数理解析研究所講究録 1425 巻 2005 年 21-39

Rotating Navier-Stokes Equations with Initial Data Nondecreasing at Infinity

乾 勝也 (Katsuya Inui)　北海道大学大学院 理学研究科数学専攻 博士課程 3 年
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

Abstract

This is a supplementary note of the paper [12] by Yoshikazu Giga, Alex Mahalov, Shin'ya Matsui and me. In [12] local-in-time unique existence of strong solutions was obtained for the rotating Navier-Stokes equations in $\mathbb{R}^3$ for a class of initial data that contains some nondecreasing functions at space infinity. The rotating Navier-Stokes equations has the Coriolis term of the form $e_3 \times u$, where e_3 denotes the vertical unit vector. The Coriolis solution operator is estimated uniformly in the Coriolis parameter $\Omega \in \mathbb{R}$, using its skew-symmetry. Then it is shown that local existence time estimate for the rotating Navier-Stokes equations is uniform in $\Omega \in \mathbb{R}$.

1 Introduction

We consider the rotating Navier-Stokes equations in $\mathbb{R}^3$:

$$\text{(RNS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = -\Omega e_3 \times u & \text{for } x \in \mathbb{R}^3, \quad 0 < t < T, \\ \operatorname{div} u = 0 & \text{for } x \in \mathbb{R}^3, \quad 0 < t < T, \\ u|_{t=0} = u_0 & \text{for } x \in \mathbb{R}^3, \end{cases}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ is the unknown velocity vector field and $p = p(x,t)$ is the unknown scalar pressure field, while $u_0 = u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x))$ is the given initial velocity vector field. Besides, $T > 0$, $\Omega \in \mathbb{R}$ is a scalar fixed constant, $e_3 = (0,0,1)$, and $\times$ represents the outer product, hence, $-\Omega e_3 \times u = (\Omega u^2, -\Omega u^1, 0)$.
The equations (RNS) are the Navier-Stokes equations with the term $-\Omega e_3 \times u$. The constant Ω is called the Coliolis parameter and the term $-\Omega e_3 \times u$ is called the Coliolis term, which represents the Coliolis force when the fluid is rotating with angular velocity $\Omega/2$ around x_3-axis. The Coliolis term has an another expression:

$$-\Omega e_3 \times u = -\Omega \mathrm{J} u,$$

with the skew-symmetric matrix J defined by

$$\mathrm{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For (RNS) in the case of periodic and cylindrical domains, Babin-Mahalov-Nicolaenko [3] and Mahalov-Nicolaenko [19] proved local existence and uniqueness of solutions uniformly in the

Coriolis parameter Ω. Moreover, they proved global in time regularity of solutions when Ω is sufficiently large. The method of proving global regularity for large fixed Ω is based on the analysis of fast singular oscillating limits (singular limit $\Omega \to +\infty$), nonlinear averaging and cancellation of oscillations in the nonlinear interactions for the vorticity field. It uses harmonic analysis tools of lemmas on restricted convolutions and Littlewood-Paley dyadic decomposition to prove global regularity of the limit resonant three-dimensional Navier-Stokes equations which holds without any restriction on the size of initial data and strong convergence theorems for large Ω.

Our aim is to prove local existence with its existence time estimate uniformly in $\Omega \in \mathbb{R}$ for nondecreasing initial data u_0 at space inifinity. For this purpose we transpose the Coriolis term $-\Omega e_3 \times u = -\Omega J u$ to rewrite (RNS) in the form

$$\begin{cases} u_t - \Delta u + \Omega \mathbf{J} u + (u, \nabla) u + \nabla p = 0 & \text{for } x \in \mathbb{R}^3,\ \ 0 < t < T, \\ \operatorname{div} u = 0 & \text{for } x \in \mathbb{R}^3,\ \ 0 < t < T, \\ u|_{t=0} = u_0 & \text{for } x \in \mathbb{R}^3, \end{cases}$$

so that the Coliolis term is dealt with the diffusion term Δu as a linear problem. Then we multiply the Helmholtz operator $\mathbf{P} = (\delta_{i,j} + R_i R_j)_{i,j},\ 1 \le i, j \le 3$ formally to get the abstract ordinary differential equation

$$\text{(A)} \qquad u_t - \Delta u + \Omega \mathbf{P}\mathbf{J} u + \mathbf{P}(u, \nabla) u = 0 \qquad \text{for } t > 0.$$

Here, $\delta_{i,j}$ is the Kronecker delta and R_j is the scalar Riesz operator whose symbol is $i\xi_j/|\xi|$. To get (A) we used the fact that

$$\mathbf{P} u = u \quad \text{for divergence free vector field } u \tag{1.1}$$

and that $\mathbf{P}\Delta = \Delta \mathbf{P}$.

However, instead of (A), we consider the following equation:

$$\text{(ABS)} \qquad u_t - \Delta u + \Omega \mathbf{P}\mathbf{J}\mathbf{P} u + \mathbf{P}(u, \nabla) u = 0 \qquad \text{for } t > 0,$$

which is equivalent to (A) because $\mathbf{PJ}u = \mathbf{PJP}u$ if $\operatorname{div} u = 0$ by (1.1).

The corresponding integral equation to (ABS) is written as:

$$\text{(I)} \qquad u(t) = \exp(-\mathbf{A}(\Omega)t) u_0 - \int_0^t \exp(-\mathbf{A}(\Omega)(t-s)) \mathrm{P}\operatorname{div}(u \otimes u)(s)\, ds \quad \text{for } t > 0,$$

where $\mathbf{A}(\Omega) = -\Delta + \Omega \mathbf{PJP}$. Hence, $\exp(-\mathbf{A}(\Omega)t)$, the exponential of the operator $-\mathbf{A}(\Omega)t$, is represented by

$$\exp(-\mathbf{A}(\Omega)t) = \exp(t\Delta) \exp(-\Omega \mathbf{PJP} t) \tag{1.2}$$

and can be called the 'Heat+Coliolis' solution operator.

In the case $\Omega = 0$, that is, on the Navier-Stokes equations (NS) without the Coriolis term, unique local existence of mild solution was proved if initial data u_0 belongs to L^∞_σ, the space of bounded solenoidal functions, in Cannon-Knightly [6], Cannone [7] and Giga-Inui-Matsui [11].

Of cource, the space L^∞_σ contains nondecreasing functions. There are several related works for L^∞ initial data [7],[18]. The method in [11] is to use estimate for the derivative of the heat kernel in the Hardy space $\mathcal{H}^1$ obtained by Carpio [8]. For (NS) with initial data L^∞_σ, Giga-Matsui-Sawada [13] obtained unique global existence of strong solution $u \in L^\infty_\sigma$ in the 2-dimensional case and J. Kato [17] proved uniqueness of weak solution $(u, \nabla p)$ when $u \in L^\infty$ and $p \in BMO$ in the n-dimensional case with $n \geq 2$ (see also [14]). Here, BMO is the space of functions of bounded mean oscillations.

In the case $\Omega \neq 0$, that is, rotating case, the crucial step is to estimate the Coriolis solution operator $\exp(-\Omega \mathbf{PJ}t)$ that comes from the Coriolis term $\mathbf{PJ}u = (-R_1R_1u^2 + R_1R_2u^1, -R_2R_1u^2 + R_2R_2u^1, -R_3R_1u^2 + R_3R_2u^1)$. The difficulty is that the term contains the Riesz operator R_j which is not bounded in L^∞. Moreover, Carpio's estimate does not apply to the term since it has no derivatives.

