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Bifurcation analysis to Rayleigh-Bénard convection at degenerate critical points

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1 Introduction

In this paper we consider pattern formation problem to the classical Rayleigh-Bénard convection by the standard bifurcation analysis method. Let us consider the problem by the Bussinesq approximation (Oberbeck-Boussinesq model) with up-down symmetric boundary condition. It is already a classical fact that the hexagonal pattern appear right after the critical Rayleigh number and it is unstable. See for example [6]. In [2] they obtained general bifurcational structure under the up-down symmetry including the Boussinesq approximation. However, both of them have not obtained the eigenvalues about the mixed mode solutions such as the hexagonal pattern. On the other hand, recent 3D numerical simulation shows that it is rather easy to obtain the hexagonal pattern under the same up-down symmetric situation (eg. [3]). Therefore we have calculated the cubic normal form about the critical point where both the roll and hexagonal patterns appear and study the dynamics of them in our previous study([5]). By the cubic normal form we can study the invariant torus which includes the fixed point corresponding to the hexagonal pattern inside. To determine the motion on the torus we need to calculate the normal form up to higher order. But we only discuss the stability of the invariant torus and calculate the eigenvalues for the transversal direction to the torus. One of our previous results shows that it is true that the invariant torus of the hexagonal pattern has positive eigenvalues but they are small compared to the absolute value of the negative ones. The invariant torus is a saddle for its transversal directions and it will take quite a long time to observe unstable dynamics. It is consistent to the classical theoretical results and also the numerical simulations. Notice that a hexagonal pattern can be stable in the case when the two boundary condition are different so that it breaks the up-down symmetry. In fact the normal form has quadratic terms which correspond to the hexagonal resonance.

In these previous works the fluid tank with particular size is mainly considered. There exist 3 roll solutions which have the same critical wave length and cross each other by the angle of 120 degrees. Here, in this study, we shall consider more general situations by changing the size of the tank. It is still difficult to study the reduce dynamics for all the variations of the system size. However, we found a stable mixed mode solution (patchwork quilt type) by taking the size and the Prandtl number appropriately.

2 Formulation

We consider the Boussinesq approximation for the Rayleigh-Bénard convection. Variation equations about the conductive state can be written in the following non-dimensional form.

$$\left\{\begin{array}{rcl} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &=& -\nabla p + R\theta \mathrm{e}_z + \Delta \boldsymbol{u} \\ \theta_t + (\boldsymbol{u}\cdot\nabla)\theta &=& (w+\Delta\theta)/P \\ \nabla\cdot\boldsymbol{u} &=& 0 \end{array}\right. \tag{2.1}$$

Here, $\boldsymbol{u} = (u, v, w)$ is a velocity vector and $\mathrm{e}_z = (0,0,1)$. Two constants R and P are the Rayleigh and the Prandtl numbers, respectively.

Let the boundary conditions be free-slip for both the top and the bottom:

$$u_z = v_z = w = \theta = 0 \quad (z = 0, 1). \tag{2.2}$$

Therefore solutions to this problem have up-down symmetry which means that they are invariant under the mapping:

$$(u, v, w, \theta, p)(t, x, y, z) \mapsto (u, v, -w, -\theta, p)(t, x, y, 1 - z).$$

Moreover let us assume the periodicity for both x and y directions with the periods $(2\pi/\alpha, 2\pi/\beta)$. We represent each unknown variables by the Fourier expansion.

$$u = \sum_{(m,n,l)\in \mathbf{Z}^3} u_{m,n,l} e^{i(m\alpha x + n\beta y + l\pi z)}$$

We use an abrebiated notation $\mathbf{m} = (m, n, l)$ for the mode vector and $u_{\mathbf{m}} = u_{m,n,l}$.

Since all the unknown variables are real valued and their Fourier coefficients have the Hermitian symmetry: $u_{\mathbf{m}} = u_{-\mathbf{m}}$. They also satisfy the following properties which correspond to the up-down symmetry.

$$\begin{aligned} u_{m,n,l} &= u_{m,n,-l}, \\ v_{m,n,l} &= v_{m,n,-l}, \\ w_{m,n,l} &= -w_{m,n,-l}, \\ \theta_{m,n,l} &= -\theta_{m,n,-l}, \\ p_{m,n,l} &= p_{m,n,-l} \end{aligned} \tag{2.3}$$

Now we rewrite the equations (2.1) by using the Fourier coefficients as follows.

