). \tag{3}$$

- *The Stokes-Sibuya coefficients $C_k(\underline{a})$ and $\widetilde{C}_k(\underline{a})$ associated respectively with Φ_k and Φ_{k+1} are entire functions in $\underline{a}$ and,*

$$\begin{cases} C_k(\underline{a}) = C_0(\omega^k.\underline{a}), & C_k = C_{k \bmod m} \\ \widetilde{C}_k(\underline{a}) = \widetilde{C}_0(\omega^k.\underline{a}) = \omega^{m-2+2r(\omega^k.\underline{a})}, & \widetilde{C}_k = \widetilde{C}_{k \bmod m}. \end{cases} \tag{4}$$

Besides the Stokes structure at infinity, described by the previous theorem, equation $(\mathfrak{E}_m)$ develops a none trivial monodromy at infinity:

Definition 2.10. The 2×2 matrices $\mathfrak{M}_k^\infty(\underline{a})$, $k \in \mathbb{Z}$, defined by:

$$\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(\omega^m x, \underline{a}) = \mathfrak{M}_k^\infty(\underline{a}) \begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x, \underline{a}). \tag{5}$$

are called the ∞-monodromy matrices.

These ∞-monodromy matrices enjoy the following properties (cf. [8]):

Theorem 2.11. *For all $k \in \mathbb{Z}$, the ∞-monodromy matrix $\mathfrak{M}_k^\infty(\underline{a})$ is invertible, entire in $\underline{a}$, and*

$$\mathfrak{M}_k^\infty(\underline{a}) = \mathfrak{M}_0^\infty(\omega^k.\underline{a}). \tag{6}$$

Furthermore, the Stokes-Sibuya connection matrices satisfy the functional relation:

$$\mathfrak{S}_0(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) = (\mathfrak{M}_0^\infty(\underline{a}))^{-1}. \tag{7}$$

Relation (7) generalized a functional relation due to Sibuya [17].

3 Solutions of $(\mathfrak{E}_m)$ in the neighbourhood of the origin

To get more information about the Stokes-Sibuya coefficients C_k, we now localize near the origin. Since this point is a regular singular point of $(\mathfrak{E}_m)$, we use the classical Fuchs theory to describe new "canonical" systems of solutions of $(\mathfrak{E}_m)$ near the origin. The characteristic equation is $s(s-1) - a_m = 0$ so that $\frac{1 \pm p}{2}$ are the characteristic values, with $p = (1+4a_m)^{\frac{1}{2}}$.

Notation 3.1. In what follows, $p = (1+4a_m)^{\frac{1}{2}}$ and $s(p) = \frac{1+p}{2}$.
We note $\underline{a}' := (a_1, \cdots, a_{m-1})$, and for all $\tau \in \mathbb{C}$,

$$\tau.\underline{a}' := (\tau a_1, \cdots, \tau^{m-1} a_{m-1}).$$

One has the following results, see [8].

Theorem 3.2. *There exist two unique linearly independent solutions f_1, f_2 of $(\mathfrak{E}_m)$ such that*

$$\begin{cases} f_1(x, \underline{a}', p) = x^{s(p)} g_1(x, \underline{a}', p) = x^{s(p)} \left(1 + \displaystyle\sum_{k=1}^{\infty} A_k(\underline{a}', p) x^k\right) \\ \\ f_2(x, \underline{a}', p) = \lambda(\underline{a}', p) f_1(x, \underline{a}', p) \ln(x) + x^{s(-p)} g_2(x, \underline{a}', p) \end{cases}$$

where g_1, g_2 are entire functions in x and $\underline{a}'$, while λ is entire in $\underline{a}'$. Moreover, g_1 is meromorphic in p with at most simple poles when $-p \in \mathbb{N}^\star$.

1. *When $p \notin \mathbb{Z}$, then $\lambda(\underline{a}', p) = 0$ and $g_2(x, \underline{a}', p) = g_1(x, \underline{a}', -p)$.*
2. *When $p \in \mathbb{N}^\star$, then $g_2(x, \underline{a}, p) = \left(1 + \displaystyle\sum_{k=1}^{\infty} B_k(\underline{a}, p) x^k\right)$ with $B_p = 0$. Moreover, for all $k \in \mathbb{N}^\star$, $\lambda(\underline{a}', p)$, $B_k(\underline{a}', p) \in \mathbb{C}[\underline{a}']$, and $\lambda(\omega.\underline{a}', p) = \omega^{-p}\lambda(\underline{a}', p)$.*

3. When $p=0$, *then* $\lambda(\underline{a}',p)=1$ *and* $g_2(x,\underline{a}',p)=\sum_{k=1}^{\infty}B_k(\underline{a}',p)x^k$ *with, for all* $k\in\mathbb{N}^\star$, $B_k(\underline{a}',p)\in\mathbb{C}[\underline{a}']$.

