数理解析研究所講究録 1424 巻 2005 年 103-117

Aspects of the ODE/IM correspondence

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Abstract

We review a surprising correspondence between certain two-dimensional integrable models and the spectral theory of ordinary differential equations. Particular emphasis is given to the relevance of this correspondence to certain problems in $\mathcal{PT}$-symmetric quantum mechanics.

Contribution to the Proceedings "Recent Trends in Exponential Asymptotics", June 28 - July 2 (2004), RIMS, Kyoto.

1 Prelude

This short review is about a surprising link between integrable models and ordinary differential equations [1, 2, 3, 4], its relevance in the study of the novel kind of non-Hermitian quantum mechanics studied by Carl Bender and many others [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the application of techniques from the theory of integrable models to answer a long-standing question concerning the reality of the energy levels of a particular set of non-Hermitian operators.

In the theory of ordinary differential equations the adjective "exactly solvable" usually indicates that the problem is fully soluble, with solutions expressible in terms of some previously-defined mathematical functions. In the context of spectral theory of second-order differential equations the classical example is a Schrödinger problem with harmonic potential:

$$-\psi''(x) + x^2\psi(x) = E_n\psi(x) \ , \qquad \psi(x) \in L^2(\mathbb{R}) \ . \tag{1.1}$$

This equation is mathematically soluble in the sense that the eigenfunctions are given in terms of elementary functions. On the other hand, a mathematician would be reluctant to call a Schrödinger equation with potential x^α with $\alpha \in \mathbb{R}^+$ exactly solvable. However we shall shortly see that this family of quantum systems is somehow equivalent to a two dimensional exactly solvable model: the six-vertex model in its scaling limit. There is clearly a mismatch between the definitions of exact solvability in the two fields. Naively speaking, in two dimensions there is a special class of statistical models for which at least one physically-interesting quantity, the density of free-energy in the thermodynamic limit, can be computed exactly. For this reason these class of systems are given the name of "exactly solvable" or "integrable" models. (The existence of a complete set of independent, commuting conservation laws is a more precise characterisation of an integrable system, but a full discussion of this would go well beyond the scope of this review.)

Notice that this does not in general mean that other quantities, such as, for example, the finite lattice free energy or the correlation functions, can be exactly determined. This sounds like a negative note (and it is!!), but during more than 30 years of integrability the scientific community has partially overcome this negative aspect by introducing a number of powerful tools which allow an integrable system to be studied in great detail – works particularly relevant to the current story include [23, 24, 25]. This review is about the use of just one these tools in the ODE context. Of course many techniques have also been developed in the ODE framework (see for example [26, 27, 28]), and integrable models can, and to some extent already do [2, 29], profit from these as well.

2 The IM side: the six-vertex model

Let us start from the integrable model side, and consider an $N \times M$ lattice model with periodic boundary conditions and, for an irrelevant technical reason, $N/2$ even. On each link of the lattice we place a spin taking one of two values: this is conveniently denoted by placing arrows on the links, as in figure 1.

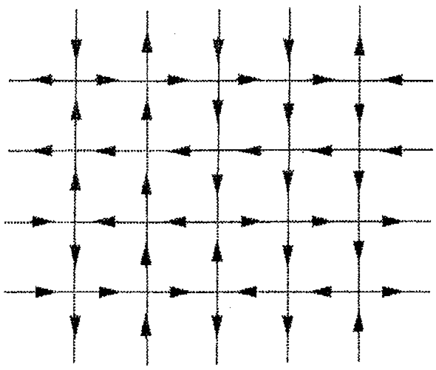

Figure 1: A configuration in the six-vertex model.

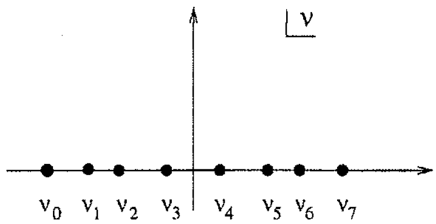

Figure 2: The ν_i's are real for the ground state.

We shall constrain the set of configurations as follows:

- Only those configurations of spins which preserve the 'flux' of arrows through each vertex are allowed;
- We only consider the 'zero field' six-vertex model, which has an additional '4-spin reversal' symmetry implying the model is invariant under simultaneous reversal of all arrows.