$$\begin{pmatrix} \dot{u}_{\mathbf{m}} \\ \dot{v}_{\mathbf{m}} \\ \dot{w}_{\mathbf{m}} \\ \dot{\theta}_{\mathbf{m}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 & 0 & 0 & -im\alpha \\ 0 & -\omega^2 & 0 & 0 & -in\beta \\ 0 & 0 & -\omega^2 & R & -il\pi \\ 0 & 0 & 1/P & -\omega^2/P & 0 \\ im\alpha & in\beta & il\pi & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{\mathbf{m}} \\ v_{\mathbf{m}} \\ w_{\mathbf{m}} \\ \theta_{\mathbf{m}} \\ p_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)v\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)w\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)\theta\}_{\mathbf{m}} \\ 0 \end{pmatrix} \tag{2.4}$$

Here, $\omega^2 = m^2\alpha^2 + n^2\beta^2 + l^2\pi^2$.

(2.4) with the symmetry (2.3) is equivalent to (2.1) with (2.2). Moreover the Fourier coefficients for the pressure p and w can be eliminated by the fifth equation of (2.4) and finally we obtain the following system of ordinary differential equations for $\mathbf{m} \neq 0$.

$$\begin{pmatrix} \dot{u}_{\mathbf{m}} \\ \dot{v}_{\mathbf{m}} \\ \dot{\theta}_{\mathbf{m}} \end{pmatrix} = M_{\mathbf{m}} \begin{pmatrix} u_{\mathbf{m}} \\ v_{\mathbf{m}} \\ \theta_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)v\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + k_{\mathbf{m}} \begin{pmatrix} m\alpha \\ n\beta \\ 0 \end{pmatrix} \quad (l \neq 0) \tag{2.5}$$

$$\begin{pmatrix} \dot{u}_{\mathbf{m}} \\ \dot{v}_{\mathbf{m}} \end{pmatrix} = M_{(m,n,0)} \begin{pmatrix} u_{\mathbf{m}} \\ v_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)v\}_{\mathbf{m}} \end{pmatrix} + k_{(m,n,0)} \begin{pmatrix} m\alpha \\ n\beta \end{pmatrix} \quad (l = 0 \text{ and } \mathbf{m} \neq 0) \tag{2.6}$$

Here,

$$k_{\mathbf{m}} = \frac{1}{\omega^2}(m\alpha\{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} + n\beta\{(\boldsymbol{u}\cdot\nabla)v\}_{\mathbf{m}} + l\pi\{(\boldsymbol{u}\cdot\nabla)w\}_{\mathbf{m}}),$$

$$M_{\mathbf{m}} = \begin{pmatrix} -\omega^2 & 0 & -ml\pi\alpha R/\omega^2 \\ 0 & -\omega^2 & -nl\pi\beta R/\omega^2 \\ -m\alpha/l\pi P & -n\beta/l\pi P & -\omega^2/P \end{pmatrix}, \quad (l \neq 0),$$

$$M_{\mathbf{m}} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}, \quad (l = 0 \text{ and } \mathbf{m} \neq 0)$$

Notice that the mean flow should be zero, that is $u_0 = v_0 = p_0 = 0$ and it holds that $w_0 = \theta_0 = 0$ by the symmetry (2.3). It is easy to see that the linearized matrix $M_{\mathbf{m}}$ has 0-eigenvalue if and only if $l \neq 0$ and $R = R(k) = (k^2 + l^2\pi^2)^3/k^2$ where k is the wave number with $k^2 = m^2\alpha^2 + n^2\beta^2$. $R(k)$ takes its minimum value $R_c = 27\pi^4/4$ at the critical wavelength $k_c = \sqrt{2}\pi/2$. R_c is called the critical Rayleigh number. We are interested in the case when the first instability takes place as we increase the Rayleigh number. Therefore, we have only to consider the case $l = 1$ and $R = R(k) = (k^2 + \pi^2)^3/k^2$.

Let us similarly consider the 2 dimensional case where the solution depends only on (x, z)-direction. Now the unknown variables are u, w, θ, p and their time evolution can be described as follows. Notice that the mode vector is $\mathbf{m} = (m, l) \in \mathbf{Z}^2$ and $\omega^2 = m^2\alpha^2 + l^2\pi^2$.

$$\begin{pmatrix} \dot{u_{\mathbf{m}}} \\ \dot{\theta_{\mathbf{m}}} \end{pmatrix} = M_{\mathbf{m}} \begin{pmatrix} u_{\mathbf{m}} \\ \theta_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + k_{\mathbf{m}} \begin{pmatrix} m\alpha \\ 0 \end{pmatrix} \quad (l \neq 0) \tag{2.7}$$

$$\dot{u_{\mathbf{m}}} = -\omega^2 u_{\mathbf{m}} \quad (l = 0 \text{ and } \mathbf{m} \neq 0) \tag{2.8}$$