Hieber-Sawada [15] and Sawada [20] constructed a local solution for (RNS) with generalized Corilolis term $\mathbf{M}u$ with 3×3 matrix $\mathbf{M}$ whose trace is zero for the solenoidal initial data $u_0 \in \dot{B}^0_{\infty,1}$. Here, $\dot{B}^0_{\infty,1}$ is a homogeneous Besov space including various periodic and almost periodic functions, that do not decay at space infinity. The space $\dot{B}^0_{\infty,1}$, which is a subspace of L^∞, was first used to solve Boussinesq equations by Sawada-Taniuchi [21] (see Taniuchi [22] for recent improvement). The advantage of the Besov space is boundedness of the Riesz operator in it. They are successful in estimating the Coriolis term in the Besov space.

However, their existence time estimate depends on Ω, since the equations (RNS) were transformed to the integral equation

$$u(t) = \exp(t\Delta)u_0 - \int_0^t \exp((t-s)\Delta)\mathbf{P}\{\operatorname{div}(u \otimes u)(s) + \Omega e_3 \times u(s)\}\, ds \quad \text{for } t > 0$$

to regard the Coriolis term as a perturbation. In this paper, we transformed (RNS) into (I) to estimate the linear "Heat+Coriolis" term uniformly in the Coriolis parameter Ω by using skew-symmetric structure of the operator $\mathbf{PJP}$. That is the reason that we deal rather the equation (ABS) instead of (A). We estimate the Coliolis solution operator in the form $\exp(-\Omega \mathbf{PJP}t)$ as in (1.2) instead of the form $\exp(-\Omega \mathbf{PJ}t)$. Smallness of the Coriolis term is not assumed. This is a major difference between our and their approach.

In the integral equation (I), the unboundedness problem in L^∞ arises again in the linear term. Since the Coriolis solution operator $\exp(-\Omega \mathbf{PJP}t)$ contains the Riesz transforms, one cannot expect its boundedness in L^∞. There was still a possibility that the "Heat+Coriolis" operator $\exp(t\Delta)\exp(-\Omega \mathbf{PJP}t)$ is bounded in L^∞, even if $\exp(-\Omega \mathbf{PJP}t)$ is unbounded in L^∞. Unfortunately, our exact calculation of the symbol arrived at conclusion that the solution operator is not bounded in L^∞ (see [[12] ; Appendix A]).

In this situation we are forced to restrict initial data to a subspace of L^∞_σ. To introduce our new subspace we split initial data into 2D3C (2 dimensional 3 components) vector field part and x_3–dependent part by taking vertical average.

Definition 1.1. (Vertical averaging)
Let $u \in L^\infty_\sigma(\mathbb{R}^3)$. We say that u admits vertical averaging if

$$\lim_{L \to +\infty} \frac{1}{2L} \int_{-L}^{L} u(x_1, x_2, x_3) dx_3 \equiv \overline{u}(x_1, x_2)$$

exists almost everywhere. The vector field $\overline{u}(x_1, x_2)$ is called *vertical average* of $u(x_1, x_2, x_3)$.

Definition 1.2. (Space for initial data) We define a subspace of L^∞_σ of the form

$$L^\infty_{\sigma,a} = L^\infty_{\sigma,a}(\mathbb{R}^3) = \{u \in L^\infty_\sigma(\mathbb{R}^3);\ u \text{ admits vertical averaging and } u^\perp \in \dot{B}^0_{\infty,1}\}.$$

Here $\dot{B}^0_{\infty,1}$ is a homogeneous Besov space (see subsection 3.2 on details of its definition and properties). The space $L^\infty_{\sigma,a}(\mathbb{R}^3)$ is a Banach space with the norm

$$||u||_{L^\infty_{\sigma,a}} = ||\overline{u}||_{L^\infty(\mathbb{R}^2;\mathbb{R}^3)} + ||u^\perp||_{\dot{B}^0_{\infty,1}(\mathbb{R}^3;\mathbb{R}^3)}.$$

Now we introduce theorems obtained in [12].

Theorem 1.1. (Existence and uniqueness of mild solution u)
Suppose that $u_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. Then
(1) *There exist $T_0 > 0$ independent of Ω and a unique solution $u = u(t)$ of* (I) *such that*

$$u \in C([\delta, T_0]; L^\infty_\sigma) \cap C_w([0, T_0]; L^\infty_\sigma) \qquad \textit{for any } \delta > 0. \tag{1.3}$$

(2) *The solution u satisfies*

$$\sup_{t \in (0,T_0)} ||t^{1/2}\nabla u||_{L^\infty_\sigma} < \infty \qquad \textit{and} \qquad \nabla u \in C([\delta, T_0]; L^\infty_\sigma) \qquad \textit{for any } \delta > 0. \tag{1.4}$$

Remark 1.1. (i) For a lower estimate for $T_0 > 0$ we get

$$T_0 \geq C/||u_0||^2_{L^\infty_{\sigma,a}}$$

with C independent of Ω.
(ii) If in addition we assume that $\overline{u_0} \in BUC$, then the solution $u \in C([0, T_0]; BUC)$. Here, BUC denotes the space of all bounded uniformly continuous functions in $\mathbb{R}^3$.
(iii) Let $u_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$ be uniformly continuous. Then the solution u of (I) obtained in Theorem 1.1 satisfies

$$\lim_{t \downarrow 0} t^{1/2} ||\nabla u(t)||_{L^\infty(\mathbb{R}^3)} = 0.$$

Theorem 1.2. (Existence of classical solution u)
Suppose that $u_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. Let $u = u(t)$ be a solution of (I) *satisfying (1.3) and (1.4). If we set*

$$\nabla p(t) = \nabla \sum_{j,k=1}^{3} R_j R_k u^j u^k(t) - \Omega \begin{pmatrix} R_1\ (R_2 u^1 - R_1 u^2) \\ R_2\ (R_2 u^1 - R_1 u^2) \\ R_3\ (R_2 u^1 - R_1 u^2) \end{pmatrix} \qquad \textit{for } t > 0, \tag{1.5}$$

then the pair $(u, \nabla p)$ is a classical solution of (RNS).

Such a solution (satisfying (1.3)-(1.5)) is unique. In fact a stronger version is available.

Theorem 1.3. (Uniqueness of classical solution u)
Suppose that $u_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. Let

$$u \in L^\infty((0,T)\times\mathbb{R}^3), \quad p \in L^1_{loc}([0,T); BMO)$$

be a solution of (RNS) *in a distributional sense for some $T>0$. Then the pair $(u,\nabla p)$ is unique. Furthermore, the relation (1.5) holds.*

Remark 1.2. (i) The space $L^\infty_{\sigma,a}$ has a topological direct sum decomposition of the form $L^\infty_{\sigma,a} = \mathcal{W}\oplus\mathcal{B}^0$, where

$$\mathcal{W} = \{f \in L^\infty_\sigma;\ \partial f^i/\partial x_3 \equiv 0 \quad \text{in distributional sense } \mathbb{R}^3 \text{ for } i=1,2,3\},$$
$$\mathcal{B}^0 = \{f \in \dot{B}^0_{\infty,1}\cap L^\infty_\sigma;\ \overline{f}(x_1,x_2)\equiv 0 \quad \text{a.e. } (x_1,x_2)\in\mathbb{R}^2\}.$$

(ii) Existence of vertical average of initial data is not needed for the thorems, but the following representation is needed:

$$u_0 = \phi(x_1,x_2) + \psi(x_1,x_2,x_3) \tag{1.6}$$

with $\phi \in \mathcal{W}$ and $\psi \in \mathcal{B}^0$, that is, u_0 belongs to the space $\mathcal{W}+\mathcal{B}^0$, which is larger than $L^\infty_{\sigma,a} = \mathcal{W}\oplus\mathcal{B}^0$ (see Remark 3.4).

