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The Jeans instability for a one-dimensional model system of compressible viscous fluids

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1. Introduction. The Jeans instability is an astrophysical idea associated with the collapse of a fluctuated infinite homogeneous self-gravitating gas. If a fluctuation ideally takes the form of a plane wave, then the growth of the wavelength is a factor of the instability. According to the idea the gravitation reduces the speed of the plane wave as the wavelength grows. Furthermore, once the wavelength exceeds a threshold, the gravitation cuts off the wave propagation and amplifies the plane wave instead. This curious observation comes from a dispersion relation for plane waves derived from the compressible Euler equations coupled with Poisson's equation. See, for instance, [2], [13], where a linear approximation of the coupled system is introduced. A rigorous analysis of the Euler-Poisson system would reveal the actual flow of a self-gravitating gas to improve our understanding of the Jeans instability. A difficulty here is that Poisson's equation of gravity is incompatible with any equilibrium of an infinite homogeneous gas, which is troublesome also in the linear approximation theory. Some modifications of the model are necessary.

We understand that the motion of a plane wave forms a one-dimensional periodic flow. Suppose the mass density is everywhere positive. Taking account of the dissipation due to viscosity, we give a model for the one-dimensional periodic motion of a self-gravitating viscous gas. In the Lagrange material coordinates the model is described as follows:

$$
(1.1)\qquad \begin{cases} \partial_t v(t,x) - \partial_x u(t,x) = 0, \\ \partial_t u(t,x) + \partial_x\Big(\dfrac{a}{v(t,x)^\gamma}\Big) - \nu\partial_x\Big(\dfrac{\partial_x u(t,x)}{v(t,x)}\Big) \\ = \dfrac{4\pi G}{\bar v}\Big(\dfrac{1}{L}\displaystyle\int_0^x y(v(t,y)-\bar v)dy + \dfrac{1}{L}\int_x^L (y-L)(v(t,y)-\bar v)dy\Big). \end{cases}
$$

In (1.1) the specific volume v and the velocity u are unknown functions of the time and space variables t, x in the Lagrange material coordinates. We assume the L-periodicity in x of the unknown functions. The constants γ, ν and G are the ratio of the specific heats, the viscosity constant, and the gravitational constant, respectively. We assume that $1 \leq \gamma < 2$, $\nu > 0$, and $G > 0$. We also assume that a in pressure $av^{-\gamma}$ is a positive constant. We denote by $\bar v$ the mean value of v, and by $\bar{u}$ that of u. If (v, u) satisfies (1.1) in an appropriate sense, then both $\bar{v}$ and $\bar{u}$ are preserved in time, so that the variable t of $\bar{v}$ and $\bar{u}$ may be omitted.

The right-hand side of the second equation of (1.1) is a gravitational field. As easily seen, this gives a forcing term L-periodic in x. In the Euler coordinates the field of force due to an l-periodic mass density ρ is expressed as

$$4\pi G\Big(-\frac{1}{l}\int_0^{\xi}\eta(\rho(\eta)-\bar{\rho})d\eta-\frac{1}{l}\int_{\xi}^{l}(\eta-l)(\rho(\eta)-\bar{\rho})d\eta\Big) \tag{1.2}$$

with $\bar{\rho} = \int_0^l \rho(\eta)d\eta/l$. This is a potential field $-\partial_\xi\phi$ with $\partial_\xi^2\phi = 4\pi G(\rho-\bar{\rho})$. In comparison with Poisson's equation the source term is corrected so that the modified equation admits an l-periodic potential ϕ. The modification of Poisson's equation is adopted in the linear approximation theory. Another way to approach the formula (1.2) is applying a summation technique of oscillatory integration to the following divergent expression of Newton's gravitational field:

$$4\pi G\Big(-\frac{1}{2}\int_{-\infty}^{\xi}\rho(\eta)d\eta+\frac{1}{2}\int_{\xi}^{+\infty}\rho(\eta)d\eta\Big).$$

See [7]. We can derive the model system (1.1) from the compressible Navier-Stokes equations written in the Euler coordinates (τ, ξ); for the flow with velocity $\Upsilon(\tau, \xi)$ in the field (1.2) with mass density $\rho(\tau, \xi)$ we have only to carry out the change of variables $t = \tau$, $x = \int_{\xi(\tau)}^{\xi}\rho(\tau, \eta)d\eta$ with $\xi(\cdot)$ solving $(d\xi/d\tau)(\tau) = \Upsilon(\tau, \xi(\tau))$, that of unknown functions $v = 1/\rho$, $u = \Upsilon$, and that of physical quantities $\bar{v} = 1/\bar{\rho}$, $L = l\bar{\rho}$. Note that (1.1) admits a trivial solution $(v, u) \equiv (\bar{v}, \bar{u})$ representing an infinite homogeneous equilibrium.