Here, $M_{\mathbf{m}}$ and $k_{\mathbf{m}}$ are defined as follows although we use the same notation as the 3-D case.

$$k_{\mathbf{m}} = \frac{1}{\omega^2}(m\alpha\{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} + l\pi\{(\boldsymbol{u}\cdot\nabla)w\}_{\mathbf{m}}),$$

$$M_{\mathbf{m}} = \begin{pmatrix} -\omega^2 & -ml\pi\alpha R/\omega^2 \\ -m\alpha/l\pi P & -\omega^2/P \end{pmatrix}, \quad (l \neq 0)$$

Now let us calculate the convolution terms.

$$\{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1\neq 0}} \frac{i\alpha(m_2 l_1 - m_1 l_2)}{l_1} u_{\mathbf{m}_1} u_{\mathbf{m}_2} \tag{2.9}$$

$$\{(\boldsymbol{u}\cdot\nabla)\theta\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1\neq 0}} \frac{i\alpha(m_2 l_1 - m_1 l_2)}{l_1} u_{\mathbf{m}_1} \theta_{\mathbf{m}_2} \tag{2.10}$$

$$\{(\boldsymbol{u}\cdot\nabla)w\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1 l_2\neq 0}} -\frac{i m_2\alpha^2(m_2 l_1 - m_1 l_2)}{l_1 l_2 \pi} u_{\mathbf{m}_1} u_{\mathbf{m}_2} \tag{2.11}$$

Here, we denote $\mathbf{m}_i = (m_i, l_i)$.

These coupled systems ((2.5) (2.6) and (2.7) (2.8)) of countably many ordinary differential equations have a trivial zero solution. We study the local bifurcation about the trivial solution. It is necessary to calculate the normal form on the center manifold which is locally spanned by the critical eigenvectors of $M_{\mathbf{m}}$ for each set of parameter values of (R, α, β).

There are two possibilities of critical points in the 2-D case. In fact if a critical point is degenerate then two adjacent modes , n and $n+1$ modes, are critical. In the 3-D case, if we choose the domain with $(\alpha, \beta) = k_c(1/2, \sqrt{3}/2)$ where k_c is the critical wave length, 3 critical modes appear and they correspond to the roll patterns which differ by 120 degrees each other.

3 2-D problem and stability of mixed mode solutions

We review our previous results for the stability of mixed mode solutions for a 2-D fluid tank. It might be easier to explain our analysis in the 2-D problem and we basically take similar strategy for the 3-D problem.

When $R = R(\alpha; m) := (m^2\alpha^2 + \pi^2)^3/m^2\alpha^2$ holds, $(m, 1)$-mode becomes critical in the linearized problem about zero to (2.7)(2.8). Therefore at most two critical modes become critical at the same time. More precisely, for a given α there exists a number R^* such that all the eigenvalues about zero is negative for $R < R^*$, and moreover one of the following holds. (See also Figure 1.) :

- **simple critical case**: There exists a natural number n such that $R^* = R(\alpha; \pm n, 1)$ and if $|m| \neq n$ then $R^* < R(\alpha; m, 1)$. We call the pair of parameter values (α, R^*) a simple critical point.
- **multiple critical case**: There exists a natural number n such that $R^* = R(\alpha; \pm n, 1) = R(\alpha; \pm(n+1), 1)$ and if $|m| \neq n, n+1$ then $R^* < R(\alpha; m, 1)$. We call the pair of parameter values (α, R^*) a multiple critical point.