This manuscript is organized as follows. In section 2, 3 and 4, we give a brief sketch of the proof of the theorems for readers' convenience although it is given in [12]. In section 2 and 3, we estimate the nonlinear term and the linear term of the integral equation (I), respectively. In section 4, we introduce Mikhlin-type theorem in the Hardy space and a homogeneous Besov space, which is crucial for uniform boundedness of the Coriolis solution operator.
In section 5, we show Remark 1.1(ii) and (iii). In [12], detailed proof of Remark 1.1(ii) is not written and the assertion (iii) is not mentioned.

2 Estimate for nonlinear term

In this section we prepare estimate for the nonlinear term of the equation (I) using an estimate for derivative of the heat kernel in the Hardy space $\mathcal{H}^1$ obtained by Carpio.

Lemma 2.1 ([8]). *Let $G_t = G_t(x)$ be the heat kernel $(4\pi t)^{-n/2}\exp(\frac{-|x|^2}{4t})$ for $t>0$. Then there exists a constant $C>0$ (depending only on space dimension n) that satisfies*

$$||\nabla G_t||_{\mathcal{H}^1(\mathbb{R}^n)} \le Ct^{-1/2} \quad \textit{for } t>0.$$

Since it is well known that the dual space of the Hardy space $\mathcal{H}^1$ is BMO, the space of functions of bounded mean oscillations, we immediately have

Lemma 2.2. *There exists a constant $C>0$ (depending only on space dimensions) that satisfies*

$$||\nabla e^{t\Delta} f||_{L^\infty} \le Ct^{-1/2}||f||_{BMO} \quad \textit{for } t>0, f\in BMO.$$

By the above two lemmas and Corollary 3.1, which will be given later, we get the following estimtes for the nonlinear term.

Proposition 2.1. (Estimates for the nonlinear term)
There exists a constant C (independent of Ω, t and f) that satisfies

$$||\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\mathrm{div}(f\otimes f)||_{L^\infty} \le Ct^{-1/2}||f||^2_{L^\infty}, \quad t>0, \quad \textit{and}$$
$$||\nabla\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\mathrm{div}(f\otimes f)||_{L^\infty} \le Ct^{-1/2}||\nabla f||_{L^\infty}||f||_{L^\infty}, \quad t>0$$

for all $f \in L^\infty$ with $\nabla f \in L^\infty$.

Proof. The proof is given in [12](Lemma 4.3) using symbol calculation of the operators, however, here we give proof again without symbol expression. For the first statement we have

$$\begin{aligned}
&||\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\mathrm{div}F||_{L^\infty}\\
&\le ||\nabla\exp(t\Delta)||_{BMO\to L^\infty}||\exp(-\Omega t\mathbf{PJP})||_{BMO\to BMO}||\mathbf{P}||_{BMO\to BMO}||f\otimes f||_{BMO}\\
&\le Ct^{-1/2}||f\otimes f||_{BMO}\\
&\le Ct^{-1/2}||f\otimes f||_{L^\infty} \le Ct^{-1/2}||f||^2_{L^\infty}.
\end{aligned}$$

Here, in the second inequality we used Lemma 2.2, Proposition 3.2 and the boundedness of the operator $\mathbf{P}$ in BMO since the Riesz transform is bounded in BMO. In the third inequality we also used the embedding $L^\infty \hookrightarrow BMO$. For the second assertion one sees similarly

$$\begin{aligned}
&||\nabla\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\mathrm{div}F||_{L^\infty}\\
&\le ||\nabla\exp(t\Delta)||_{BMO\to L^\infty}||\exp(-\Omega t\mathbf{PJP})||_{BMO\to BMO}||\mathbf{P}||_{BMO\to BMO}||\mathrm{div}(f\otimes f)||_{BMO}\\
&\le Ct^{-1/2}||\mathrm{div}(f\otimes f)||_{BMO}\\
&\le Ct^{-1/2}||\mathrm{div}(f\otimes f)||_{L^\infty} \le Ct^{-1/2}||\nabla f||_{L^\infty}||f||_{L^\infty}.
\end{aligned}$$

□

3 Estimate for linear term

In this section we show boundedness of the solution operator for the linearlized equation for nondecreasing initial data. By virtue of skew-symmetry of the operator $\mathbf{PJP}$, that we use instead of $\mathbf{PJ}$, boundedness problem of $\exp(-\Omega\mathbf{PJP}t)$ is reduced to boundedness of $\exp(\omega R_3)$ for some $\omega \in \mathbb{R}$. By $\sigma(T)$ we denote the symbol of a operator T.

3.1 Poincaré-Sobolev equations

The linealized equations of the Rotating Navier-Stokes equations is called the Poincaré-Sobolev equations and has the form:

$$(\mathrm{PS})\begin{cases} u_t - \Delta u + \Omega\mathrm{J}u + \nabla p = 0 & \text{for } x\in\mathbb{R}^3,\ \ 0<t<T,\\ \mathrm{div}\ u = 0 & \text{for } x\in\mathbb{R}^3,\ \ 0<t<T,\\ u|_{t=0} = u_0 & \text{for } x\in\mathbb{R}^3.\end{cases}$$

Multiplying the Helmholtz operator $\mathbf{P}$, the equations (PS) are transformed into

$$u_t - \Delta u + \Omega \mathbf{PJ} u = 0 \qquad \text{for } t > 0, \qquad u|_{t=0} = u_0. \tag{3.1}$$

Instead of (3.1), as mentioned in introduction we deal rather

$$u_t - \Delta u + \Omega \mathbf{PJP} u = 0 \qquad \text{for } t > 0, \qquad u|_{t=0} = u_0, \tag{3.2}$$

whose solution operator is expressed by (1.2).

Before calculating the symbol of the solution operator $\exp(-\Omega t \mathbf{PJP})$, we define the operator $\mathbf{R}$ by

$$\sigma(\mathbf{R}) \equiv \mathbf{R}(\xi) = \begin{pmatrix} 0 & -\frac{\xi_3}{|\xi|} & \frac{\xi_2}{|\xi|} \\ \frac{\xi_3}{|\xi|} & 0 & -\frac{\xi_1}{|\xi|} \\ -\frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & 0 \end{pmatrix}. \tag{3.3}$$

We note that the symbol $\mathbf{R}(\xi)$ is a 3×3 skew-symmetric matrix. Since the operator $\mathbf{R}$ has the property

$$\mathbf{R}^2 = -\mathbf{I} \qquad \text{in divergence free vector fields,} \tag{3.4}$$

we call $\mathbf{R}$ the vector Riesz operator. Here, $\mathbf{I}$ denotes the identity operator.

Simple matrix multiplication and (3.4) give the following expression of the operator $\mathbf{PJP}$.