Remark 3.3. When $-p\in\mathbb{N}^\star$, just change p into $-p$ in theorem 3.2.

Remark 3.4. In the special case when $\underline{a}'=0$, the function g_1 is meromorphic in p with at most simple poles when $-p\in m\mathbb{N}^\star$.

Remark 3.5. For every $p\in\mathbb{N}^\star$, $\lambda(\underline{a}',p)$ can be computed exactly.

¿From theorem 3.2 and from the quasi-homogeneity of equation $(\mathfrak{E}_m)$, one easily obtains:

Corollary 3.6. *We consider the fundamental system of solutions* (f_1,f_2) *of theorem 3.2. Then,*

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\omega^m x,\underline{a}',p)=\mathfrak{M}(\underline{a}',p)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(x,a) \tag{8}$$

where

$$\mathfrak{M}(\underline{a}',p)=\begin{pmatrix} e^{2i\pi s(p)} & 0 \\ 2i\pi\lambda(\underline{a}',p)e^{2i\pi s(-p)} & e^{2i\pi s(-p)} \end{pmatrix} \tag{9}$$

is the monodromy matrix at the origin.

4 The 0∞ connection matrices

To compare the set of fundamental systems of solutions (Φ_{k-1},Φ_k) of $(\mathfrak{E}_m)$ introduced in section 2 with the fundamental system of solutions (f_1,f_2), we introduce, for all $k\in\mathbb{Z}$:

$$\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x,\underline{a})=M_k(\underline{a}',p)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(x,\underline{a}',p) \tag{10}$$

where the matrices $M_k(\underline{a}',p)$ are invertible.

Definition 4.1. The matrices $M_k(\underline{a}',p)$ are called the 0∞-*connection matrices.*

We now give some properties of the M_k, see [8].

Theorem 4.2. *a) For all* $k\in\mathbb{Z}$,

$$\det M_k(\underline{a}',p)=\begin{cases} 2(-1)^k\dfrac{\omega^{(k-1)(1-\frac{m}{2})+r(\omega^k.\underline{a})}}{p} & \text{for } p\neq 0 \\[2ex] 2(-1)^{k-1}\omega^{(k-1)(1-\frac{m}{2})+r(\omega^k.\underline{a})} & \text{for } p=0. \end{cases} \tag{11}$$

b) For all $k\in\mathbb{Z}$, *the matrix* $M_k(\underline{a}',p)$ *is entire in* $\underline{a}'$. *More precisely,*

$$M_k(\underline{a}',p)=$$
$$\begin{pmatrix} L_k(\underline{a}',p) & \widetilde{L}_k(\underline{a}',p) \\ \omega^{s(p)}L_k(\omega.\underline{a}',p)+\frac{2i\pi}{m}\lambda(\underline{a}',p)\omega^{s(-p)}\widetilde{L}_k(\omega.\underline{a}',p) & \omega^{s(-p)}\widetilde{L}_k(\omega.\underline{a}',p) \end{pmatrix} \tag{12}$$

where $L_k(\underline{a}',p)$ *and* $\widetilde{L}_k(\underline{a}',p)$ *are entire in* $\underline{a}'$.

c) For all $k \in \mathbb{Z}$, the matrix $M_k(\underline{a}',p)$ is holomorphic in $p \notin \mathbb{Z}$, and

$$\forall p \notin \mathbb{Z}, \forall \underline{a}' \in \mathbb{C}^{m-1},\ \widetilde{L}_k(\underline{a}',p) = L_k(\underline{a}',-p).$$

Moreover, $\widetilde{L}_k$ extends analytically at $p \in \mathbb{N}^\star$.

d) We have:

$$M_m(\underline{a}',p) = M_0(\underline{a}',p)\mathfrak{M}(\underline{a}',p). \tag{13}$$

In addition to theorem 4.2, it is easy to show the following proposition (the special case where $\underline{a}' = 0$ follows from remark 3.4):

Proposition 4.3. *The restriction to $p \notin \mathbb{Z}$ of the function $L_k(\underline{a}',p)$ (resp. $\widetilde{L}_k(\underline{a}',p)$) has a meromorphic continuation in p, with at most simple poles when $p \in \mathbb{N}$ (resp. $-p \in \mathbb{N}$).*
In the special case where $\underline{a}' = 0$, the restriction to $p \notin \mathbb{Z}$ of the function $L_k(a)$ (resp. $\widetilde{L}_k(a)$) has a meromorphic continuation in p, with at most simple poles at $p \in m\mathbb{N}$ (resp. $-p \in m\mathbb{N}$).