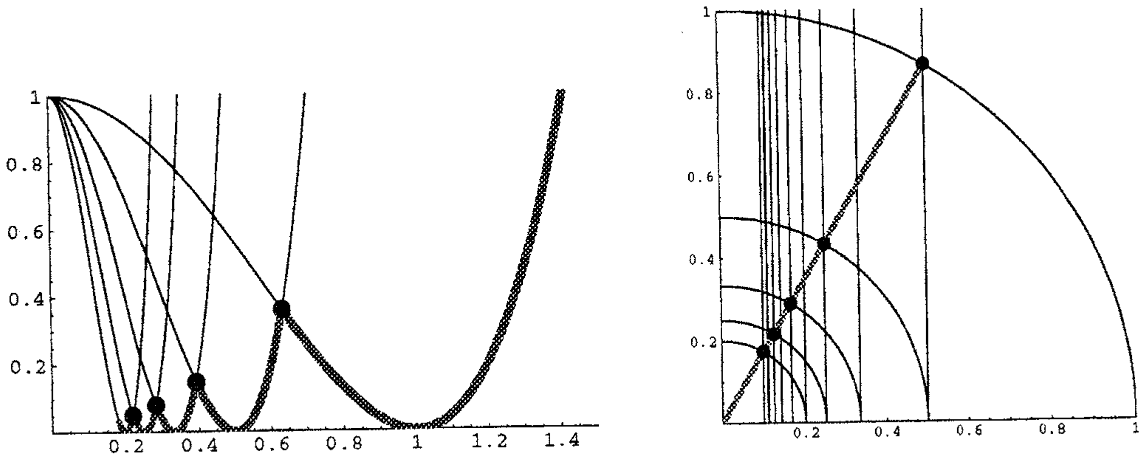

Figure 1: [Left figure]:Neutral stability curves drawn in (α, R)-plane. They correspond the critical curves for $R = R(\alpha; 1), \cdots, R(\alpha, 4)$ respectively from the right. Thick curve means the first instability and the black dots are mutiple critical points. [Right figure]:$C_{m,n}$ drawn in (α, β)-plane for the 3-D problem (See section 4 in detail) for $(m, n) = (1,1), (2,2), (3,3), (4,4), (5,5), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0), (8,0), (9,0)$ and $(10, 0)$. Black dots correspond to the hexagonal critical points.

It is easy to see that a roll solution bifurcates at a simple critical point as a supercritical pitchfork bifurcation. In fact, $M_{\mathbf{m}}$ has simple 0-eigenvalue if and only if $\mathbf{m} \in S := \{(\pm n, \pm 1), (\pm n, \mp 1)\}$. The critical eigenvectors are not $u_{\mathbf{m}}$ but $\tilde{u}_{\mathbf{m}}, \mathbf{m} \in S$ as follows. The linear transformation:

$$\begin{pmatrix} \tilde{u}_{\mathrm{m}} \\ \tilde{\theta}_{\mathrm{m}} \end{pmatrix} = T \begin{pmatrix} u_{\mathrm{m}} \\ \theta_{\mathrm{m}} \end{pmatrix}, \quad T = \frac{1}{(1+P)m\alpha l\pi\omega^2} \begin{pmatrix} m\alpha & -lP\pi\omega^2 \\ m\alpha & l\pi\omega^2 \end{pmatrix}$$

make the linear part of the equation for $\mathbf{m} \in S$ diagonal as

$$\begin{pmatrix} \dot{\tilde{u}}_{\mathbf{m}} \\ \dot{\tilde{\theta}}_{\mathbf{m}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1+P}{P}\omega^2 \end{pmatrix} \begin{pmatrix} \tilde{u}_{\mathbf{m}} \\ \tilde{\theta}_{\mathbf{m}} \end{pmatrix} - T \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + Tk_{\mathbf{m}} \begin{pmatrix} m\alpha \\ 0 \end{pmatrix} \tag{3.1}$$

Now the center manifold about the simple critical point can be described by $\tilde{u}_{\mathbf{m}}(\mathbf{m} \in S)$. The other modes: $\tilde{\theta}_{\mathbf{m}}(\mathbf{m} \in S)$ and $u_{\mathbf{m}}$, $\theta_{\mathbf{m}}(\mathbf{m} \notin S)$ are the slave modes. Moreover, it holds that $\tilde{u}_{(n,1)} = \tilde{u}_{(n,-1)}$ by the up-down symmetry. Therefore $\tilde{u}_{(n,1)}, \overline{\tilde{u}_{(n,1)}}$ gives the local coordinate on the center manifold. We are interested in the small solutions $|\tilde{u}_{(n,1)}| < \delta$ near the critical point. Then all the slave modes are $O(\delta^2)$ by the center manifold theorem. To obtain the effective normal form on the center manifold we pick up the nonlinear terms from the equations of the critical modes in (3.1) up to $O(\delta^3)$. The nonlinear terms for the equation of $\tilde{u}_{(n,1)}$ in (3.1) consist of $u_{\mathbf{m}_1}u_{\mathbf{m}_2}$ and $u_{\mathbf{m}_1}\theta_{\mathbf{m}_2}$ with $\mathbf{m}_1 + \mathbf{m}_2 = (n,1)$. The combinations of $(\mathbf{m}_1, \mathbf{m}_2)$ which give nonlinear terms up to $O(\delta^3)$ are $((n,1),(0,0))$, $((-n,1),(2n,0))$, $((n,-1),(0,2))$ and $((-n,-1),(2n,2))$. Since $\theta_{(m,0)} = 0$ holds by the up-down symmetry and $u_{(m,0)} = 0$ holds in the 2-D setting, the nonlinear terms come from the first two combinations of above are zero. It is also zero for $(\mathbf{m}_1, \mathbf{m}_2) = ((-n,-1),(2n,2))$ by (2.9), (2.10) and (2.11). Therefore the slave modes up to $O(\delta^3)$ which relate to the equation for $A := \tilde{u}_{(n,1)}$ are only $B := u_{(0,2)}$ and $C := \theta_{(0,2)}$. We can write the corresponding equations as follows.