Lemma 3.1 ([4]). **(Symbol of the operator PJP)**
(1) *We have*

$$\sigma(\mathbf{PJP}) = \frac{\xi_3}{|\xi|} \begin{pmatrix} 0 & -\frac{\xi_3}{|\xi|} & \frac{\xi_2}{|\xi|} \\ \frac{\xi_3}{|\xi|} & 0 & -\frac{\xi_1}{|\xi|} \\ -\frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & 0 \end{pmatrix} \left(= \frac{\xi_3}{|\xi|}\mathbf{R}(\xi)\right). \tag{3.5}$$

(2) *In particular, in divergence free vector fields*

$$\sigma((\mathbf{PJP})^2) = -\frac{\xi_3^2}{|\xi|^2}\mathbf{I} \left(= (i\frac{\xi_3}{|\xi|})(i\frac{\xi_3}{|\xi|})\mathbf{I} \right), \quad \textit{i.e.,} \quad (\mathbf{PJP})^2 = R_3^2 \mathbf{I}. \tag{3.6}$$

Remark 3.1. The matrix $\sigma(\mathbf{PJP})$ is a 3×3 skew-symmetric matrix. This fact is key in the argument of the subsection 3.3.

By (3.5) and (3.6) we can calculate the symbol of the exponential of the operator $\mathbf{PJP}$ defined by

$$\exp(-\Omega t \mathbf{PJP}) = \sum_{j=0}^{\infty} \frac{(-\Omega t)^j}{j!} (\mathbf{PJP})^j$$

to get

Proposition 3.1 ([4]). **(Symbol of the operator** $\exp(-\Omega t \mathbf{PJP})$**)** *There holds*

$$\exp(-\Omega t \mathbf{PJP}) = \cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} - \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi) \qquad \text{for } t > 0. \tag{3.7}$$

3.2 Homogeneous Besov spaces

In order to estimate the linear term $\exp(t\Delta)\exp(-\Omega\mathbf{PJP}t)u_0$ in L^∞, the difficulty is that the Coriolis solution operator $\exp(-\Omega\mathbf{PJP}t)$ contains the Riesz operator that is not bounded in L^∞. Moreover, Carpio's estimate does not apply the linear term since it has no derivatives.
It is possible that the "Heat+Coriolis" solution operator is bounded in L^∞_σ even if the Coriolis solution operator is not bounded in L^∞_σ. However, the calculation of the kernel $K(x)$ (see Appendix A in [12]), that is, the function $K(x)$, defined by the identity

$$\exp(t\Delta)\exp(-\Omega\mathbf{PJP}t)f = F^{-1}\left(e^{-t|\xi|^2}\cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} - e^{-t|\xi|^2}\sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi)\right) * f =: K * f,$$

turned out to have the asymptotic behavior

$$K(x) \sim C\frac{1}{|x|^3} \qquad \text{for large } |x|.$$

The corresponding integral operator cannot be viewed as a bounded operator in $L^\infty(\mathbb{R}^3)$ since a characteristic function of the outside of a large ball is always mapped to ∞ by this operator.
In this situation we are forced to restrict initial data to a subspace of L^∞_σ, in which, the Coriolis solution operator (in particular, the Riesz transform) is bounded. We follow the idea to use a homogeneous Besov space $\dot{B}^0_{\infty,1}$, that was first used to solve Boussinesq equations by Sawada-Taniuchi [21].
Before introducing the homogeneous Besov spaces, we prepare some notations. By $\mathcal{S}$ we denote the class of rapidly decreasing functions. The dual of $\mathcal{S}$, the space of tempered distributions is denoted by $\mathcal{S}'$. Let $\{\phi_j\}_{j=-\infty}^\infty$ be the Littlewood-Paley dyadic decomposition satisfying

$$\widehat{\phi_j}(\xi) = \widehat{\phi_0}(2^{-j}\xi) \in C_c^\infty(\mathbb{R}^n), \quad \mathrm{supp}\widehat{\phi_0} \subset \{1/2 < |\xi| < 2\}, \quad \sum_{j=-\infty}^{\infty} \widehat{\phi_j}(\xi) = 1 \quad (\xi \neq 0). \tag{3.8}$$

Definition 3.1. (See, e.g. [5] page 146)
The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for $n \in \mathbb{N}$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^n) := \{f \in \mathcal{Z}'; ||f; \dot{B}^s_{p,q}|| < \infty\}$$

for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, where

$$||f||_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \begin{cases} \left[\sum_{j=-\infty}^\infty 2^{jsq}||\phi_j * f; L^p(\mathbb{R}^n)||^q\right]^{1/q} & \text{if} \quad q < \infty, \\ \sup_{-\infty \le j \le \infty} 2^{js}||\phi_j * f; L^p(\mathbb{R}^n)|| & \text{if} \quad q = \infty. \end{cases}$$

Here $\mathcal{Z}'$ is the topological dual space of the space $\mathcal{Z}$, which is defined by $\mathcal{Z} \equiv \{f \in \mathcal{S};\ D^\alpha \hat{f}(0) = 0 \text{ for all multi-indices } \alpha = (\alpha_1, \dots, \alpha_n)\}$.

The above definition yields that all polynomials vanish in $\dot{B}^s_{p,q}(\mathbb{R}^n)$, however, it is well known that

$$\dot{B}^s_{p,q}(\mathbb{R}^n) \cong \{f \in \mathcal{S}';\ ||f||_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty \text{ and } f = \sum_{j=-\infty}^{\infty} \phi_j * f \text{ in } \mathcal{S}'\} \tag{3.9}$$

$$\text{if} \quad s < n/p \quad \text{or} \quad (s = n/p \text{ and } q = 1). \tag{3.10}$$

Since indices of our target space $\dot{B}^0_{\infty,1}(\mathbb{R}^n)$ satisfy (3.10), the space $\dot{B}^s_{p,q}$ can be regarded as (3.9). It is known that the inclusion $\dot{B}^0_{\infty,1}(\mathbb{R}^n) \subset BUC(\mathbb{R}^n)$ and the embedding $\dot{B}^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ hold. For the details and examples one can consult e.g. [20],[21],[22].

3.3 Uniform estimate of the Coriolis solution operator

In this subsection we show boundedness of the Coriolis solution operator $\exp(-\Omega t\mathbf{PJP})$ in BMO and the Besov space $\dot{B}^0_{\infty,1}$ defined in the previous subsection uniformly in $\Omega \in \mathbb{R}$ and $t > 0$. For the purpose it is sufficient to show boundedness of the operator of the form $\exp(\omega R_3)$ uniformly in $\omega \in \mathbb{R}$. In fact, noting that $\cos x = (\exp(ix) + \exp(-ix))/2$, we see

$$\sigma(\cos(\frac{\xi_3}{|\xi|}\Omega t)) = \sigma(\cos(-ii\frac{\xi_3}{|\xi|}\Omega t)) = \cos(-iR_3\Omega t) = \frac{1}{2}\{\exp(\Omega t R_3) + \exp(-\Omega t R_3)\}$$

and similarly

$$\sigma(\sin(\frac{\xi_3}{|\xi|}\Omega t)) = \frac{1}{2i}\{\exp(\Omega t R_3) - \exp(-\Omega t R_3)\}.$$

Besides, the vector Riesz operator $\mathbf{R}$ appeared in the symbol (3.7) of $\exp(-\Omega t\mathbf{PJP})$ is bounded in $\dot{B}^0_{\infty,1}$ and BMO.

The boundedness in the one space $\dot{B}^0_{\infty,1}$ is sufficient to get unique local existence, however, we could obtain boundedness in $\dot{B}^0_{\infty,q}$ for all $1 \le q \le \infty$ as follows;

Proposition 3.2. (Uniform boundedness of the operator $\exp(\omega R_j)$) *Let $X = \dot{B}^0_{\infty,q}$ for $1 \le q \le \infty$ and BMO. Then there holds*

$$||\exp(\omega R_j)f||_X \le ||f||_X$$

for $f \in X$, $\omega \in \mathbb{R}$ and $j = 1, 2, 3$.