In the model system (1.1) the Jeans instability is associated with the exchange of stability of trivial solutions in the stationary problem involving bifurcation. While the mean value of v is small, the stationary problem admits only a trivial solution, being stable. As the mean value crosses a critical value, however, non-trivial and stable stationary solutions branch off with the loss of stability of the trivial solution. In the critical case the spatial period of the flow is related to both the mass density and the sound speed in an infinite homogeneous gas, as suggested in astronomy.

This note is a summary of the works [7], [8], [9]. We only sketch proofs of the results. For details, see [8], [9].

2. Results. Let us briefly mention the Cauchy problem for (1.1). For a non-negative integer m we denote by H^m_{per} the usual Sobolev space of real-valued L-periodic functions on $(-\infty, \infty)$, which is equipped with a complete norm

$$\|f\|_{H^m} = \Big(\sum_{k=0}^{m}\int_0^L|\partial_x^k f(x)|^2dx\Big)^{1/2}.$$

For a non-negative integer s and a positive number T the space of s times continuously differentiable functions on an interval $[0, T]$ with values in H^m_{per} is denoted by $C^s([0,T]; H^m_{\mathrm{per}})$. The space $H^s(0,T; H^m_{\mathrm{per}})$ is the set of all square integrable functions on $(0,T)$ with values in H^m_{per} whose distributional derivatives of order up to s are also square integrable on $(0,T)$. We follow the usual notations $H^0_{\mathrm{per}} = L^2_{\mathrm{per}}$ and $H^0(0,T; H^m_{\mathrm{per}}) = L^2(0,T; H^m_{\mathrm{per}})$.

The unique global solvability of the Cauchy problem is stated as follows.

Theorem 1. *Assume $(v_0, u_0) \in H^1_{\mathrm{per}} \times H^1_{\mathrm{per}}$ with $v_0 > 0$. Then the Cauchy problem for (1.1) with initial condition $(v(0,\cdot), u(0,\cdot)) = (v_0, u_0)$ has a unique global solution (v, u) on $[0, \infty)$ with the following properties; for any positive number T, $v \in C^1([0,T]; L^2_{\mathrm{per}}) \cap C^0([0,T]; H^1_{\mathrm{per}})$, $v(T,\cdot) > 0$, and $u \in H^1(0,T; L^2_{\mathrm{per}}) \cap L^2(0,T; H^2_{\mathrm{per}})$.*

Sketch of Proof: The system (1.1) takes the form of a viscous p-system but contains an unfamiliar self-gravitating force. Since the mean value of v is eventually preserved in time, we can replace it partly with that of the initial data and rewrite the forcing term as a bounded linear operator on the Sobolev spaces. This enables us to construct a local solution in the same way as we solve the usual viscous p-system. Once a local solution with $v > 0$ is obtained, the magnitude of the forcing term has an upper bound independent of v. Thus we can continue the local solution to a global one as if (1.1) were a viscous p-system with a bounded forcing term. ∎

We are interested in the stability of stationary solutions of (1.1). Since the mean value $(\bar{v}, \bar{u})$ of a solution of (1.1) is preserved in time, it is natural to consider the stability in every section of the product space $H^1_{\mathrm{per}} \times H^1_{\mathrm{per}}$ on which both $\bar{v} > 0$ and $\bar{u}$ take prescribed values. By the change of functions $u \mapsto u - \bar{u}$ we may assume $\bar{u} = 0$. Let V be a positive parameter. The task will be first to study the structure of the whole stationary solutions lying in the manifold

$$M_V = \{(v,u) \in H^1_{\mathrm{per}} \times H^1_{\mathrm{per}}; v > 0, \bar{v} = V, \bar{u} = 0\},$$

and then to decide whether the stationary solutions are stable or not in M_V.

For any $V > 0$ a constant function $(V, 0)$ clearly solves the stationary problem. The other solutions, if exist, must posses the least period L/n for some positive integer n. The following theorem gives a necessary and sufficient condition on V for which the stationary problem admits in M_V a solution with least period L/n. Also the structure of the whole stationary solutions with least period L/n is given. Here and in what follows we abuse the notations $A\infty$, ∞/A, ∞^A, $A/0$, and 0^{-A}, all of which mean ∞ for $A > 0$.