$$\dot{A} = \lambda A + \frac{n^2\alpha^2 - \pi^2}{n^2\alpha^2 + \pi^2} n\alpha i AB + \pi\omega^2 iAC + O(\delta^4) \tag{3.2}$$

$$\dot{B} = -4\pi^2 B + O(\delta^4) \tag{3.3}$$

$$\dot{C} = -4\pi^2 C + 4\pi\omega^2 n^2\alpha^2 i|A|^2 + O(\delta^4) \tag{3.4}$$

Here, we assume $\lambda = R - R^* = O(\delta^2)$. To obtain the normal form on the center manifold we need to calculate the approximation of the center manifold by the coordinate A or we take an appropriate near-identity transformation before the center manifold reduction as follows. More precisely we determine unknown constants p and q so that we can eliminate the quadratic terms in the equation after taking the near-identity transformation $\tilde{A} = A + pAB + qAC$. Finally we obtain

$$\dot{\tilde{A}} = \lambda\tilde{A} - \omega^4 n^2\alpha^2|\tilde{A}|^2\tilde{A} + O(\delta^4). \tag{3.5}$$

It shows that a supercritical pitchfork bifurcation to a roll solution occurs at the critical point.

Next, we shall consider the multiple critical case. $M_{\mathbf{m}}$ has simple 0-eigenvalue if and only if $\mathbf{m} \in S := \{(\pm n, \pm 1), (\pm n, \mp 1), (\pm n', \pm 1), (\pm n', \mp 1)\}$, where $n' = n + 1$. After taking the similar linear transformation which diagonalize the matrix $M_{\mathbf{m}}$ the 4 critical modes are represented by $A := \tilde{u}_{(n,1)}$, $\overline{A} = \tilde{u}_{(-n,-1)}$, $B := \tilde{u}_{(n',1)}$ and $\overline{B} = \tilde{u}_{(-n',-1)}$.

Moreover, the slave modes coming into the equation for $A = \tilde{u}_{(n,1)}$ are $C := u_{(0,2)}$, $D := \theta_{(0,2)}$, $E := u_{(n-n',2)}$, $F := \theta_{(n-n',2)}$, $G := u_{(n+n',2)}$ and $H := \theta_{(n+n',2)}$ up to $O(\delta^3)$. We obtain the following normal form when $n \geq 2$ by taking the similar near-identity transformation as above. Notice that it has quadratic resonance terms when $n = 1$ and we need a different approach as one can see in [1].

$$\begin{cases} \dot{\tilde{A}} = \tilde{A}(\lambda - a|\tilde{A}|^2 - b|\tilde{B}|^2) + O(\delta^4) \\ \dot{\tilde{B}} = \tilde{B}(\lambda' - c|\tilde{A}|^2 - d|\tilde{B}|^2) + O(\delta^4) \end{cases} \tag{3.6}$$

It can be separated into the equation for the modulus(amplitude) and the argument(angle) by the polar coordinate. And the equations for the amplitudes are

$$\begin{cases} \dot{r} &= r(\lambda - ar^2 - bs^2) + O(\delta^4), \\ \dot{s} &= s(\lambda' - cr^2 - ds^2) + O(\delta^4). \end{cases} \tag{3.7}$$

Here, we denote $r = |\tilde{A}|, s = |\tilde{B}|$. The equations (3.7) have at most 4 equilibriums $(0,0)$, $(r_1, 0)$, $(0, s_1)$ and (r^*, s^*) when $r \geq 0, s \geq 0$. We call one of these equilibriums (r^*, s^*) a mixed mode solution if $r^* s^* \neq 0$. Since the linearized matrix for (3.7) about the mixed mode solution is given by $N = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a mixed mode solution is stable in the sense of (3.7) if $\det N > 0$. It means that (3.6) has stable invariant torus which include a mixed mode stationary solution. Here, we don't consider the motion on the invariant torus but the stability of the invariant torus, since we need higher order normal form to determine the whole dynamics on the center manifold. We showed that there exist a stable mixed mode by taking the Prandtl number appropriately. In fact Figure 2 shows the relation between $\det N$ and the Prandtl number.