5 Functional relations

The different results we have described open on a set of interesting functional relations.

5.1 First functional equation

Comparing the definitions of the 0∞ connection matrices M_k and of the Stokes-Sibuya connection matrices, we see that, for all $k \in \mathbb{Z}$:

$$\mathfrak{S}_k(\underline{a}) = M_k(\underline{a}',p)M_{k+1}^{-1}(\underline{a}',p). \tag{14}$$

Therefore,

$$\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a})\cdots\mathfrak{S}_{m-1}(\underline{a}) = M_0(\underline{a}',p)M_m^{-1}(\underline{a}',p).$$

Using (13), we obtain the following theorem:

Theorem 5.1. *The Stokes-Sibuya connection matrices satisfy the following functional relation:*

$$\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a})\cdots\mathfrak{S}_{m-1}(\underline{a}) = M_0(\underline{a}',p)\mathfrak{M}^{-1}(\underline{a}',p)M_0^{-1}(\underline{a}',p). \tag{15}$$

This new functional relation, which is equivalent to formula (7), induces the following two corollaries.

Corollary 5.2. *We have $Tr\left(\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a})\cdots\mathfrak{S}_{m-1}(\underline{a})\right) = -2\cos(\pi p)$ where Tr is the Trace.*

Corollary 5.3. *We assume that $p \in \mathbb{N}^\star$. Then, with the notations of theorem 3.2,*

$$\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a})\cdots\mathfrak{S}_{m-1}(\underline{a})\,|_{\lambda(\underline{a}',p)=0} = (-1)^{p+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5.2 Second functional equation

Theorem 5.4. *We use the notations of theorem 4.2.*

- *1. We assume $p \notin \mathbb{Z}$. We assume furthermore that $\underline{a}$ is chosen so that, for all $k = 0, \cdots, m-1$, $\widetilde{L}_0(\omega^k.\underline{a}', p) \neq 0$*[4]*. Then*

$$\frac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} = -i\frac{\omega^{-\frac{3}{2}}\omega^{-(m+1)\frac{p}{2}}}{p\sin(\pi p)}\sum_{k=0}^{m-1}\frac{\omega^{r(\omega^k.\underline{a})+(k+1)p}}{\widetilde{L}_0(\omega^k.\underline{a}', p)\widetilde{L}_0(\omega^{k+1}.\underline{a}', p)}. \tag{16}$$

- *2. We assume $p \in \mathbb{N}^\star$. Assuming also that $\underline{a}$ is chosen so that, for all $k = 0, \cdots, m-1$, $\widetilde{L}_0(\omega^k.\underline{a}', p) \neq 0$, then*

$$i\pi p\omega^{\frac{3}{2}+\frac{p}{2}}\lambda(\underline{a}', p) = \sum_{k=0}^{m-1}\frac{\omega^{r(\omega^k.\underline{a},p)+(k+1)p}}{\widetilde{L}_0(\omega^k.\underline{a}', p)\widetilde{L}_0(\omega^{k+1}.\underline{a}', p)}. \tag{17}$$

The proof of this theorem relies on elementary linear algebra. For instance, to get point 1), one just uses formulas (11) and (12) to obtain, for $\underline{a}$ generic,

$$\mathfrak{L}\begin{pmatrix} \frac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} \\ \vdots \\ \frac{L_0(\omega^{m-1}.\underline{a}', p)}{\widetilde{L}_0(\omega^{m-1}.\underline{a}', p)} \end{pmatrix} = -\frac{2\omega^{-1}}{p}\begin{pmatrix} \frac{\omega^{r(\underline{a})}}{\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)} \\ \vdots \\ \frac{\omega^{r(\omega^{m-1}.\underline{a})}}{\widetilde{L}_0(\omega^{m-1}.\underline{a}', p)\widetilde{L}_0(\underline{a}', p)} \end{pmatrix},$$

where

$$\mathfrak{L} = \begin{pmatrix} \omega^{s(-p)} & -\omega^{s(p)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{s(-p)} & -\omega^{s(p)} \\ -\omega^{s(p)} & 0 & \cdots & 0 & \omega^{s(-p)} \end{pmatrix}$$

is a $m \times m$ invertible circulant matrix. This gives the result. Point 2. of the theorem is shown in a similar way, see [8].