Remark 3.2. (i) The uniform boundedness in BMO is used in Proposition 2.1 to get uniform estimate of the nonlinear term. The uniform boundedness in $\dot{B}^0_{\infty,1}$ is used to estimate the linear term.

(ii) The boundedness in $\dot{B}^0_{\infty,\infty}$ (i.e., $q = \infty$) is used in the proof of the regularity result, Remark 1.1 (ii) (see section 5).

Proof. By spectrum mapping theorem we have for $j = 1, 2, 3$

$$\begin{aligned} ||\exp(\omega R_j)||_{X\to X} &= \sup\{|z|; z \in \operatorname{Spec}(\exp(\omega R_j))\} \\ &= \sup\{|z|; z \in \exp(-i\omega \operatorname{Spec}(iR_j))\} \\ &= \sup\{|\exp(-i\omega z)|; z \in \operatorname{Spec}(iR_j)\}. \end{aligned}$$

Here, $\operatorname{Spec}(T)$ denotes the spectrum set of an operator T. Now consider the resolvent operator of iR_j, that is, $(\lambda - iR_j)^{-1}$ for $\lambda \in \mathbb{C}$. Since its symbol $m(\xi) = 1/(\lambda + \frac{\xi_j}{|\xi|})$ satisfies the assumption of Mikhlin-type theorem (4.1) if λ is not real, it follows that $\operatorname{Spec}(iR_j) \subset \mathbb{R}$, which gives

$$||\exp(\omega R_j)||_{X\to X} \le \sup\{|\exp(-i\omega z)|; z \in \mathbb{R}\} = 1$$

since $|\exp(-i\omega z)| = 1$ when $z \in \mathbb{R}$. □

Corollary 3.1. *Let $X = \dot{B}^0_{\infty,q}$ for $1 \le q \le \infty$ and BMO. There exists a constant $C > 0$ independent of Ω and t such that*
(1) $||\exp(-\Omega t\mathbf{PJP})f||_X \le C||f||_X$ *for* $t > 0,\ f \in X$,
(2) $||\exp(-\mathbf{A}(\Omega)t)f||_{L^\infty} \le C||f||_{\dot{B}^0_{\infty,1}}$ *for* $t > 0,\ f \in \dot{B}^0_{\infty,1}$.

Proof. The statement (1) is obvious from Proposition 3.2 and and the argument in the beginning of this subsection. For (2) one sees from $||G_t||_1 = 1$, $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$ and (1) that

$$\begin{aligned} ||\exp(-\mathbf{A}(\Omega)t)f||_{L^\infty} &= ||\exp(t\Delta)\exp(-\Omega t\mathbf{PJP})f||_{L^\infty} \\ &\le ||\exp(-\Omega t\mathbf{PJP})f||_{L^\infty} \\ &\le ||\exp(-\Omega t\mathbf{PJP})f||_{\dot{B}^0_{\infty,1}} \\ &\le C||f||_{\dot{B}^0_{\infty,1}}. \end{aligned}$$

□

3.4 Vertical average

By combining Corollary 3.1(2) and the nonlinear estimate Proposition 2.1 at least for initial data $u_0 \in \dot{B}^0_{\infty,1}$ with $\mathrm{div}u_0 = 0$ local-in-time existence of (RNS) is guaranteed with its existence time estimate is uniform in Ω.
However, we can see the following property of the Corilois solution operator:

Remark 3.3. Let f be a 2D3C(2-dimensional 3-components) vector field, that is,

$$f = (f^1(x_1,x_2), f^2(x_1,x_2), f^3(x_1,x_2)),$$

Then,

$$\exp(-\Omega t\mathbf{PJP})f = f \quad \text{for } t > 0.$$

In fact, the symbol matrix of the operator $\mathbf{PJP}$, (3.5), has a ξ_3 in all elements, hence, $\mathbf{PJP}$ has ∂/∂_{x_3} in all components. Then there holds $\mathbf{PJP}f = 0$ for a 2D3C vector field f. Hence its exponential operator becomes the identity operator, i.e., $\exp(-\Omega t\mathbf{PJP})f = f$ for a 2D3C vector field f.
If we care about the structure of the operator $\mathbf{PJP}$, the class $L^\infty_{\sigma,a}$, which was defined in introduction, is allowed for local-in-time existence for initial data.

Proposition 3.3. *There exists a constant $C > 0$ independent of Ω such that*

$$||\exp(-\mathbf{A}(\Omega)t)f||_{L^\infty} \le C||f||_{L^\infty_{\sigma,a}}, \quad \text{for } t > 0,\ f \in L^\infty_{\sigma,a}.$$

Proof. Since by Remark 3.3 we see for $f \in L^\infty_{\sigma,a}$ that

$$\exp(-\mathbf{A}(\Omega)t)f = \exp(t\Delta)\overline{u_0} + \exp(t\Delta)\exp(-\Omega t\mathbf{PJP})u_0^\perp,$$

One has

$$\begin{aligned}
\|\exp(-\mathbf{A}(\Omega)t)f\|_{L^\infty} &= \|e^{t\Delta}\overline{f}+e^{t\Delta}\exp(-\Omega t\mathbf{PJP})f^{\perp}\|_{L^\infty}\\
&\le \|e^{t\Delta}\overline{f}\|_{L^\infty}+\|e^{t\Delta}\exp(-\Omega t\mathbf{PJP})f^{\perp}\|_{L^\infty}\\
&\le \|\overline{f}\|_{L^\infty}+\|\exp(-\Omega t\mathbf{PJP})f^{\perp}\|_{L^\infty}\\
&\le \|\overline{f}\|_{L^\infty}+\|\exp(-\Omega t\mathbf{PJP})f^{\perp}\|_{\dot{B}^0_{\infty,1}}\\
&\le \|\overline{f}\|_{L^\infty}+C\|f^{\perp}\|_{\dot{B}^0_{\infty,1}}\\
&\le C\|f\|_{L^\infty_{\sigma,a}}.
\end{aligned}$$

Here, we used Corollary 3.1 and $\|\cdot\|_{L^\infty}\le\|\cdot\|_{\dot{B}^0_{\infty,1}}$. □

Remark 3.4. (i) The above proof does not require the existence of vertical averege of f but require only the representation of f as in (1.6).
(ii) Similarly, we can get the derivative estimate of the linear term

$$\|\nabla\exp(-\mathbf{A}(\Omega)t)f\|_{L^\infty}\le Ct^{-1/2}\|f\|_{L^\infty_{\sigma,a}};\quad t>0,$$

for $f\in L^\infty_{\sigma,a}$ (see Lemma 4.2 in [12]).

The estimates Proposition 3.3 and Proposition 2.1 yield Theorem 1.1 by the following iteration;

$$\begin{cases} u_1(t) &= \exp(-\mathbf{A}(\Omega)t)u_0,\\ u_{j+1}(t) &= \exp(-\mathbf{A}(\Omega)t)u_0-\int_0^t\exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\mathrm{div}(u_{j-1}\otimes u_{j-1})(s)\,ds\end{cases}\tag{3.11}$$

for $j\ge 1$. Lower estimate of existence time T_0 (Remark 1.1(i)) comes from uniform estimate for

$$K_j=K_j(T)=\sup_{0<s<T}\|u_j(s)\|_{L^\infty}\quad\text{and}\quad K'_j=K'_j(T)=\sup_{0<s<T}s^{1/2}\|\nabla u_j(s)\|_{L^\infty}\quad\text{for } T>0.$$

We note that Theorem 1.2 follows from Theorem 1.1 as observed in [11], where the case $\Omega=0$ is discussed. We also note that Theorem 1.3 can be proved along the line of [17], where the case $\Omega=0$ is discussed. We won't repeat the proofs.