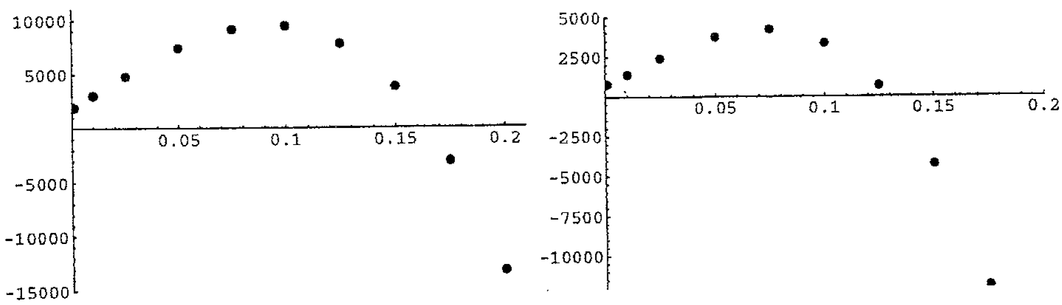

Figure 2: The stability of $n-(n+1)$ mixed mode. Values of $\det N$ with respect to different values of P are drawn. Two of the figures correspond to $n = 2$ and $n = 3$ from the left, respectively.

4 Neutral stability surfaces for 3-D problem

The 3 critical roll modes become unstable exactly at the critical Rayleigh number R_c when $(\alpha, \beta) = k_c(1/2, \sqrt{3}/2)$. While, on the other hand, the first instability occurs at $R > R_c$ in general. It is convenient to define the neutral stability surface for each mode $(m, n, 1)$(or simply we denote (m, n)) as follows.

$$G_{m,n} = \left\{ (\alpha, \beta, R) \ ; \ R = R_{m,n}(\alpha, \beta) := R\left(\sqrt{m^2\alpha^2 + n^2\beta^2}\right), \alpha, \beta \in (0, +\infty) \right\}$$

The (m, n)-mode instability occurs on the surface $G_{m,n}$. Remember that we have set $l = 1$ since we are interested in the first instability. Therefore, for a given (α, β), the first instability occurs as (m_*, n_*)-mode where $R_{m_*,n_*}(\alpha, \beta) \leq R_{m,n}(\alpha, \beta)$ for any $(m, n) \in \mathbf{Z}^2$. There can be multiple critical modes. In fact when $(\alpha, \beta) = k_c(1/2, \sqrt{3}/2)$, both $(2, 0)$ and $(1, 1)$-modes are critical at $R = R_c$. More precisely, the set of critical modes are

$\{(\pm 2, 0), (\pm 1, \pm 1), (\pm 1, \mp 1)\}$ in this case. By using the Hermitian symmetry we have essentially 3 critical modes: $\tilde{u}_{(2,0,1)}, \tilde{u}_{(-1,1,1)}$ and $\tilde{u}_{(-1,-1,1)}$. $R_{m,n}(\alpha, \beta)$ attains its minimum R_c on

$$C_{m,n} = \left\{ (\alpha, \beta)\ ;\ m^2\alpha^2 + n^2\beta^2 = k_c^2, \alpha, \beta \in (0, +\infty) \right\}.$$

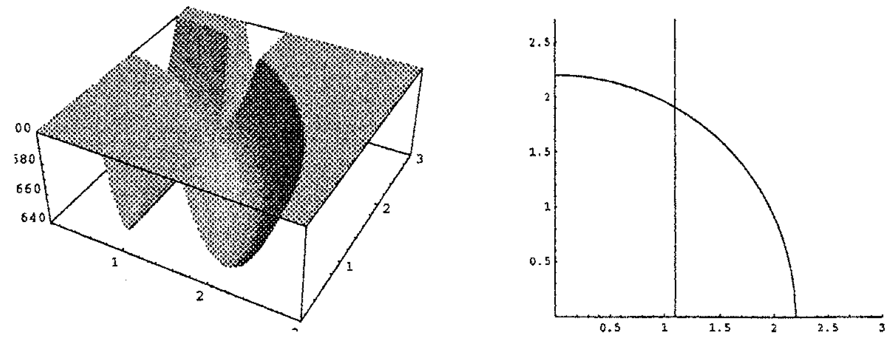

Figure 3: Neutral stability surfaces for $(2, 0)$ and $(1, 1)$(left) and $C_{m,n}$ (right).