Theorem 5.4 provides the following corollary:

Corollary 5.5. *The Stokes-Sibuya multiplier $C_0(\underline{a})$ satisfies:*

- *when $m = 1$, for all $\underline{a} \in \mathbb{C}$:*

$$C_0(\underline{a}) = -2\cos(\pi p),$$

- *when $m = 2$, for all $\underline{a}' \in \mathbb{C}$ and $p \notin -\mathbb{N}$:*

$$C_0(\underline{a})\widetilde{L}_0(\omega.\underline{a}', p) = -2ie^{-i\pi\frac{a_1}{2}}\cos\left(\frac{\pi}{2}(p + a_1)\right)\widetilde{L}_0(\underline{a}', p),$$

[4] This is a generic hypothesis on $\underline{a}$ since $\widetilde{L}_0(\underline{a}', p)$ cannot be identically zero.

- *when $m \geq 3$, for all $\underline{a}' \in \mathbb{C}^{m-1}$ and $p \notin -\mathbb{N}$:*

$$C_0(\underline{a})\widetilde{L}_0(\omega.\underline{a}', p) =$$

$$\omega^{r(\underline{a})-1+\frac{m}{4}}\left(\widetilde{L}_0(\underline{a}', p)\omega^{-r(\underline{a})+\frac{1}{2}-\frac{m}{4}+\frac{p}{2}} + \widetilde{L}_0(\omega^2.\underline{a}', p)\omega^{r(\underline{a})-\frac{1}{2}+\frac{m}{4}-\frac{p}{2}}\right).$$

Proof. One deduces from (14) with $k = 0$ that

$$C_0(\underline{a}) = -\frac{p}{2}\omega^{1-r(\omega.\underline{a})}\left(\omega^{-p}\frac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} - \omega^{p}\frac{L_0(\omega^2.\underline{a}', p)}{\widetilde{L}_0(\omega^2.\underline{a}', p)}\right)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p). \tag{18}$$

As shown in [8], the corollary is then a consequence of formula (16). □

6 Higher symmetries

6.1 Reduction to lower orders

In this subsection, we study a class of differential equations $(\mathfrak{E}_m)$ with higher symmetries. Namely, we shall consider the following differential equation:

$$(\mathfrak{E}^n_{nm}) \qquad x^2\frac{d^2}{dx^2}\Phi(x, \underline{a}_n) = P_{nm}(x, \underline{a}_n)\Phi(x, \underline{a}_n).$$

with the following notations:

Notation 6.1. For $m, n \in \mathbb{N}^\star$, we define

$$\underline{a}_n = (a_j)_{1 \leq j \leq nm} \text{ so that } a_j = 0 \text{ if } j \neq 0 \mod m.$$

For such a $\underline{a}_n$, we also define:

$$\underline{a}'_n := (a_j)_{1 \leq j \leq nm-1}$$

and

$$\begin{cases} \widetilde{\underline{a}}_n := \left(\frac{a_n}{n^{\frac{2}{m}}}, \frac{a_{2n}}{n^{\frac{4}{m}}}, \dots, \frac{a_{n(m-1)}}{n^{\frac{2(m-1)}{m}}}, -\frac{1}{4} + \frac{1+4a_{nm}}{4n^2}\right) \\ \widetilde{\underline{a}}'_n := \left(\frac{a_n}{n^{\frac{2}{m}}}, \frac{a_{2n}}{n^{\frac{4}{m}}}, \dots, \frac{a_{n(m-1)}}{n^{\frac{2(m-1)}{m}}}\right). \end{cases}$$

The key point in what follows is the following easy lemma:

Lemma 6.2. *If Φ satisfies the differential equation $(\mathfrak{E}^n_{nm})$ with $n, m \in \mathbb{N}^\star$, then Ψ defined by*

$$\Psi(x, \widetilde{\underline{a}}_n) := x^{\frac{n-1}{2n}}\Phi\left((n^{\frac{2}{m}}x)^{\frac{1}{n}}, \underline{a}_n\right)$$

satifies the differential equation $(\mathfrak{E}_m)$ with $\underline{a} = \widetilde{\underline{a}}_n$, that is:

$$x^2\frac{d^2}{dx^2}\Psi(x, \widetilde{\underline{a}}_n) = P_m(x, \widetilde{\underline{a}}_n)\Psi(x, \widetilde{\underline{a}}_n). \tag{19}$$

This lemma allows to compare the connection matrices of equation $(\mathfrak{E}^n_{nm})$ with those of $(\mathfrak{E}_m)$, associated with the polynomial $P_m(x, \widetilde{\underline{a}}_n)$ of lower order. We refer to [8] for the proofs of the following results.