4 Mikhlin-type theorems

We introduce Mikhlin-type theorems in 3 kinds of spaces– the Hardy space $\mathcal{H}^1$, the space of functions of bounded mean oscillations BMO, and the Besov spaces $\dot{B}^0_{\infty,q}$ for $1\le q\le\infty$. The Hardy space version theorem is applied to estimate of nonlinear term, and the Besov space version is for linear term. All statements in this section are valid for general space dimension $n\in\mathbb{N}$.

Lemma 4.1. ((1),(2):Theorem 7.30 in [10], [16])
Let $m(\xi)\in C^k(\mathbb{R}^n\setminus\{0\})$ for some integer $k>n/2$ satisfy

$$|D^\alpha m(\xi)|\le C_\alpha|\xi|^{-|\alpha|}\quad(\xi\ne 0)\qquad\textit{for all }\ |\alpha|=\alpha_1+\cdots+\alpha_n\le k.\tag{4.1}$$

Then the operator defined by $T_m = F^{-1}mF$ is bounded
(1) *from $\mathcal{H}^1(\mathbb{R}^n)$ to itself,*
(2) *from $BMO(\mathbb{R}^n)$ to itself, and*
(3) *from $\dot{B}^0_{\infty,q}(\mathbb{R}^n)$ to itself for all $1 \le q \le \infty$.*

In [12], the statement (3) is proved by a Lemma on boundedness of convolution-type operator (see [[12];Lemma B.1 and Remark B.1]), however, here we will give another proof of (3) in the case $k = n+1$ when $n > 2$, using the following lemma by Amann [1].

Lemma 4.2 ([1];Lemma 4.2(i)). *Assume $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Let $m \in C^{n+1}(\mathbb{R}^n \setminus \{0\})$ satisfy*

$$\mu_j := \max_{|\alpha| \le n+1} \sup_{2^{j-1} \le |\xi| \le 2^{j+1}} |\xi|^{|\alpha|} |D^\alpha m(\xi)| < \infty \qquad \text{for some} \quad j \in \mathbb{Z}. \tag{4.2}$$

Then $F^{-1}(m\widehat{\phi}_j) \in L^1(\mathbb{R}^n)$ and

$$||F^{-1}(m\widehat{\phi}_j)||_{L^1(\mathbb{R}^n)} \le C\mu_j,$$

where $C = C(n) > 0$ is independent of m and j.

Although we deal with only the scalar-valued Besov spaces with specific indices $p = \infty, q \in [1, \infty]$ and $s = 0$, that is, $\dot{B}^0_{\infty,q}$, Amann [1] proved Mikhlin-type theorem in the vector-valued Besov spaces $B^s_{p,q}(\mathbb{R}^n, E)$. Here, E is a Banach space without any restriction such as UMD nor HT spaces (see also [2], [9]), and $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Though he mentions only the inhomogeneous Besov spaces, his proof can be adapted to the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n, E)$.

Proof of Lemma 4.1(3): For $f \in \dot{B}^0_{\infty,q}$ with $1 \le q \le \infty$ it follows from $\phi_j * (F^{-1}mFf) = \phi_j * (F^{-1}m) * f = (F^{-1}(m\widehat{\phi}_j)) * f$ that

$$||F^{-1}mFf||_{\dot{B}^0_{\infty,q}} = (\sum_{j\in\mathbb{Z}} ||(F^{-1}(m\widehat{\phi}_j)) * f||^q_{L^\infty})^{1/q}.$$

By (3.9) and Young's inequality we get

$$||F^{-1}mFf||_{\dot{B}^0_{\infty,q}} = (\sum_{j,k\in\mathbb{Z},|j-k|\le 2} ||(F^{-1}(m\widehat{\phi}_j)) * f * \phi_k||^q_{L^\infty})^{1/q}$$

$$\le (\sum_{j,k\in\mathbb{Z},|j-k|\le 2} ||F^{-1}(m\widehat{\phi}_j)||^q_{L^1} ||f * \phi_k||^q_{L^\infty})^{1/q} \le C(\sum_{j,k\in\mathbb{Z},|j-k|\le 2} \mu^q_j ||f * \phi_k||^q_{L^\infty})^{1/q}.$$

Since the assumption (4.1) (the case $k = n+1$) yields that

$$\sup_{j\in\mathbb{Z}} \mu_j \le \max_{|\alpha|\le n+1} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{|\alpha|} |D^\alpha m(\xi)| \le C$$

for some $C > 0$ independent of j, one sees

$$||F^{-1}mFf||_{\dot{B}^0_{\infty,q}} \le C(\sup_{j\in\mathbb{Z}} \mu_j)(\sum_{k\in\mathbb{Z}} ||f * \phi_k||^q_{L^\infty})^{1/q} \le C||f||_{\dot{B}^0_{\infty,q}}.$$

□

5 Regularity of mild solution

In this section we prove Remark 1.1(ii) and (iii). All lemmas in this section hold for general space dimension $n \in \mathbb{N}$ although the Remark 1.1 is valid only for $n = 3$.

Lemma 5.1. *There exists a constant $C > 0$ independent of f and g such that*

$$||f * g||_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \le C||f||_{\dot{B}^0_{1,1}(\mathbb{R}^n)}||g||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$$

for $f \in \dot{B}^0_{1,1}(\mathbb{R}^n)$ and $g \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$.

Proof. By Young's inequality we have

$$\begin{aligned}
||f * g||_{\dot{B}^0_{\infty,1}} &\le \sum_{j\in\mathbb{Z}} ||\phi_j * (f * g)||_{L^\infty} \le \sum_{j,k\in\mathbb{Z}} ||\phi_j * (f * g) * \phi_k||_{L^\infty} \\
&\le \sum_{j,k\in\mathbb{Z},|j-k|\le 2} ||\phi_j * f||_{L^1}||g * \phi_k||_{L^\infty} \\
&\le 3\sup_{k\in\mathbb{Z}} ||g * \phi_k||_{L^\infty} \sum_{j\in\mathbb{Z}} ||\phi_j * f||_{L^1} \le 3||g||_{\dot{B}^0_{\infty,\infty}}||f||_{\dot{B}^0_{1,1}}
\end{aligned}$$

□

Lemma 5.2. *Let G_t be the heat kernel $(4\pi t)^{-n/2}\exp(\frac{-|x|^2}{4t})$ for $t > 0$. Then*
(1) $||\nabla G_t(x)||_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \le Ct^{-1/2}$.
(2) $||\nabla e^{t\Delta} f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \le Ct^{-1/2}||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$ *for $f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$.*

Proof. (1) Since $\phi_j(x) = 2^{jn}\phi_0(2^j x)$, we see

$$\begin{aligned}
||\phi_j * \nabla G_t||_1 &= ||\nabla(\phi_j) * G_t||_1 = 2^j|| \int_{\mathbb{R}^n} |2^{jn}(\nabla\phi_0)(2^j y)G_t(x-y)|dy||_1 \\
&\le 2^j||2^{jn}(\nabla\phi_0)(2^j\cdot)||_1||G_t||_1 \le 2^j||\nabla\phi_0||_1||G_t||_1. \qquad (5.1)
\end{aligned}$$