Thus the 'local Boltzmann weights', numbers assigned to each vertex depending on the spins next to that vertex, can be parameterised in terms of just three quantities:

$$W\left[\begin{array}{ccc} & \uparrow & \\ \rightarrow & & \rightarrow \\ & \uparrow & \end{array}\right] = W\left[\begin{array}{ccc} & \downarrow & \\ \leftarrow & & \leftarrow \\ & \downarrow & \end{array}\right] = a \ ,$$

$$W\left[\begin{array}{ccc} & \downarrow & \\ \rightarrow & & \rightarrow \\ & \downarrow & \end{array}\right] = W\left[\begin{array}{ccc} & \uparrow & \\ \leftarrow & & \leftarrow \\ & \uparrow & \end{array}\right] = b \ ,$$

$$W\left[\begin{array}{ccc} & \uparrow & \\ \rightarrow & & \leftarrow \\ & \downarrow & \end{array}\right] = W\left[\begin{array}{ccc} & \downarrow & \\ \leftarrow & & \rightarrow \\ & \uparrow & \end{array}\right] = c \ .$$

In fact the overall normalisation factors out trivially from all calculations, and the remaining two degrees of freedom can be parametrised using the two variables

$$\nu : \ \text{the spectral parameter} \ , \qquad \eta : \ \text{the anisotropy} \tag{2.1}$$

as

$$a = \sinh(i\eta - \nu) \ , \quad b = \sinh(i\eta + \nu) \ , \quad c = \sinh(2i\eta) \ . \tag{2.2}$$

The relative probability of finding any given configuration is simply given by the product of the Boltzmann weights at each vertex. The six-vertex model introduced above is a classic example of an integrable lattice model.

The first quantity of interest - the partition function Z - is the sum of these numbers over all possible configurations:

$$Z = \sum_{\text{arrows}} \prod_{\text{sites}} W\left[\cdot \ \vdots \ \cdot\right] \ . \tag{2.3}$$

One popular technique to calculate Z makes use of the so-called transfer matrix $\mathbf{T}$, which performs the sum over one set of horizontal links:

$$\mathbf{T}^{\{\alpha'\}}_{\{\alpha\}}(\nu) \;=\; \sum_{\{\beta_i\}} W\!\begin{bmatrix}\beta_1 \overset{\alpha'_1}{\underset{\alpha_1}{}} \beta_2\end{bmatrix} W\!\begin{bmatrix}\beta_2 \overset{\alpha'_2}{\underset{\alpha_2}{}} \beta_3\end{bmatrix} \ldots W\!\begin{bmatrix}\beta_N \overset{\alpha'_N}{\underset{\alpha_N}{}} \beta_1\end{bmatrix} .$$

In terms of $\mathbf{T}$ the partition function is given by

$$Z = \mathrm{Trace}\left[\mathbf{T}^M\right] . \tag{2.4}$$

In the limit $M \to \infty$ with N finite the free energy per site can then be obtained as

$$f = \tfrac{1}{NM} \ln Z = \tfrac{1}{NM} \ln \mathrm{Trace}\left[\mathbf{T}^M\right] \sim \tfrac{1}{N} \ln t_0 , \tag{2.5}$$

where t_0 is the ground-state eigenvalue of $\mathbf{T}$. The problem is thus reduced to the determination of t_0; and since the model is integrable, there are many methods to achieve this end. The detailed description of these methods goes beyond the scope of this review, but it is worth mentioning that they usually lead to a set of non-linear constraints (functional relations or non-linear integral equations) on $t_0(\nu)$, or, more simply, constraints on the zeroes of a related function $q_0(\nu)$, which we shall introduce shortly. These constraints are very powerful even at finite N but they lead to a closed expression for the free energy only in the thermodynamic limit $M, N \to \infty$.

Here we shall sketch the logical flow of a method developed by Baxter (for more details, see [23]) leading to a relation known as the TQ-system that can also be easily deduced starting from the ODE side. This should provide a first clear hint of what we mean by an ODE/IM correspondence.

Baxter began by showing that there exists an auxiliary function $q_0(\nu)$,

$$q_0(\nu) = \prod_{i=0}^{n-1} \sinh(\nu - \nu_i) , \tag{2.6}$$

such that the following 'TQ relation' holds

$$t_0(\nu) q_0(\nu) = a(\nu,\eta)^N q_0(\nu + 2i\eta) + b(\nu,\eta)^N q_0(\nu - 2i\eta) . \tag{2.7}$$

This is perhaps a puzzling step to take: we want to find $t_0(\nu)$ and we now claim that the relation (2.7), which just defines $t_0(\nu)$ in terms of another unknown function $q_0(\nu)$, will somehow help us in this task. The explanation is simple: the constraint (2.7) should be combined with a knowledge of the analytic properties of both t_0 and q_0. In particular the fact that $t_0(\nu)$ and $q_0(\nu)$ are entire functions of ν means that a powerful constraint, the Bethe ansatz system, is immediate. By definition

$$q_0(\nu_i) = 0 , \tag{2.8}$$

and combining this with the relation (2.7) and the fact that $t_0(\nu)$ is entire leads to the following set of Bethe ansatz equations (BAE):

$$-1 = \frac{a^N(\nu_i,\eta)}{b^N(\nu_i,\eta)} \frac{q_0(\nu_i + 2i\eta)}{q_0(\nu_i - 2i\eta)} \quad , \quad i = 0, 1, \ldots, n-1 . \tag{2.9}$$

This already looks to be a more serious constraint than (2.7), but this is not yet sufficient since (2.9) has many sets of solutions $\{\nu_i\}$. However, one can argue that there is a particular solution, with $n = N/2$ and all of the ν_i real (depicted in figure 2), which uniquely corresponds to the ground-state eigenvalue of $\mathbf{T}$, i.e. t_0.