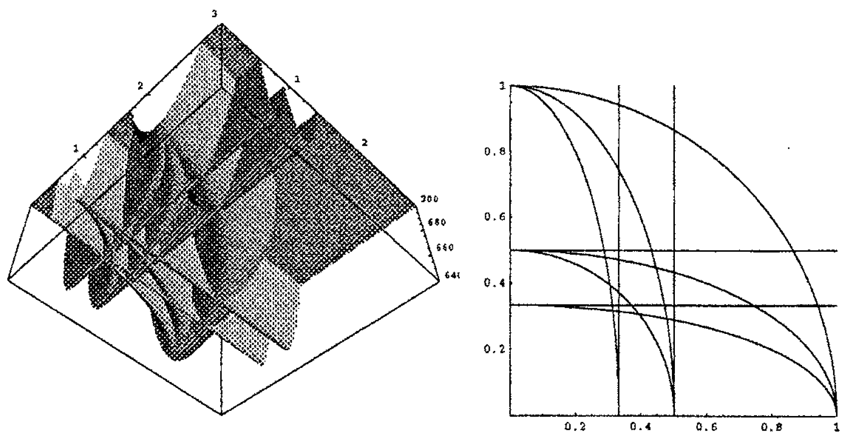

Figure 4: Neutral stability surfaces (left) and $C_{m,n}$ (right) for $(m, n) = (1, 1), (2, 0), (0, 2), (3, 0), (2, 1), (1, 2), (0, 3), (3, 1), (2, 2)$ and $(1, 3)$.

Next, if we proportionally increase or decrease the size (α, β) to some extent then we have the same set of critical modes but the critical value of R is larger than R_c. If we further decrease (α, β) then another set of modes replace this set of critical modes. There are so many possibilities of multiple critical points. In this article we are interested in the critical points where the set of critical modes are $\{(\pm n, 0), (\pm m, \pm l), (\pm m, \mp l)\}$, where $n > m$ and $l \neq 0$. We call the point (α, β, R) which satisfies this property the pseudo hexagonal critical point since it has essentially 3 critical roll modes. In fact the center manifold can be described by 3 critical modes $\tilde{u}_{(n,0,1)}, \tilde{u}_{(-m,l,1)}$ and $\tilde{u}_{(-m,-l,1)}$.

By using the similar argument to the previous section we can obtain the normal form on

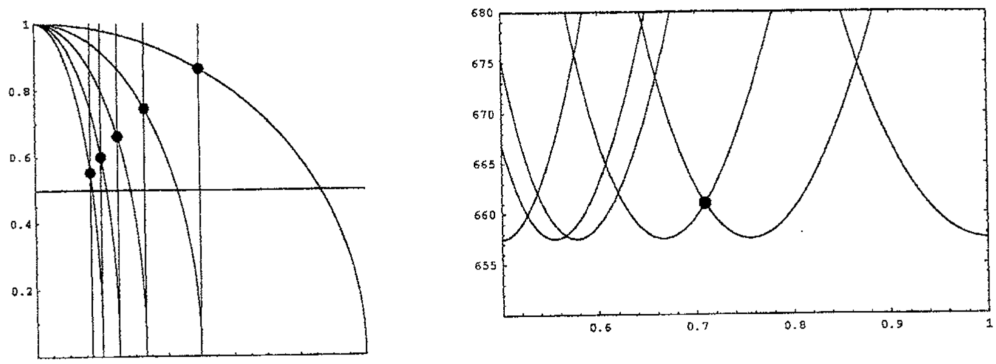

Figure 5: Pseudo hexagonal points of $(n,1)$ and $(n+1,0)$ (left) and the vertical section of neutral critical surfaces on the line $\{(\alpha,\beta) = sk_c(1/2,\sqrt{3}/2); s \in (1/2,1)\}$ (right). Each curves in the right figure corresponds to the section of $S_{1,1}, S_{2,1}, S_{3,0}, S_{3,1}, S_{1,2}$ and $S_{2,2}$ from the right, respectively. Notice that the pair of surfaces $\{S_{1,1}, S_{2,0}\}$ as well as $\{S_{0,2}, S_{3,1}\}$ coincide each other on this line. Black dot is the point where the patchwork-quilt is stable.

the center manifold as follows. Here we denote $A_1 = \tilde{u}_{(n,0,1)}, A_2 = \tilde{u}_{(-m,l,1)}, A_3 = \tilde{u}_{(-m,-l,1)}$.

$$\left\{ \begin{array}{l} \dot{A}_1 = A_1(\mu - a|A_1|^2 - b|A_2|^2 - b|A_3|^2) \\ \dot{A}_2 = A_2(\mu - c|A_1|^2 - d|A_2|^2 - e|A_3|^2) \\ \dot{A}_3 = A_3(\mu - c|A_1|^2 - e|A_2|^2 - d|A_3|^2) \end{array} \right. \tag{4.1}$$

In the case of hexagonal critical point it holds that $a = d, b = c = e$. Also it has generally quadratic terms as the hexagonal resonance, however, in this case quadratic terms should vanish by the up-down symmetry.