Theorem 2. *Let n be a positive integer. Within the manifold M_V the stationary problem for (1.1) admits a non-trivial solution with least period L/n if and only* *if* $\{(n^2\pi a\gamma)/(GL^2)\}^{1/\gamma} < V < \{(n^2\pi a\gamma c(\gamma))/(GL^2)\}^{1/\gamma}$ *with* $c(1) = \infty$ *and a constant* $c(\gamma) > 1$ *depending only on* γ *for* $1 < \gamma < 2$. *We have* $\lim_{\gamma\to 1+0} c(\gamma) = \infty$ *and* $\lim_{\gamma\to 2-0} c(\gamma) = 1$. *The set of all stationary solutions of* (1.1) *in* M_V *with least period* L/n *is of the form*

$$\{(\tilde{v}(\cdot - \alpha), 0); 0 \le \alpha < L/n\},$$

where $(\tilde{v}, 0)$ *is one of such stationary solutions.*

Sketch of Proof: Let $(\tilde{v}, \tilde{u}) \in M_V$ be a stationary solution of (1.1). Then we have

$$\begin{cases} -\partial_x \tilde{u} = 0, \\ \partial_x\left(\dfrac{a}{\tilde{v}^\gamma}\right) - \nu\partial_x\left(\dfrac{\partial_x \tilde{u}}{\tilde{v}}\right) \\ = \dfrac{4\pi G}{\bar{\tilde{v}}}\left(\dfrac{1}{L}\displaystyle\int_0^x y(\tilde{v} - \bar{\tilde{v}})dy + \dfrac{1}{L}\int_x^L (y - L)(\tilde{v} - \bar{\tilde{v}})dy\right) \end{cases}$$

From the first equation $\tilde{u}$ is a constant function, and must be zero because the mean value of $\tilde{u}$ vanishes. Differentiating the second equation and changing the unknown function $\tilde{v} \mapsto r = (\tilde{v}/V)^{-\gamma} - 1$, we obtain

$$r''(x) + \lambda\{1 - (1 + r(x))^{-1/\gamma}\} = 0, \quad r(x) > -1, \tag{2.1}$$

with $\lambda = 4\pi G V^\gamma / a$. The stationary problem is reduced to finding the L/n-periodic orbits of solutions to (2.1).

Note that the energy of an orbit:

$$E = \frac{1}{2}r'(x)^2 + \lambda\int_0^{r(x)} \{1 - (1+p)^{-1/\gamma}\}dp$$

admits the values between 0 and $\lambda/(\gamma - 1)$. The period of the orbit is given by $(2/\lambda)^{1/2} I_\gamma((E/\lambda)^{1/2})$, where

$$I_\gamma(\mu) = \mu\int_0^1 \frac{dy}{(1-y)^{1/2} f_+(p_+(\mu^2 y))} + \mu\int_0^1 \frac{dy}{(1-y)^{1/2} f_-(p_-(\mu^2 y))}.$$

The functions $f_\pm$ are given by $f_\pm(p) = \pm\{1 - (1 \pm p)^{-1/\gamma}\}$, and $p_\pm(z)$, $z \ge 0$, are the non-negative solutions of $\int_0^p f_\pm(\tilde{p})d\tilde{p} = z$. By calculating the derivative of I_γ we see that I_γ is strictly increasing on the interval $(0, (\gamma-1)^{-1/2})$. Hence for a positive integer n the equation (2.1) admits a solution with least period L/n if and only if

$$\left(\frac{2}{\lambda}\right)^{1/2} I_\gamma(+0) < \frac{L}{n} < \left(\frac{2}{\lambda}\right)^{1/2} I_\gamma((\gamma-1)^{-1/2} - 0).$$

Moreover, the orbit is uniquely determined by the period. By elementary calculus we have

$$I_\gamma(+0) = (2\gamma)^{1/2}\pi, \quad I_\gamma((\gamma-1)^{-1/2}-0)\begin{cases} =\infty, & \gamma = 1, \\ <\infty, & 1<\gamma<2. \end{cases}$$

Hence we obtain the assertion of the theorem with $c(\gamma) = (I_\gamma((\gamma-1)^{-1/2} - 0)/I_\gamma(+0))^2$. ∎

The following theorem refers to the exchange of stability of the trivial solutions at the first bifurcation point and the stability of stationary solutions arising at each bifurcation point.