To prepare the ground for the connection with differential equations, let us define

$$\lambda_i = e^{2\nu_i} \quad , \quad \omega = -e^{i2\eta} \; , \tag{2.10}$$

so that the Bethe ansatz equations (2.9) become

$$-1 = \left(\frac{1+\lambda_i/\omega}{1+\lambda_i\omega}\right)^N \prod_{n=0}^{N/2-1} \frac{(\lambda_n - \lambda_i\omega^2)}{(\lambda_n - \lambda_i\omega^{-2})} \; . \tag{2.11}$$

We should mention one final detail: currently correspondences with ordinary differential equations have only been established with integrable models in their continuum, conformal limits. For the six-vertex model this limit is found by sending

$$N \to \infty \quad \text{and} \quad a = e^{\pi\nu/2\eta} \to 0 \; , \tag{2.12}$$

with aN kept finite. Sending

$$a \to 0 \quad \text{as} \quad a \to a/N \tag{2.13}$$

one discovers that the λ_i's with $i \ll \ln N$ rescale to zero as

$$\lambda_i \sim E_i a^{4\eta/\pi} \sim E_i N^{-4\eta/\pi} \; . \tag{2.14}$$

For $\pi/4 < \eta < \pi/2$ the limiting product converges with no need for extra regulating factors, and one obtains

$$-1 = \prod_{n=0}^{\infty} \frac{(E_n - E_i\omega^2)}{(E_n - E_i\omega^{-2})} \; . \tag{2.15}$$

Finally, the six-vertex model can be generalised to incorporate twisted boundary conditions without losing integrability. The twist is introducing by making the following replacement in the definition of the original model:

$$W\left[\beta_N \, {}^{\alpha'_N}_{\alpha_N} \, \beta_1\right] \to e^{i(\delta_{\beta_1,\uparrow} - \delta_{\beta_1,\downarrow})\phi} \, W\left[\beta_N \, {}^{\alpha'_N}_{\alpha_N} \, \beta_1\right] \; . \tag{2.16}$$

In the conformal limit described above the introduction of the twist leads to the more general BAE

$$-1 = e^{i2\phi} \prod_{n=0}^{\infty} \frac{(E_n - E_i\omega^2)}{(E_n - E_i\omega^{-2})} \; . \tag{2.17}$$

We shall shortly see how to derive the same set of equations starting from a differential equation.

3 $\mathcal{PT}$-symmetric quantum mechanics

Before we go into the details of the precise link between ODEs and integrable models we shall first give a description of the non-Hermitian quantum mechanical problems mentioned in the prelude.

Researchers working on integrable 1+1 dimensional massive quantum field theories have become quite accustomed to systems described by non-Hermitian Hamiltonians which nevertheless possess, at least in a certain range of the parameters, a real and positive energy spectrum. A standard example, which has been studied since the mid-1990s, is the scaling Lee-Yang model (see, for example, [30]). This is the theory of a massive relativistic bosonic field ϕ with cubic interaction $g\phi^3$. Along with the $\lambda\phi^4$ case, it is one of the standard text-book examples used for practising Feynman diagram technology [31]. However, since the potential is unbounded from below, the Lee-Yang model with real g suffers from obvious pathologies. Surprisingly, at least in the 1+1 dimensional integrable case, these pathologies disappear in the fully renormalised scaling limit at $g = i|g|$. We invite the interested reader to consult [30, 32, 33] to appreciate various aspects of this phenomenon, and to note in particular the striking similarity between the plots in figures 6 and 7 in Ref. [33] and figure 6a below, which shows the energy spectrum of a non-Hermitian quantum-mechanical problem, (3.2).

Indeed, it was the study of the 1+1 dimensional Lee-Yang model that led Bessis and Zinn-Justin to the following, rather simpler, question: is the energy spectrum associated with the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + ix^3\psi(x) = E_n\psi(x) , \quad \psi(x) \in L^2(\mathbb{R}) \tag{3.1}$$

real? Perturbative and numerical studies led Bessis and Zinn-Justin to conjecture that the spectrum $\{E_n\}$ is indeed real, and positive [34]. But how does one prove this statement analytically?