Notation 6.3. We note $C^n_k(\underline{a}_n)$ and $\widetilde{C}^n_k(\underline{a}_n)$, $k \in \mathbb{Z}$, the Stokes-Sibuya coefficients associated with equation $(\mathfrak{E}^n_{nm})$.

Corollary 6.4. *The Stokes-Sibuya coefficients $C^n_0(\underline{a}_n)$ and $\widetilde{C}^n_0(\underline{a}_n)$ associated with equation $(\mathfrak{E}^n_{nm})$ are related to the Stokes-Sibuya coefficients C_0 and $\widetilde{C}_0$ of equation $(\mathfrak{E}_m)$ by:*

$$\begin{aligned} C^n_0(\underline{a}_n) &= \omega^{\frac{n-1}{2n}} C_0\left(\widetilde{\underline{a}}_n\right) \\ \widetilde{C}^n_0(\underline{a}_n) &= \omega^{\frac{n-1}{n}} \widetilde{C}_0\left(\widetilde{\underline{a}}_n\right) \end{aligned} \tag{20}$$

where $\omega = e^{\frac{2i\pi}{m}}$.

Notation 6.5. We note $\widetilde{L}^n_k(\underline{a}'_n, p(a_{mn}))$ and $L^n_k(\underline{a}'_n, p(a_{mn}))$, $k \in \mathbb{Z}$, the coefficients of the 0∞ connection matrices associated with equation $(\mathfrak{E}^n_{nm})$ with $p(a_{mn}) = (1 + 4a_{mn})^{\frac{1}{2}}$.

Corollary 6.6. *When $-\dfrac{p(a_{mn})}{n} \notin \mathbb{N}$,*

$$\widetilde{L}^n_0(\underline{a}'_n, p(a_{mn})) = e^{\frac{i\pi}{m}(1-\frac{1}{n})} n^{-\frac{2}{m}r(\widetilde{\underline{a}}_n) + \frac{p(a_{mn})}{mn} + \frac{1}{m} - 1} \widetilde{L}_0\left(\widetilde{\underline{a}}'_n, \frac{p(a_{mn})}{n}\right). \tag{21}$$

6.2 Quasi-exact solvable cases

In quantum mechanics, a special class of spectral problems admits partial (or even complete) algebraization, that is part of the energy spectrum and associated eigenfunctions is calculable algebraically. These systems are said to be quasi-exactly solvable, after Turbiner [19].

Here we discuss what can be thought of as an analog of the quasi-exact-solvability. The consequence will be informations about the location of the zeros of the coefficient $\widetilde{L}_0$.

We assume that $m = 2k$ is even. We look for solutions of $(\mathfrak{E}_m)$ having the following form:

$$\Phi(x, \underline{a}) = x^{s(p)} e^{-S(x,\underline{a})} \sum_{n=0}^{\infty} \frac{\Gamma(p+1) Q_n(\underline{a}', p)}{\Gamma(n+p+1)} x^n$$

with $S(x, \underline{a})$ as in theorem 2.1, $s(p) = \dfrac{1+p}{2}$ and $p = (1 + 4a_m)^{\frac{1}{2}}$. We assume that $p \notin -\mathbb{N}^\star$. We normalize Φ by imposing the condition $Q_0(\underline{a}', p) = 1$.
Demanding that Φ is a solution of $(\mathfrak{E}_m)$, one sees that the coefficients $Q_n(\underline{a}', p)$ has to satisfy a $(k+1)$ term recursion relation:

$$\begin{cases} \sum\limits_{j=0}^{k-1} \alpha_{n+j}(\underline{a}', p) Q_{n+j}(\underline{a}', p) = (n+k) Q_{n+k}(\underline{a}', p), \ n \in \mathbb{Z} \\ Q_0(\underline{a}', p) = 1 \\ Q_n(\underline{a}', p) = 0 \text{ for } n < 0. \end{cases} \tag{22}$$

In (22), the coefficients α_n are polynomial functions in $(\underline{a}', p)$ so that the recursion relation (22) determines the $Q_n(\underline{a}', p) \in \mathbb{C}[\underline{a}', p]$ uniquely. We exemplify (22) for $m = 2, 4, 6$:

1. $m=2$: $\begin{cases} S(x,\underline{a}) = x \\ (a_1+p+1+2n)Q_n(a_1,p) = (n+1)Q_{n+1}(a_1,p), \ n \geq 0 \\ Q_0(a_1,p) = 1. \end{cases}$

2. $m=4$: $\begin{cases} S(x,\underline{a}) = \frac{1}{2}x^2 + \frac{1}{2}a_1 x \\ (n+p+1)\left[\left(a_2 - \frac{1}{4}a_1^2\right) + 2n+p+2\right] Q_n - \left[a_3 + (n+\frac{p}{2}+\frac{3}{2})a_1\right] Q_{n+1} \\ = (n+2)Q_{n+2}, \ n \geq -1 \\ Q_{-1}(\underline{a}',p) = 0, Q_0(\underline{a}',p) = 1. \end{cases}$

3. $m=6$: $\begin{cases} S(x,\underline{a}) = \frac{1}{3}x^3 + \frac{1}{4}a_1x^2 + \frac{1}{2}\left(a_2 - \frac{1}{4}a_1^2\right)x \\ (n+p+1)(n+p+2)\left[-\frac{1}{2}a_1\left(a_2-\frac{1}{4}a_1^2\right) + a_3 + 2n+p+3\right] Q_n \\ +(n+p+2)\left[-\frac{1}{4}\left(a_2-\frac{1}{4}a_1^2\right)^2 + a_4 + \left(n+\frac{p}{2}+2\right)a_1\right] Q_{n+1} \\ + \left[\left(n+\frac{p}{2}+\frac{5}{2}\right)\left(a_2-\frac{1}{4}a_1^2\right) + a_5\right] Q_{n+2} = (n+3)Q_{n+3}, \ n \geq -2 \\ Q_{-2}(\underline{a}',p) = 0, Q_{-1}(\underline{a}',p) = 0, Q_0(\underline{a}',p) = 1. \end{cases}$

In the case $m=2$, the condition $p+a_1+1=-2N$ with $N \in \mathbb{N}$ is a easily seen to be a sufficient condition of quasi-exact-solvability since, for $p \notin -\mathbb{N}^\star$,

$$\Phi(x,\underline{a}) = x^{s(p)}e^{-x}\sum_{n=0}^{N}\frac{\Gamma(p+1)Q_n(a_1,p)}{\Gamma(n+p+1)}x^n$$

is an exact solution of $(\mathfrak{E}_2)$. In such a case, $\Phi(x,\underline{a})$ coincide with $f_1(x,a_1,p)$, and moreover $\Phi(x,\underline{a}) = (-1)^N 2^N \dfrac{\Gamma(p+1)}{\Gamma(N+p+1)}\Phi_0(x,\underline{a})$. This means that $\widetilde{L}_0(\omega.\underline{a}',p) = 0$ when $\begin{cases} p+a_1+1 \in -2\mathbb{N} \\ p \notin -\mathbb{N}^\star \end{cases}$, so that:

Lemma 6.7. *For $m=2$, $\widetilde{L}_0(\underline{a}',p) = 0$ when* $\begin{cases} p-a_1+1 \in -2\mathbb{N} \\ p \notin -\mathbb{N}^\star \end{cases}$.

The same kind of result can be otained for higher values of m. For instance, when $m=4$, assuming that there exists $N \in \mathbb{N}$ such that $a_2 - \frac{1}{4}a_1^2 + 2N + p + 2 = 0$, then the three term recursion relation (22) implies that for all $n \geq N+1$, Q_n is a multiple of Q_{N+1}. Therefore, $\begin{cases} a_2 - \frac{1}{4}a_1^2 + 2N + p + 2 = 0, \ N \in \mathbb{N} \\ Q_{N+1}(\underline{a}',p) = 0 \end{cases}$ is a sufficient condition for quasi-exact-solvability (for instance, when $N=0$, this yields $\begin{cases} a_2 - \frac{1}{4}a_1^2 + p + 2 = 0 \\ 2a_3 + (p+1)a_1 = 0 \end{cases}$). In such a case, $\widetilde{L}_0(\omega.\underline{a}',p) = 0$ again.

To end this subsection, note that some quasi-exact-solvability conditions for odd m can be deduced from the even case by using lemma 6.2 (for instance, the $m=3$ case can be deduced from the $m=6$ case).