On the other hand, we get by the mean value theorem and $\int \phi_0(z)dz = 0$

$$\begin{aligned}
(\phi_j * \nabla G_t)(x) &= \int_{\mathbb{R}^n} \phi_j(y)(\nabla G_t)(x-y)dy \\
&= \int_{\mathbb{R}^n} 2^{jn}\phi_0(2^j y)(\nabla G_t)(x-y)dy = \int_{\mathbb{R}^n} \phi_0(z)(\nabla G_t)(x - 2^{-j}z)dz \\
&= \int_{\mathbb{R}^n} \phi_0(z)\{(\nabla G_t)(x-2^{-j}z) - (\nabla G_t)(x)\}dz \\
&= \int_{\mathbb{R}^n} \phi_0(z)2^{-j}z(\int_0^1 (\nabla^2 G_t)(x - \theta 2^{-j}z)d\theta)dz.
\end{aligned}$$

Hence,

$$\begin{aligned}
||\phi_j * \nabla G_t||_1 &\le 2^{-j}\int_{\mathbb{R}^n} |\phi_0(z)z\int_0^1 (\nabla^2 G_t)(x-\theta 2^{-j}z)d\theta|dz \\
&\le 2^{-j}\int_{\mathbb{R}^n} |\phi_0(z)||z|||\nabla^2 G_t||_1 dz \le 2^{-j}||\phi_0(z)|z|||_1||\nabla^2 G_t||_1. \qquad (5.2)
\end{aligned}$$

Putting $C_0 = ||\nabla\phi_0||_1, C_1 = ||\phi_0(z)|z|||_1$, the inequalities (5.1), (5.2) and $||G_t||_1 = 1$ yield

$$||\phi_j * \nabla G_t||_1 \leq \begin{cases} C_0 2^j, \\ C_2 2^{-j} t^{-1}. \end{cases}$$

Here, $C_2 = C_1||\nabla^2 G_t||_1 t$ is indepencent of t. Thus we get for any $N \in \mathbb{Z}$

$$\begin{aligned} ||\nabla G_t(x)||_{\dot{B}^0_{1,1}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{\infty} ||\phi_j * \nabla G_t(x)||_1 = (\sum_{j=-\infty}^{N} + \sum_{j=N}^{\infty})||\phi_j * \nabla G_t(x)||_1 \\ &\leq C_0 \sum_{j=-\infty}^{N} 2^j + C_2 t^{-1} \sum_{j=N}^{\infty} 2^{-j} = C_0 2^{N+1} + C_2 2^{-N} t^{-1}. \end{aligned}$$

Taking $N \in \mathbb{Z}$ such that $(C_2/C)t^{-1/2} \leq 2^N \leq (1/2C_0)t^{-1/2}$, we derive the result.
(2) This is a direct consequence of (1) and Lemma 5.1. $\square$

Proof of Remark 1.1(iii): Let $u_0^\eta = G_\eta * u_0$ for small $\eta > 0$ where G_η is the heat kernel $(4\pi\eta)^{-3/2}\exp(\frac{-|x|^2}{4\eta})$. Then $u_0^\eta \in L^\infty$ and $\nabla u_0^\eta \in \dot{B}^0_{\infty,1}$ since $||u_0^\eta||_\infty \leq ||G_\eta||_1||u_0||_\infty \leq ||u_0||_\infty$ and $||\nabla u_0^\eta||_\infty \leq C\eta^{-1/2}||u_0||_\infty$ by Lemma 2.2. It is easy from the second inequality of Proposition 2.1 to see the nonlinear term

$$t^{1/2}\int_0^t ||\nabla \exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\mathrm{div}(u\otimes u)(s)||_\infty ds$$

tends to 0 as $t \downarrow 0$. On the linear term we get by Lemma 2.2 and Corollary 3.1 that

$$\begin{aligned} & t^{1/2}||\nabla \exp(-\mathbf{A}(\Omega)(t))u_0||_\infty \\ \leq\ & t^{1/2}(||\nabla \exp(t\Delta)\exp(-\Omega t\mathbf{PJP})(u_0 - u_0^\eta)||_\infty + ||\nabla \exp(t\Delta)\exp(-\Omega t\mathbf{PJP})u_0^\eta||_\infty) \\ \leq\ & Ct^{1/2}t^{-1/2}||\exp(-\Omega t\mathbf{PJP})(u_0 - u_0^\eta)||_{BMO} + t^{1/2}||\exp(t\Delta)\exp(-\Omega t\mathbf{PJP})\nabla u_0^\eta||_\infty \\ \leq\ & C||u_0 - u_0^\eta||_{BMO} + t^{1/2}||\exp(-\Omega t\mathbf{PJP})\nabla u_0^\eta||_\infty. \end{aligned}$$

By $||\cdot||_{BMO} \leq ||\cdot||_{L^\infty} \leq ||\cdot||_{\dot{B}^0_{\infty,1}}$ and uniform boundedness of $\exp(-\Omega t\mathbf{PJP})$ (Corollary 3.1) we estimate

$$\begin{aligned} & t^{1/2}||\nabla \exp(-\mathbf{A}(\Omega)(t))u_0||_\infty \\ \leq\ & C||u_0 - u_0^\eta||_\infty + t^{1/2}||\exp(-\Omega t\mathbf{PJP})\nabla u_0^\eta||_{\dot{B}^0_{\infty,1}} \\ \leq\ & C||u_0 - u_0^\eta||_\infty + Ct^{1/2}||\nabla u_0^\eta||_{\dot{B}^0_{\infty,1}} \\ \leq\ & C||u_0 - u_0^\eta||_\infty + Ct^{1/2}\eta^{-1/2}||u_0||_{\dot{B}^0_{\infty,\infty}}, \end{aligned}$$

where we used Lemma 5.2(2). After taking $\eta = t^{1/2}$, send $\eta \downarrow 0$. Then the first term in the RHS $C||u_0 - u_0^\eta||_\infty \to 0$ since u_0 is assumed to be uniformly continuous (see Lemma 5 in [11]). The second term also tends to 0 since $||u_0||_{\dot{B}^0_{\infty,\infty}} \leq ||u_0||_{L^\infty} \leq ||u_0||_{L^\infty_{\sigma,a}}$ is finite. $\square$

Lemma 5.3. *Let $0 < \alpha \le 1$. Then there exists a constant $C_\alpha = C(\alpha) > 0$ such that*
(1) $||(-\Delta)^\alpha G_t||_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \le C_\alpha t^{-\alpha}$ *for* $t > 0$,
(2) $||(-\Delta)^\alpha \exp(t\Delta) f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \le C_\alpha t^{-\alpha} ||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$ *for* $t > 0$, $f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$,
(3) $||(\exp(t\Delta) - \mathbf{I})f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \le C_\alpha t^{-\alpha} ||(-\Delta)^\alpha f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$ *for* $t > 0$, $f \in D((-\Delta)^\alpha)$,
(4) $||(\exp(s\Delta) - \exp(t\Delta))f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \le C_\alpha (s-t)^\alpha t^{-\alpha} ||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$ *for* $s > t > 0$, $f \in \dot{B}^0_{\infty,1}(\mathbb{R}^n)$.
Here, $D((-\Delta)^\alpha) = \{f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n); (-\Delta)^\alpha f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)\}$.