Later, in 1997, Bender and Boettcher [7] considered the spectrum of the following generalisation of Bessis and Zinn-Justin's problem:

$$-\frac{d^2}{dx^2}\psi(x) - (ix)^N\psi(x) = E_n\psi(x) , \qquad (N \text{ real}, > 0) \tag{3.2}$$

with the associated boundary conditions to be specified shortly. Again, extensive numerical checks led the authors of [7] to a precise reality conjecture. They also remarked that both equations (3.1) and (3.2) are invariant under the action of the operator $\mathcal{PT}$:

$$\begin{array}{lll} \mathcal{P} & \text{(parity reflection):} & x \to -x \ , \ p \to -p \ , \\ \mathcal{T} & \text{(time reversal):} & x \to \ x \ , \ p \to -p \ , \ i \to -i \ . \end{array}$$

The special class of non-hermitian quantum systems with $\mathcal{PT}$ symmetry has only recently started to be studied in depth. These theories are mathematically interesting, and they may have an important rôle in physics [35].

Returning to the ODE (3.2), we note that there appear to be a couple of problems with the generalisation. Firstly, for non-integer values of N the 'potential' $-(ix)^N$ is not single-valued and we need to add a branch cut, which we shall put along the positive imaginary x-axis:

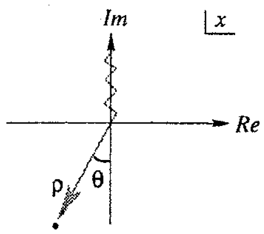

Figure 3: The branch cut, and a ray in the complex x-plane.

Secondly, when N reaches 4 the 'potential' is $-x^4$, and the eigenvalue problem runs into difficulties since all the solutions to (3.2) decay algebraically as $|x| \to \infty$ on the real axis. To overcome this problem it is necessary to enlarge the perspective and treat x as a genuinely complex variable. Consider solutions to the ODE in the complex x-plane. For most values of $\arg x$, up to an overall normalisation there will be a unique exponentially-decaying solution at large $|x|$ (this solution is called "subdominant") and any other linearly independent solution will be exponentially growing (these solutions are called "dominant"). However, whenever

$$\arg x = \pm\frac{\pi}{N+2}\,, \quad \pm\frac{3\pi}{N+2}\,, \quad \pm\frac{5\pi}{N+2}\,, \quad \dots \tag{3.3}$$

all solutions decay algebraically. The lines along which this occurs are known as 'anti-Stokes lines' and the wedges between the anti-Stokes lines are the 'Stokes sectors'. Since all functions involved are analytic, we can continue the wavefunction along some other contour in the complex plane, and as long as it does not cross any anti-Stokes lines the spectrum will remain the same. At $N = 4$ an anti-Stokes line coincides with the real axis and the correct analytic continuation of the original problem is achieved by bending the wavefunction contour down into the complex plane, as shown in figure 4.

We conclude from this discussion that each pair of Stokes sectors in which the wavefunction is required to decay at large $|x|$ defines a different eigenvalue problem. In the terminology of the WKB method, these are related to 'lateral' connection problems. Contours which instead join $x = 0$ to $x = \infty$ lead to what are called 'radial' (or 'central') connection problems. Figure 5 depicts a sample of the possible lateral and radial wavefunction contours.

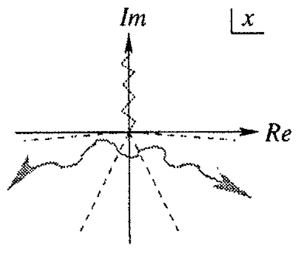

Figure 4: A possible wavefunction contour for $N > 4$.

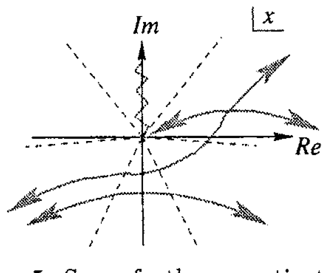

Figure 5: Some further quantization contours.

Questions in $\mathcal{PT}$-symmetric quantum mechanics are all related to lateral problems, with one particular pair of Stokes sectors selected. The first eigenproblem to arise in the ODE/IM correspondence was of radial type [1], with the contour defined along the positive real axis. However it turns out that the $\mathcal{PT}$-symmetric quantum mechanical problems (3.2) are intimately related with certain radial problems, in a way that will be discussed in the next-but-one section.

4 Numerical evidence

We now present some numerical evidence concerning $\mathcal{PT}$-symmetric problems of (3.2). In figure 6a part of the spectrum of (3.2) is plotted. This plot, together with figures 6b-d, was obtained in [10] via a non-linear integral equation. The results shown in the first plot had previously been obtained in [7, 8] by a direct numerical treatment of the differential equation in the complex plane.