7 Applications

We end this paper by showing what kind of information we can extract from our analysis, refering to[8] for the proofs.

First application Particular simplifications occur when $\underline{a}' = 0$:

Proposition 7.1. *We consider $(\mathfrak{E}_m)$ on restriction to $\underline{a}' = 0$. Then*

$$\mathfrak{S}_0(0, a_m) = \begin{pmatrix} 2e^{-\frac{i\pi}{m}} \cos\left(\pi \frac{p}{m}\right) & -e^{-\frac{2i\pi}{m}} \\ 1 & 0 \end{pmatrix}$$

where $p = (1 + 4a_m)^{1/2}$. Furthermore, for $\frac{p}{m} \notin \mathbb{Z}$, the 0∞ connection matrix M_0 is given by

$$M_0(0,p) = \begin{pmatrix} e^{\beta_m(-p)} \frac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}} \Gamma\left(-\frac{p}{m}\right) & e^{\beta_m(p)} \frac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}} \Gamma\left(\frac{p}{m}\right) \\ \omega^{s(p)} e^{\beta_m(-p)} \frac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}} \Gamma\left(-\frac{p}{m}\right) & \omega^{s(-p)} e^{\beta_m(p)} \frac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}} \Gamma\left(\frac{p}{m}\right) \end{pmatrix}$$

where $s(p) = \frac{1+p}{2}$, while $\beta_m(p)$ is an odd function, entire in p, such that for all $k \in \mathbb{N}^\star$, $e^{\beta_m(km)} = \pm m^k$.

This means that almost everything can be deduced from the symmetries in this case, apart from the β function in the 0∞ connection matrix M_0. Using special functions (Bessel), one can derive from our analysis that $\widetilde{L}_0(0,p) = e^{-\frac{i\pi}{m}} m^{\frac{p}{m}} e^{i\pi\frac{p}{m}} \frac{\Gamma(\frac{p}{m})}{\sqrt{m\pi}}$.

The $m = 2$ case In this case, the tools developed in sections 5 and 6 yield the following result.

Proposition 7.2. *We assume $m = 2$. Then the Stokes-Sibuya mutiplier C_0 may be written as*

$$C_0(\underline{a}) = -2ie^{-i\pi\frac{a_1}{2}} \cos\left(\frac{\pi}{2}(p + a_1)\right) \frac{\Gamma(\frac{p}{2} + \frac{a_1}{2} + \frac{1}{2})}{\Gamma(\frac{p}{2} - \frac{a_1}{2} + \frac{1}{2})} e^{2\beta(a_1,p)} \quad \textit{and} \quad \widetilde{C}_0(\underline{a}) = e^{-i\pi a_1} \tag{23}$$

where β is an entire function satisfying $\beta(a_1, p) = \beta(a_1, -p) = -\beta(-a_1, p)$.
Moreover, the coefficients of the 0∞ connection matrix M_0 of theorem 4.2 satisfy, for $p \notin \mathbb{Z}$,

$$\begin{cases} \widetilde{L}_0(a_1, p) = -i2^{-\frac{p-1}{2}} e^{i\pi\frac{p}{2}} \dfrac{\Gamma(p)}{\Gamma(\frac{p}{2} - \frac{a_1}{2} + \frac{1}{2})} e^{\beta(a_1,p)} \\ L_0(a_1,p)\widetilde{L}_0(\omega a_1, p) = 2\dfrac{\cos\left(\frac{\pi}{2}(p + a_1)\right)}{p \sin(\pi p)} \end{cases} \tag{24}$$

Using the Whittaker special functions, it is possible to show that the unknown function β is simply $\beta(a_1, p) = -a_1$. With this addendum and corollaries 6.4 and 6.6, proposition 7.2 implies the following corollary:

Corollary 7.3. *For $n \in \mathbb{N}^\star$, we consider the differential equation*

$$(\mathfrak{E}^n_{2n}) \qquad x^2 \frac{d^2}{dx^2}\Phi = \left(x^{2n} + a_n x^n + a_{2n}\right)\Phi$$