Theorem 3. *Assume $V \neq \{(\pi a\gamma)/(GL^2)\}^{1/\gamma}$. The stability of stationary solutions of* (1.1) *in M_V is stated as follows.*
(i) *For $0 < V < \{(\pi a\gamma)/(GL^2)\}^{1/\gamma}$ the trivial solution $(V,0)$ is asymptotically stable in M_V; for $(v_0,u_0) \in M_V$, if $\|v_0 - V\|_{H^1} + \|u_0\|_{L^2}$ is sufficiently small, then the solution (v,u) of the Cauchy problem for* (1.1) *with initial data (v_0,u_0) tends to $(V,0)$ exponentially in $H^1_{\mathrm{per}} \times H^1_{\mathrm{per}}$ as $t \to \infty$.*
(ii) *For $\{(\pi a\gamma)/(GL^2)\}^{1/\gamma} < V < \{(\pi a\gamma c(\gamma))/(GL^2)\}^{1/\gamma}$ the set S_V of all stationary solutions in M_V with least period L is asymptotically stable in M_V; for $(v_0,u_0) \in M_V$, if $\min_{(\tilde v,0)\in S_V} \|v_0 - \tilde v\|_{H^1} + \|u_0\|_{L^2}$ is sufficiently small, then the solution (v,u) of the Cauchy problem for* (1.1) *with initial data (v_0,u_0) tends to a stationary solution $(\tilde v_\infty, 0) \in S_V$ exponentially in $H^1_{\mathrm{per}} \times H^1_{\mathrm{per}}$ as $t \to \infty$.*
(iii) *Let $(\tilde v,0)$ be either the trivial solution $(V,0)$ with $V > \{(\pi a\gamma)/(GL^2)\}^{1/\gamma}$ or a non-trivial stationary solution in M_V with period less than L. Then $(\tilde v,0)$ is unstable in M_V; there exists an open neighborhood $\mathcal{O}$ of $(\tilde v,0)$ in M_V such that in any small open neighborhood of $(\tilde v,0)$ in M_V we can find an initial data (v_0,u_0) for which the solution of the Cauchy problem for* (1.1) *eventually escapes from $\mathcal{O}$.*

In the Euler coordinates the relation $V > \{(\pi a\gamma)/(GL^2)\}^{1/\gamma}$ of the part (iii) is written as $l > \{(\pi a\gamma\bar\rho^{\gamma-1})/(G\bar\rho)\}^{1/2}$, where l is the spatial period of the flow and $\bar\rho$ the mean value of mass density. This implies that the period of the flow at which an infinite homogeneous gas has just got unstable is estimated by the mass density $\bar\rho$ and the sound speed $\{(d/d\rho)(a\rho^\gamma)\}^{1/2}|_{\rho=\bar\rho} = (a\gamma\bar\rho^{\gamma-1})^{1/2}$ free from any field of force and viscosity. This is in agreement with Jeans' criterion in astronomy. The viscosity disappears in the criterion but plays a substantial role in the stability of stationary solutions.

Several ideas to show the stability of stationary solutions will be given in the next section.

Theorem 3 refers only to the local dynamics near the set of stationary solutions. In the isothermal case ($\gamma = 1$) we can say more about the global dynamics.

In this case, since the forcing term in (1.1) has an upper bound independent of t, according to the argument of Matsumura and Nishida [10] we can show the boundedness in M_V of the orbit of each solution to the Cauchy problem. This enables us to take the ω-limit of the orbit in some topology weaker than that of H^1, e.g. the topology of L^∞:

$$\omega((v_0,u_0)) \equiv \bigcap_{s\geq 0} \overline{\{(v(t,\cdot),u(t,\cdot));t\geq s\}}^{L^\infty\times L^\infty} \neq \emptyset.$$

We are concerned with the relation between the ω-limit set and the set of stationary solutions lying in M_V.