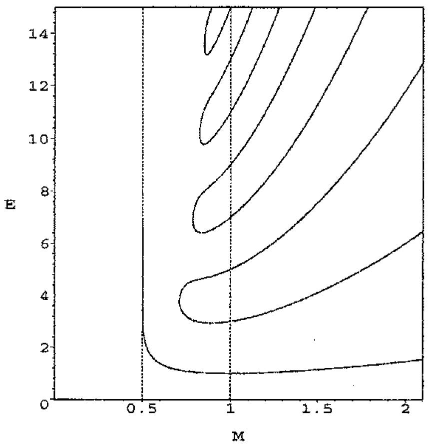

Fig. 6a: $l = 0$.

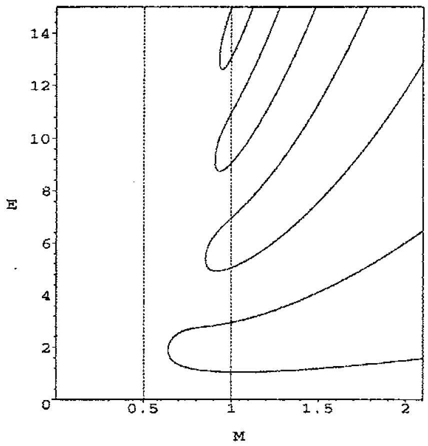

Fig. 6b: $l = -0.025$.

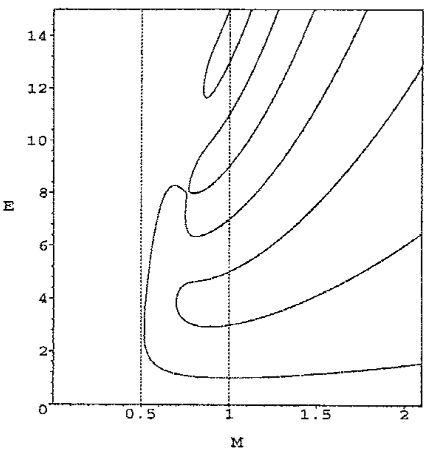

Fig. 6c: $l = -0.001$.

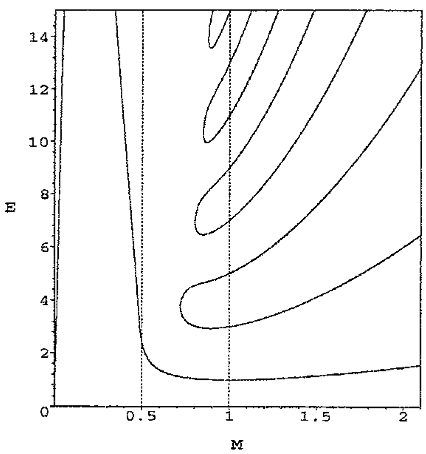

Fig. 6d: $l = 0.001$.

Figure 6: Eigenvalues of the Hamiltonian $p^2 - (ix)^{2M} + l(l+1)x^{-2}$.

From figure 6a, for $N \equiv 2M \geq 2$ the spectrum is, within our numerical precision, real and positive, while as N moves below 2 an infinite number of energy levels pair off and become complex. The transition to infinitely-many complex eigenvalues was interpreted in [8] as a spontaneous breaking of $\mathcal{PT}$ symmetry*.

*Since $[\mathcal{PT}, H] = 0$ it follow that if an eigenfunction Ψ of H is also an eigenfunction of $\mathcal{PT}$ with

We shall tackle the reality question for $N \geq 2$ analytically. For this purpose it is convenient to enlarge the perspective by including two extra parameters, α and l, a generalisation that does not add any extra technical difficulties but gives a wider phenomenology. It is also convenient to trade the parameter N for $M \equiv N/2$. The more general theory is [10, 11]

$$-\frac{d^2}{dx^2}\psi(x) - \Big((ix)^{2M} + \alpha(ix)^{M-1} + \frac{l(l+1)}{x^2}\Big)\psi(x) = E_n\psi(x) \ . \qquad (4.1)$$

Even with $\alpha = 0$, for $-1 < l < 0$ the additional angular-momentum term has a remarkable effect on the connectivity of the spectrum, as can be seen in the middle two plots of figure 6. Moreover, we shall show in the next sections, as in [11, 12], that the spectrum of (4.1) is

- *real* if $\alpha < M + 1 + |2l{+}1|$
- *positive* if $\alpha < M + 1 - |2l{+}1|$.

The proof makes use of integrable model technology. While there will not be space to go into details below, we remark that these ideas can also be used to study the way that the energy levels merge to become complex [36].