Then,
$$\begin{cases} C^n_0(\underline{a}_n) = 2e^{-\frac{i\pi}{2n}} e^{-i\pi \frac{a_n}{2n}} 2^{-\frac{a_n}{n}} \dfrac{\Gamma(\frac{p}{2n}+\frac{a_n}{2n}+\frac{1}{2})}{\Gamma(\frac{p}{2n}-\frac{a_n}{2n}+\frac{1}{2})} \cos\left((\dfrac{p}{2n}+\dfrac{a_n}{2n})\pi\right) \\ \widetilde{C}^n_0(\underline{a}_n) = -e^{-\frac{i\pi}{n}} e^{-i\pi\frac{a_n}{n}} \end{cases}$$
where $p = (1 + 4a_{2n})^{\frac{1}{2}}$. Moreover, when $p \notin -n\mathbb{N}$,

$$\widetilde{L}^n_0(\underline{a}'_n, p) = e^{-\frac{i\pi}{2n}} e^{i\pi\frac{p}{2n}} \left(\frac{n}{2}\right)^{\frac{a_n}{2n}+\frac{p}{2n}-\frac{1}{2}} \frac{\Gamma(\frac{p}{n})}{\Gamma(\frac{p}{2n}-\frac{a_n}{2n}+\frac{1}{2})}$$

Application when $m = 3$. We end with the $m = 3$ case, where no special function solution of $(\mathfrak{E}_3)$ is known.
Corollary 5.2 yields the following functional relation between the Stokes-Sibuya multipliers:

$$C_0(\underline{a})C_1(\underline{a})C_2(\underline{a}) + \widetilde{C}_0(\underline{a})C_2(\underline{a}) + \widetilde{C}_1(\underline{a})C_0(\underline{a}) + \widetilde{C}_2(\underline{a})C_1(\underline{a}) = -2\cos(\pi p)$$

where, by (4) of theorem 2.9, $\widetilde{C}_0(\underline{a}) = \widetilde{C}_1(\underline{a}) = \widetilde{C}_2(\underline{a}) = e^{\frac{i\pi}{3}}$. Applying now corollary 5.5, we find, for all $\underline{a}' \in \mathbb{C}^2$ and $p \notin -\mathbb{N}$:

$$\widetilde{L}_0(\omega.\underline{a}', p)C_0(\underline{a}) = \omega^{-\frac{1}{2}}\left(\widetilde{L}_0(\underline{a}', p)\omega^{\frac{p}{2}} + \widetilde{L}_0(\omega^2.\underline{a}', p)\omega^{-\frac{p}{2}}\right), \tag{25}$$

Now, for $p \in \mathbb{N}^\star$, formula (17) translates into:

$$\begin{aligned} &i\pi p\omega^{\frac{7}{4}+\frac{p}{2}}\lambda(\underline{a}', p)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) = \\ &\widetilde{L}_0(\omega.\underline{a}', p) + \omega^p\widetilde{L}_0(\omega^2.\underline{a}', p) + \omega^{2p}\widetilde{L}_0(\underline{a}', p). \end{aligned} \tag{26}$$

We now add the assumption that $\underline{a}'$ has been chosen so that

$$\lambda(\underline{a}', p)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) = 0.$$

Then, from equations (26) and (25) one can derives that $C_0(\underline{a}) = C_1(\underline{a}) = C_2(\underline{a}) = -\omega^{-\frac{1}{2}-\frac{3p}{2}} = (-1)^{p+1}e^{-\frac{i\pi}{3}}$. To summarize:

Proposition 7.4. *For $m = 3$, the Stokes-Sibuya multiplier $C_0(\underline{a})$ satisfies the functional equation*

$$C_0(\underline{a})C_0(\omega.\underline{a})C_0(\omega^2.\underline{a}) + e^{\frac{i\pi}{3}}\left(C_0(\underline{a}) + C_0(\omega.\underline{a}) + C_0(\omega^2.\underline{a})\right) = -2\cos(\pi p) \tag{27}$$

with $p = (1 + 4a_3)^{\frac{1}{2}}$ and $\omega = e^{\frac{2i\pi}{3}}$, whereas $\widetilde{C}_0(\underline{a}) = e^{\frac{i\pi}{3}}$. Moreover, when $a_3 = \dfrac{p^2-1}{4}$ with $p \in \mathbb{N}^\star$, then

$$\lambda(\underline{a}', p)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) = 0$$

is equivalent to C_0 being a constant, precisely $C_0 = (-1)^{p+1}e^{-\frac{i\pi}{3}}$.

For a given $p \in \mathbb{N}^\star$, the case $\lambda(\underline{a}', p) = 0$ can be seen as an isomonodromic deformation condition, since both the monodromy at the origin and the Stokes structure are fixed. In other words:

Corollary 7.5. *For $m = 3$ and $p \in \mathbb{N}^\star$, the condition $\lambda(\underline{a}', p) = 0$ is an isomonodromic deformation condition.*

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