Before stating the result we revisit the stationary problem from the viewpoint of the following energy of states: for $v\in H^1_{\mathrm{per}}$ with $v>0$, $\bar v = V$,

$$\mathcal{E}_V(v) = -a\int_0^L \log\frac{v}{V}dx - \frac{2\pi G}{V}\int_0^L K_L(v-V)\cdot(v-V)dx,$$

where K_L is the Green operator of $-\partial_x^2$ on the space of L-periodic functions with vanishing mean values. Note that the energy of the constant state $v\equiv V$ is zero. Let n be a positive integer. According to **Theorem 2**, for $(\pi a n^2)/(GL^2) < V \leq \{\pi a(n+1)^2\}/(GL^2)$ the set of all stationary solutions lying in M_V consists of that of the trivial solution:

$$S_V^{(\infty)} = \{(\tilde v_0^{(\infty)},0)\equiv (V,0)\}$$

and those of the non-trivial solutions with least period L/j, $j=1,\dots,n$:

$$S_V^{(j)} = \{(\tilde v_\alpha^{(j)},0); 0\leq\alpha<L/j\},$$

where $\tilde v_\alpha^{(j)} = \tilde v_0^{(j)}(\cdot-\alpha)$ with an element $(\tilde v_0^{(j)},0)\in S_V^{(j)}$. We then have the following:

Lemma.

$$\mathcal{E}_V(\tilde v_\alpha^{(j)}) = \mathcal{E}_V(\tilde v_0^{(j)}), \qquad 0\leq\alpha<L/j,\ \ j=1,\dots,n, \tag{2.2}$$

$$\mathcal{E}_V(\tilde v_0^{(j)}) < \mathcal{E}_V(\tilde v_0^{(k)}) \leq 0 = \mathcal{E}_V(\tilde v_0^{(\infty)}), \quad 1\leq j<k\leq\infty. \tag{2.3}$$

Sketch of Proof: By integration by parts

$$\begin{aligned}
&\partial_\alpha(\mathcal{E}_V(\tilde v_\alpha^{(j)}))\\
&= \int_0^L\left\{\frac{a}{\tilde v_\alpha^{(j)}} + \frac{4\pi G}{V}K_L(\tilde v_\alpha^{(j)}-V)\right\}\partial_x\tilde v_\alpha^{(j)}dx\\
&= -\int_0^L\left\{\partial_x\left(\frac{a}{\tilde v_\alpha^{(j)}}\right) - \frac{4\pi G}{V}\left(\frac{1}{L}\int_0^x y(\tilde v_\alpha^{(j)}-V)dy + \frac{1}{L}\int_x^L (y-L)(\tilde v_\alpha^{(j)}-V)dy\right)\right\}\tilde v_\alpha^{(j)}dx\\
&= 0.
\end{aligned}$$

This proves (2.2). To show (2.3) define a function on $(\{(\pi a)/(GV)\}^{1/2}, \infty)$ by

$$\varepsilon(l) = -a\int_0^l \log\frac{v_l}{V}dx - \frac{2\pi G}{V}\int_0^l K_l(v_l - V)\cdot(v_l - V)dx,$$

where $v_l = V(1+r_l)^{-1}$ and r_l is a solution of (2.1) with l-periodic orbit. Since

$$\mathcal{E}_V(\tilde{v}_0^{(j)}) = j\varepsilon(L/j),\ j = 1,\dots,n, \quad \varepsilon(\{(\pi a)/(GL)\}^{1/2} + 0) = 0,$$

(2.3) follows from $(d/dl)(\varepsilon(l)/l) < 0$, $l > \{(\pi a)/(GV)\}^{1/2}$. ∎

The following theorem shows that the asymptotic behavior of an orbit lying in M_V is almost under the control of the set of stationary solutions.

Theorem 4. *Assume $\gamma = 1$. For $(v_0, u_0) \in M_V$ we have:*
(i) *For $0 < V \le (\pi a)/(GL^2)$ we have $\omega((v_0, u_0)) = \{(V, 0)\}$. Moreover, the orbit starting from (v_0, u_0) tends to $(V, 0)$ in $H^1_{\text{per}} \times H^1_{\text{per}}$ as $t \to \infty$.*
(ii) *Let n be an integer. For $(n^2\pi a)/(GL^2) < V \le \{(n+1)^2\pi a\}/(GL^2)$ we have $\omega((v_0, u_0)) \subset S_V^{(j)}$ for some $j = 1, \dots, n, \infty$. If $j = 1$, then we have $\omega((v_0, u_0)) = \{(v_\alpha^{(1)}, 0)\}$ for some $\alpha \in [0, L)$. Moreover, the orbit starting from (v_0, u_0) tends to $(v_\alpha^{(1)}, 0)$ in $H^1_{\text{per}} \times H^1_{\text{per}}$ as $t \to \infty$.*