5 Analysis of the Schrödinger equation

As in [10, 11], we start the analysis by considering a related differential equation

$$\left(-\frac{d^2}{dx^2} + x^{2M} + \alpha x^{M-1} + \frac{l(l+1)}{x^2} - E\right)\phi(x) = 0 \qquad (5.1)$$

with x and E possibly complex and $M > 0$.

We shall need a couple of important facts. Firstly, the equation has a solution $y = y(x, \alpha, E, l)$ such that:

- y is entire in E and x (though, due to the branch point at $x = 0$, x must in general be considered to live on a suitable cover of the punctured complex plane) †
- for $M > 1$, y and $y' = dy/dx$ admit the following asymptotic representations

$$y \sim \frac{x^{-M/2-\alpha/2}}{\sqrt{2i}} \exp(-\tfrac{1}{M+1}x^{M+1}) \quad , \quad y' \sim -\frac{x^{M/2-\alpha/2}}{\sqrt{2i}} \exp(-\tfrac{1}{M+1}x^{M+1}) \qquad (5.2)$$

for $|x| \to \infty$ in any closed sector contained in the sector $|\arg x| < \frac{3\pi}{2M+2}$ (though extra terms appear for $0 < M \leq 1$.)

eigenvalue λ then $\mathcal{PT}\, H\Psi = E^*\lambda\Psi \ \equiv\ H\,\mathcal{PT}\Psi = E\lambda\Psi$ and $E = E^*$.

†The entirety of y was first proved by Sibuya (see Ref. [26]). His work concerned only the case $l = 0$, $\alpha = 0$, $2M \in \mathbb{N}$, but the result also holds for the more general situation of eq. (5.1), so long as the branching at the origin is taken into account. In this respect we should mention that the $l = 0$, $\alpha = 0$, $2M \in \mathbb{R}^+$ case was explicitly discussed by Tabara in [37], while the generalisation to a potential $P(x)/x^2$ with $P(x)$ a polynomial in x was studied by Mullin [38], and more recently in [39]. It is also worth noting that with a change of variable it is possible to map eq. (5.1) with $\alpha \in \mathbb{R}$, $l \in \mathbb{R}$, $2M \in \mathbb{Q}^+$ onto particular cases of those treated in [38].

Secondly, setting $x = \rho e^{i\theta}$ as in figure 3 with ρ real, and denoting the sector $\left|\theta - \frac{k\pi}{M+1}\right| < \frac{\pi}{2M+2}$ by $\mathcal{S}_k$ as in figure 7, we see that, as $\rho \to \infty$ with θ fixed,

- in $\mathcal{S}_0 : y \to 0$ (y is subdominant in $\mathcal{S}_0$)
- at $\theta = \pm\frac{\pi}{2M+2}$ y decays algebraically
- in $\mathcal{S}_{\pm 1} : y \to \infty$ (y is dominant in $\mathcal{S}_{\pm 1}$).

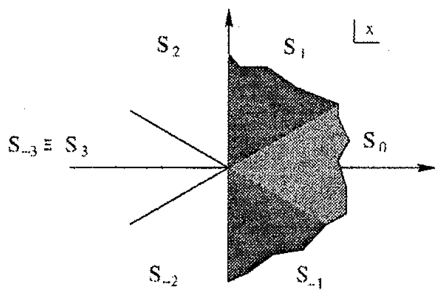

Figure 7: The sectors $\mathcal{S}_k$ for the potential x^4.

It is then easy to see that the uniquely-determined solution $y(x, E, \alpha, l)$ defines an associated set of functions

$$y_k = \omega^{k/2+k\alpha/2} y(\omega^{-k}x,\ \omega^{-2Mk}E,\ (-1)^k\alpha,\ l) \quad , \quad \omega = e^{\frac{i\pi}{M+1}} , \tag{5.3}$$

which are also solutions of (5.1) for integer k, and any pair $\{y_k, y_{k+1}\}$ forms a basis of solutions. We can therefore write y_{-1} as a linear combination of y_0 and y_1. The result has the form

$$T(E,\alpha,l) y_0(x,E,\alpha,l) = y_{-1}(x,E,\alpha,l) + y_1(x,E,\alpha,l) , \tag{5.4}$$

where the function T is called a Stokes multiplier. Keeping $\Re e\, l > -1/2$, the leading behaviour of y near $x = 0$ at generic E is

$$y(x,E,\alpha,l) \sim Q(E,\alpha,l) x^{-l} + \dots , \tag{5.5}$$

and in terms of the shorthand notation

$$T^{(\pm)} = T(E, \pm\alpha, l) , \quad Q^{(\pm)} = Q^{(\pm)}(E) = Q(E, \pm\alpha, l) \tag{5.6}$$

we find the following relations which intertwine the as-yet undetermined functions $T^{(\pm)}$ and $Q^{(\pm)}$:

$$T^{(+)}(E) Q^{(+)}(E) = \omega^{-\frac{2l+1+\alpha}{2}} Q^{(-)}(\omega^{2M}E) + \omega^{\frac{2l+1+\alpha}{2}} Q^{(-)}(\omega^{-2M}E) \tag{5.7}$$

$$T^{(-)}(E) Q^{(-)}(E) = \omega^{-\frac{2l+1-\alpha}{2}} Q^{(+)}(\omega^{2M}E) + \omega^{\frac{2l+1-\alpha}{2}} Q^{(+)}(\omega^{-2M}E) . \tag{5.8}$$