We extract the equations for the amplitudes from (4.1) by taking the polar coordinates $A_i = r_i e^{i\phi_i}$:

$$\left\{ \begin{array}{lcl} \dot{r_1} &=& r_1(\mu - a\, r_1{}^2 - b\, r_2{}^2 - b\, r_3{}^2) \\ \dot{r_2} &=& r_2(\mu - c\, r_1{}^2 - d\, r_2{}^2 - e\, r_3{}^2) \\ \dot{r_3} &=& r_3(\mu - c\, r_1{}^2 - e\, r_2{}^2 - d\, r_3{}^2) \end{array} \right. \tag{4.2}$$

These equations can have the following equilibriums:

- (O) : $(0,0,0)$
- (R) : $(r_1^\dagger, 0, 0), (0, r_2^\dagger, 0), (0, 0, r_3^\dagger)$
- (PQ) : $(r_{11}^\ddagger, r_{12}^\ddagger, 0), (0, r_{21}^\ddagger, r_{22}^\ddagger), (r_{31}^\ddagger, 0, r_{32}^\ddagger)$
- (H) : (r_1^*, r_2^*, r_3^*)

Here, each of $r^\dagger, r^\ddagger$ and r^* is non-zero. We call them (O):zero, (R):roll, (PQ):patchwork quilt and (H):pseudo hexagonal solution, respectively. As we showed in the previous study in the case of the hexagonal critical point, *i.e.*, when $(n,m,l) = (2,1,1)$, the number of positive eigenvalues about each solution in the sense of (4.2) is (O):3, (R):0, (PQ):1, (H):2, respectively. Especially, the eigenvalues about (H) are $b-a > 0, b-a > 0, -a-2b < 0$ and we can show the ratio $(b-a)/(a+2b)$ between the absolute values of the positive and negative eigenvalues is small. This means that the invariant torus corresponding to the hexagon is a saddle for the transversal direction to the torus and it will take a long time to observe unstable dynamics in general.

Now let us go back to the analysis on the pseudo-hexagonal critical point. We calculated the normal form about some of these critical points in $\{(\alpha, \beta, R_c); (\alpha, \beta) \in C_{n,1} \cap C_{n+1,0}\}$ as in Figure 5 (left). As a result it turns out that all the pseudo hexagonal and patchwork quilt solutions are unstable. We conjecture this holds for every pseudo hexagonal critical points at the critical Rayleigh number R_c.

On the other hand, a patchwork quilt pattern can be stable by taking (α, β) suitably so that the critical value for R is larger than R_c. In fact this is true, for example, at the multiple critical point of $(3,0)$ and $(2,1)$ with $(\alpha, \beta) = s(1, \sqrt{3})$ for some $s > 0$ (See also Figure 5) and P is less than certain critical value which is approximately 0.8. At that point our results on the normal form show that $a, b, d, e > 0, d < e$ while $c < 0$. We also conjecture that there are other parameter regions where the patchwork quilt pattern becomes stable.

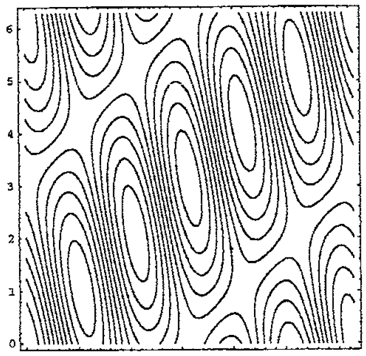

Figure 6: Schematic picture of a patchwork quilt pattern which is stable at a multiple critical points of $(3,0)$ and $(2,1)$.

5 Discussion

We are developing a code to exactly calculate normal form. The results shown in the previous section are based on these calculations. (These are joint work with my student Takashi Okuda.) We can exactly calculate normal form coefficients at a given critical point, however, it is still not easy to analyze the dynamics on the center manifold by the following reasons. One difficulty comes from the fact that some of the critical points have more than 4 critical modes. In these cases cubic resonance terms might appear and we can not separate the dynamics on the center manifold into the equations for amplitude and argument. Another trouble is that the normal form coefficients obtained by the code are generally described by tremendously long and complex formula. Therefore we can utilize the formula for only obtaining the numerical values of them. However, we believe further analysis to the normal form will give us an idea on why and how the mixed mode solutions be stabilized under a suitable setting.

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