Sketch of Proof: We use the notations $\tilde{v}_0^\infty = V$, $S_V^{(\infty)} = \{(\tilde{v}_0^\infty, 0)\}$ for any $V > 0$. By Lemma 5.1 of [8] the set $\omega((v_0, u_0))$ consists of stationary solutions of (1.1) lying in M_V. Moreover, for $(\tilde{v}_\omega, 0) \in \omega((v_0, u_0))$ we have

$$\mathcal{E}_V(\tilde{v}_\omega) + \nu\int_0^\infty\int_0^L \frac{u_x^2}{v}dxdt = \frac{1}{2}\int_0^L u_0^2 dx + \mathcal{E}_V(v_0).$$

This implies that the value $\mathcal{E}_V(\tilde{v}_\omega)$ depends only on the initial value (v_0, u_0). Since the values $\mathcal{E}_V(\tilde{v}_0^{(k)})$, $k = 1, \dots, n, \infty$, are distinct, we can conclude that $\omega((v_0, u_0)) \subset S_V^{(j)}$ for some $j = 1, \dots, n, \infty$.

In both the cases (i) and (ii) $j = 1$ we can show the convergence of the orbit as in the proof of **Theorem 3** if we take a state on the orbit sufficiently close to the ω-limit set as a new initial state. ∎

3. Energy principle of stability. Energy equalities play an essential role in the stability analysis of stationary solutions. Let $(\tilde{v}, 0) \in M_V$ be a stationary solution of (1.1), and $(v, u) \in M_V$ a solution of (1.1) having the properties as in **Theorem 1**. Then we have

$$\frac{d}{dt}\tilde{E} = -\nu\int_0^L \frac{u_x^2}{v}dx$$

with

$$\tilde{E} = \int_0^L \Big\{ \frac{1}{2}u^2 + q(\tilde{v})(v-\tilde{v}) - \int_{\tilde{v}}^{v} q(z)dz \Big\} dx - \frac{2\pi G}{V} \int_0^L K_L(v-\tilde{v}) \cdot (v-\tilde{v}) dx,$$

where $q(z) = az^{-\gamma}$. This shows the decrease of the energy $\tilde{E}$ due to viscosity. Now consider an approximation of the energy:

$$\tilde{E} = \frac{1}{2}\int_0^L u^2 dx + \frac{1}{2}\int_0^L \Phi_{\tilde{v}}[v-\tilde{v}]dx + \text{h.o.t.}$$

with

$$\Phi_{\tilde{v}}[\varphi] = a\gamma \int_0^L \frac{\varphi^2}{\tilde{v}^{\gamma+1}} dx - \frac{4\pi G}{V}\int_0^L K_L\varphi \cdot \varphi dx \;\; (\varphi \in \mathcal{H} \equiv \{\varphi \in L^2_{\text{per}}; \bar{\varphi} = 0\}).$$

We can expect the decay of the difference $(v - \tilde{v}, u)$ if the quadratic form $\Phi_{\tilde{v}}$ is positive definite. According to the spectrum of the Green operator K_L on $\mathcal{H}$, in case of the trivial solution $(V, 0)$ the quadratic form is positive definite if and only if $V < \{(\pi a\gamma)/(GL^2)\}^{1/\gamma}$. The positivity of the quadratic form gets lost as the parameter V crosses the critical value $V = \{(\pi a\gamma)/(GL^2)\}^{1/\gamma}$. This gives rise to the bifurcation of stationary solutions together with the exchange of stability of the trivial solutions.

Unfortunately, for a stationary solution lying in the first bifurcation branch the positivity of the quadratic form fails to hold. To see this consider a selfadjoint operator on $\mathcal{H}$ associated with the quadratic form:

$$\Phi_{\tilde{v}}[\varphi] = \frac{a}{V^{\gamma+1}} \int_0^L \hat{F}_w(\lambda, \tilde{w})\varphi \cdot \varphi dx,$$

where $\lambda = 4\pi G V^{\gamma}/a$, $\tilde{w} = \tilde{v}/V - 1$, and $\hat{F}_w(\lambda, w)$ is given by extending onto $\mathcal{H}$ the Fréchet derivative of the following map on the space of L-periodic continuous functions with vanishing mean values:

$$F(\lambda, w) = -(1+w)^{-\gamma} + \frac{1}{L}\int_0^L (1+w)^{-\gamma} dx - \lambda K_L w.$$