Notice the striking similarity between equations (5.7), (5.8) and the Baxter TQ-system of equation (2.7).

At the zeroes $\{E_k(l,\alpha)\}$ of $Q(E)$, the leading behaviour of y at the origin changes to

$$y(x, E_k, \alpha, l) \sim Q(E_k, \alpha, -l-1) x^{l+1} + \dots , \tag{5.9}$$

and $y(x,E,\alpha,l)$ decays at the origin as well as at infinity. This implies that $Q(E)$ is the spectral determinant encoding the eigenvalues of (5.1) for boundary conditions of radial type. For $M>1$, we can use the Hadamard factorisation theorem to write Q as:

$$Q(E,l,\alpha)=Q(0,l,\alpha)\prod_{n=0}^{\infty}\left(1-\frac{E}{E_n}\right) . \tag{5.10}$$

Both $T^{(\pm)}(E)$ and $Q^{(\pm)}(E)$ are entire in E, so the LHS of the relevant TQ equation ((5.7) or (5.8)) vanishes at

$$E=E_k^{(\pm)}=E_k(l,\pm\alpha) , \tag{5.11}$$

and the following system of equations of Bethe ansatz type for the energy spectrum is obtained

$$\begin{aligned}\prod_{n=0}^{\infty}\left(\frac{E_n^{(-)}-\omega^{-2M}E_k^{(+)}}{E_n^{(-)}-\omega^{2M}E_k^{(+)}}\right) &= -\omega^{-2l-1-\alpha} ,\\ \prod_{n=0}^{\infty}\left(\frac{E_n^{(+)}-\omega^{-2M}E_k^{(-)}}{E_n^{(+)}-\omega^{2M}E_k^{(-)}}\right) &= -\omega^{-2l-1+\alpha} .\end{aligned} \tag{5.12}$$

At $\alpha=0$, $\{E_k^{(+)}\}=\{E_k^{(-)}\}$ and the set of equations (5.12) reduces to the Bethe ansatz system (2.17) of the six-vertex model in its continuum limit. (For $\alpha\neq 0$, the mapping is instead to a theory called the three-state Perk-Schulz model [3, 40].)

The Stokes multiplier

$$T(E,\alpha,l)=W[y_{-1},y_1]=\mathrm{Det}\begin{bmatrix}y_{-1}(x) & y_1(x)\\ y'_{-1}(x) & y'_1(x)\end{bmatrix} , \tag{5.13}$$

given here as a Wronskian, vanishes if and only if

$$W[y_{-1},y_1]=0 \quad\leftrightarrow\quad y_{-1}\ \&\ y_1 \ \text{ are linearly dependent.} \tag{5.14}$$

This holds if and only if the ODE has a solution decaying in the two sectors $\mathcal{S}_{-1}$ and $\mathcal{S}_1$ simultaneously. Since (5.1) is related to the $\mathcal{PT}$-symmetric problem (4.1) by the transformation $x\to x/i$, $E\to -E$, this means that $T(-E,-\alpha,l)$ is precisely the spectral determinant for the generalised Bender-Boettcher problem (4.1). Thus the TQ relations (5.7), (5.8) encode the spectra of both radial and lateral eigenproblems.

6 A simple proof of the reality property

In order to prove the reality and positivity claims made at the end of section 4 we return to equation (5.7)

$$T^{(+)}Q^{(+)}=\omega^{-\frac{2l+1+\alpha}{2}}Q^{(-)}(\omega^{2M}E)+\omega^{\frac{2l+1+\alpha}{2}}Q^{(-)}(\omega^{-2M}E) , \tag{6.1}$$

and define the zeroes of $T^{(+)}=T(E,\alpha,l)$ to be $E\in\{-\lambda_k\}$. Setting $E=-\lambda_k$ and using the factorised form for $Q^{(-)}$ we get

$$\prod_{n=0}^{\infty}\left(\frac{E_n^{(-)}+\omega^{-2M}\lambda_k}{E_n^{(-)}+\omega^{2M}\lambda_k}\right)=-\omega^{-2l-1-\alpha} , \qquad k=0,1,\dots . \tag{6.2}$$