Note that $(\tilde{v}, 0)$ is a stationary solution of (1.1) if and only if $F(\lambda, \tilde{w}) = 0$. When this is the case, we have $F(\lambda, \tilde{w}(\cdot - \alpha)) = 0$ for any α. Differentiating this equation in α and evaluating the derivative at $\alpha = 0$, we obtain

$$F_w(\lambda, \tilde{w})\tilde{w}' = 0.$$

For a non-trivial stationary solution $(\tilde{v}, 0)$ this implies that $F_w(\lambda, \tilde{w})$ as well as its extension has the eigenvalue 0 with eigenfunction $\tilde{w}'$. In fact, if $(\tilde{v}, 0)$ has the least period L, then zero is an isolated simple eigenvalue giving the lower bound of the selfadjoint operator $\hat{F}_w(\lambda, \tilde{w})$. The energy principle of stability dose not work well!

We can overcome the difficulty by modifying the expression of the energy. For a continuously differentiable function X on $[0, \infty)$ consider the following "time-dependent" energy:

$$\begin{aligned}\tilde{E}^X(t) = &\int_0^L \left\{\frac{1}{2}u^2 + q(\tilde{v}_{X(t)})(v - \tilde{v}_{X(t)}) - \int_{\tilde{v}_{X(t)}}^v q(z)dz\right\}dx \\ &- \frac{2\pi G}{V}\int_0^L K_L(v - \tilde{v}_{X(t)}) \cdot (v - \tilde{v}_{X(t)})dx,\end{aligned}$$

where $\tilde{v}_\alpha = \tilde{v}(\cdot - \alpha)$. Recalling that the null space of $\hat{F}_w(\lambda, \tilde{v}_{X(t)}/V - 1)$ contains the function $\tilde{v}'_{X(t)}$, by direct calculation we can verify the following energy equality:

$$\frac{d}{dt}\tilde{E}^X = -\nu\int_0^L \frac{u_x^2}{v}dx.$$

We can even show that $\tilde{E}^X = \tilde{E}$ for any function X as above. By approximating the energy $\tilde{E}^X$ as before, the decay of the difference $(v(t,\cdot) - \tilde{v}_{X(t)}, u(t,\cdot))$ is expected if we can choose the function X so that, at every time t, the quadratic form $\Phi_{\tilde{v}_{X(t)}}$ acts on $v(t,\cdot) - \tilde{v}_{X(t)}$ as if it were positive definite, that is to say, $v(t,\cdot) - \tilde{v}_{X(t)}$ is orthogonal to the null space of $\hat{F}_w(\lambda, \tilde{v}_{X(t)}/V - 1)$. Since $\tilde{v}'_{X(t)}$ constitutes a basis of the null space, the requirement on X is given by

$$\int_0^L (v(t,x) - \tilde{v}_{X(t)}(x))\tilde{v}'_{X(t)}(x)dx = 0, \quad t \geq 0. \tag{3.1}$$

To construct a function X satisfying (3.1) we solve the Cauchy problem of the following functional differential equation:

$$\frac{d}{dt}X(t) = \frac{\int_0^L u(t,x)\tilde{v}''_{X(t)}(x)dx}{\int_0^L \tilde{v}'(x)^2dx - \int_0^L(v(t,x) - \tilde{v}_{X(t)}(x))\tilde{v}''_{X(t)}(x)dx} \tag{3.2}$$

with initial condition $X(0) = x_0$ satisfying the orthogonality:

$$\int_0^L (v_0 - \tilde{v}_{x_0})\tilde{v}'_{x_0}dx = 0.$$

This equation is derived from (3.1) together with the equation of continuity $\partial_t v - \partial_x u = 0$. The global solvability of the Cauchy problem, in general, needs the smallness of the difference $v(t,\cdot) - \tilde{v}_{X(t)}$ in the divisor on the right-hand side of (3.2). This follows from the decrease of the energy $\tilde{E}^X$ if we choose the initial value x_0 so that the difference $v_0 - \tilde{v}_{x_0}$ is small enough.

In order to determine the limit of (v, u) we first establish the exponential decay of the norm of u in L^2_{per}. The equation (3.2) then implies the exponential decay of the derivative dX/dt. This ensures the finite limit $x_\infty = \lim_{t\to\infty} X(t)$. Finally, the limit of (v, u) is given by $(\tilde{v}_{x_\infty}, 0)$.

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