Since the original $\mathcal{PT}$ eigenproblem is invariant under $l \to -1-l$ we can assume $l \geq -1/2$. Then each $E_n^{(-)}$ is an eigenvalue of a Hermitian eigenproblem associated with $\mathcal{H}(M, -\alpha, l)$, and hence is real. It is also easy to show that the eigenvalues are all positive, provided $\alpha < M+2l+2$ [11]. Taking the modulus^2, using the reality of the $E_k^{(-)}$, and writing $\lambda_k = |\lambda_k| \exp(i\,\delta_k)$, we have

$$\prod_{n=0}^{\infty} \left(\frac{(E_n^{(-)})^2 + |\lambda_k|^2 + 2E_n^{(-)}|\lambda_k| \cos(\frac{2\pi}{M+1} + \delta_k)}{(E_n^{(-)})^2 + |\lambda_k|^2 + 2E_n^{(-)}|\lambda_k| \cos(\frac{2\pi}{M+1} - \delta_k)} \right) = 1 . \tag{6.3}$$

For $\alpha < M + 2l + 2$, all of the $E_n^{(-)}$ are strictly positive and each single term in the product is either greater than, smaller than or equal to one depending only on the cosine terms. To match the RHS we therefore must have

$$\cos(\tfrac{2\pi}{M+1} + \delta_k) = \cos(\tfrac{2\pi}{M+1} - \delta_k) . \tag{6.4}$$

Since $M > 1$ the only possibility is

$$\delta_k = n\pi , \qquad n \in \mathbb{Z} \tag{6.5}$$

and the eigenvalues are indeed real. Relaxing the condition on l we have shown the spectrum of (4.1) is

- *real* if $\alpha < M + 1 + |2l+1|$

and using continuity in M to keep track of the signs of the eigenvalues (see [11]) it can then be seen that the spectrum is

- *positive* if $\alpha < M + 1 - |2l+1|$.

Referring to figure 8, the spectrum is entirely real for $(\alpha, l) \in B \cup C \cup D$ and positive for $(\alpha, l) \in D$. For $M < 1$, the order of $Q^{(-)}$ is greater than one, the Hadamard-factorised form for $Q^{(-)}(E)$ no longer has such a simple form, and the proof breaks down. In fact we know that most of the λ_k's become complex in this region.

Finally, we remark that while the above constraints on the parameters M, α and l are sufficient they are not necessary, as can be seen by studying figure 9 for the case of $M = 3$. The full domain of unreality obtained numerically in [12] is shown as the interior of the curved line, a proper subset of A.

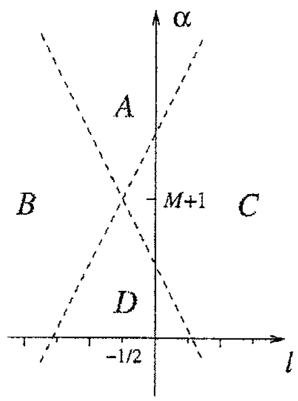

Figure 8: The 'phase diagram' at fixed M.

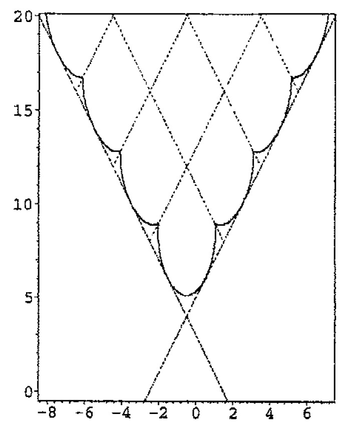

Figure 9: The domain of unreality A for $M = 3$ obtained numerically.

7 Conclusions

We hope to have shown that there is a very interesting relationship between the conformal limit of two dimensional integrable models and the spectral theory of ordinary differential equations. The instance described in this review involves the six-vertex model in its conformal limit and an interesting class of $\mathcal{PT}$-symmetric quantum mechanical systems with complex potential. As an application of this correspondence we have briefly sketched the proof of a conjecture due to Bessis, Zinn-Justin, Bender and Boettcher concerning the reality of the spectrum of a particular class of $\mathcal{PT}$-symmetric operators.

Acknowledgments

RT thanks the organisers for the invitation to speak at the conference. We would also like to thank Carl Bender, Barry McCoy, Eric Delabaere, Junji Suzuki, Yoshitsugu Takei, André Voros and Miloslav Znojil for useful conversations. This work was partly supported by the EC network "EUCLID", contract number HPRN-CT-2002-00325, and partly by a NATO grant PST.CLG.980424. PED was also supported by JSPS/Royal Society grant and RT by RIMS. CD thanks the Australian Research Council for financial support.

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