Remark 5.1. By $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$ we immediately see by (2)

$$||(-\Delta)^\alpha \exp(t\Delta) f||_{\dot{B}^0_{\infty,\infty}} \le C_\alpha t^{-\alpha} ||f||_{\dot{B}^0_{\infty,\infty}} \quad \text{for } t > 0, f \in \dot{B}^0_{\infty,\infty}. \tag{5.3}$$

Proof. The inequality (1) shall be proved in Appendix. The assertion (2) immediately follows from (1) and Lemma 5.1. For (3) we see for $f \in D((-\Delta)^\alpha)$ that

$$(\exp(t\Delta) - \mathbf{I})f = -\int_0^t (-\Delta)\exp(s\Delta) f ds = -\int_0^t (-\Delta)^{1-\alpha} \exp(s\Delta)(-\Delta)^\alpha f ds.$$

Then by (2)

$$\begin{aligned}
||(\exp(t\Delta) - \mathbf{I})f||_{\dot{B}^0_{\infty,1}} &\le \int_0^t ||(-\Delta)^{1-\alpha}\exp(s\Delta)||_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,1}} ||(-\Delta)^\alpha f||_{\dot{B}^0_{\infty,\infty}} ds \\
&\le C_{1-\alpha} \int_0^t s^{\alpha-1} ds ||(-\Delta)^\alpha f||_{\dot{B}^0_{\infty,\infty}} \le C_{1-\alpha} \frac{1}{\alpha} t^\alpha ||(-\Delta)^\alpha f||_{\dot{B}^0_{\infty,\infty}}.
\end{aligned}$$

For (4) let $f \in \dot{B}^0_{\infty,1}$. Then $\exp(t\Delta) f \in D((-\Delta)^\alpha)$ for $t > 0$. In fact, $f \in \dot{B}^0_{\infty,1} \subset L^\infty$, hence $\exp(t\Delta) f \in L^\infty \subset \dot{B}^0_{\infty,\infty}$. So, (2) implies $(-\Delta)^\alpha \exp(t\Delta) f \in \dot{B}^0_{\infty,1}$ for $t > 0$. It follows from (5.3) and (3) that

$$\begin{aligned}
&||(\exp(s\Delta) - \exp(t\Delta))f||_{\dot{B}^0_{\infty,1}} \\
&= ||(\exp((s-t)\Delta) - \mathbf{I})(-\Delta)^{-\alpha}||_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,1}} ||(-\Delta)^\alpha \exp(t\Delta) f||_{\dot{B}^0_{\infty,\infty}} \\
&\le C_\alpha (s-t)^\alpha t^{-\alpha} ||f||_{\dot{B}^0_{\infty,\infty}}.
\end{aligned}$$

□

Proof of Remark 1.1(ii): Let $\{u_j\}$ be sequence of the succesive iteration (3.11). By the assumption $\overline{u_0} \in BUC$ we see $\exp(t\Delta)\overline{u_0} \in BUC$ since $\exp(t\Delta)$ is an analytic semigroup in BUC (see e.g. Proposition A.1.1 of [11]). On the other hand, $u_0 \in L^\infty_{\sigma,a}$, hence $u_0^\perp \in \dot{B}^0_{\infty,1}$ yields $\exp(-\Omega t\mathbf{PJP})u_0^\perp \in \dot{B}^0_{\infty,1} \subset BUC$ by Corollary 3.1(1). Then $\exp(t\Delta)\exp(-\Omega t\mathbf{PJP})u_0^\perp \in BUC$ thanks to the semigroup $\exp(t\Delta)$ in BUC again. Thus $u_1 = \exp(t\Delta)\overline{u_0} + \exp(t\Delta)\exp(-\Omega t\mathbf{PJP})u_0^\perp$ belongs to BUC.

Next we show $u_j \in BUC$ for all $j \ge 2$. Since it is known that $f \in L^\infty$ is uniformly continuous if and only if $||\exp(\delta\Delta)f - f||_{L^\infty} \to 0$ as $\delta \downarrow 0$ (see e.g. Lemma 5 in [11]), it is sufficient to show that $u_j \in L^\infty$ satisfies $||\exp(\delta\Delta)u_j - u_j||_{L^\infty} \to 0$ as $\delta \downarrow 0$. We have for fixed $t \in (0, T_0]$ and any $\delta > 0$

$$\begin{aligned}
&||\exp(\delta\Delta)u_j - u_j||_{L^\infty} \\
&\leq ||\exp(\delta\Delta)\exp(-\mathbf{A}(\Omega)t)u_0 - \exp(-\mathbf{A}(\Omega)t)u_0||_{\dot{B}^0_{\infty,1}} \\
&\quad + \int_0^t ||(\exp(\delta\Delta)\exp(-\mathbf{A}(\Omega)(t-s)) - \exp(-\mathbf{A}(\Omega)(t-s)))\,\mathbf{P}\mathrm{div}(u_{j-1}\otimes u_{j-1})(s)||_{L^\infty}\, ds \\
&\leq ||\exp((t+\delta)\Delta) - \exp(t\Delta)||_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,1}} ||\exp(-\Omega t\mathbf{PJP})||_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,\infty}} ||u_0||_{\dot{B}^0_{\infty,\infty}} \\
&\quad + \int_0^t ||\nabla\cdot\exp(\frac{t-s}{2}\Delta)||_{\dot{B}^0_{\infty,\infty}\to L^\infty} ||\exp(-\Omega(t-s)\mathbf{PJP})||_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,\infty}} \\
&\qquad ||\exp((\frac{t-s}{2}+\delta)\Delta) - \exp(\frac{t-s}{2}\Delta)||_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,1}} ||u_{j-1}\otimes u_{j-1}||_{\dot{B}^0_{\infty,\infty}}(s)\, ds \\
&\leq C_\alpha\delta^\alpha t^{-\alpha}||u_0||_{\dot{B}^0_{\infty,\infty}} + C_\alpha\delta^\alpha\int_0^t (t-s)^{-\alpha-\frac{1}{2}}||u_j\otimes u_j||_{L^\infty}(s)\, ds
\end{aligned}$$

with all $0 < \alpha < 1$. Here, we used Lemma 5.3(4), Proposition 3.2 and $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$. Choose $0 < \alpha < 1/2$ and send $\delta \downarrow 0$ to see RHS tends to 0, noting that $||u_j \otimes u_j||_{L^\infty}(s) \leq ||u_j||^2_{L^\infty}(s)$ for all $0 \leq s \leq t$. Thus we have proved that $u_j \in BUC$ for all $j \geq 1$, which implies that its uniform limit $u \in BUC$. □

A Appendix: Estimate for fractional power of Laplacian of the heat kernel

In this appendix we shall prove Lemma 5.3(1).

Proof of Lemma 5.3(1): Setting $x = t^{1/2}z$, it is easy to see that

$$((-\Delta)^\alpha G_t)(x) = t^{-\frac{n}{2}-\alpha}((-\Delta)^\alpha G_1)(z) \quad \text{for} \quad t > 0. \tag{A.1}$$

Hence, it is sufficient to show only the case $t = 1$. In fact, by scaling invariance $||f(\lambda\cdot)||_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \approx \lambda^{-n}||f||_{\dot{B}^0_{1,1}(\mathbb{R}^n)}$ for $\lambda > 0$ we get

$$||((-\Delta)^\alpha G_t)(x)||_{\dot{B}^0_{1,1}} = t^{-\frac{n}{2}-\alpha}||((-\Delta)^\alpha G_1)(t^{-1/2}x)||_{\dot{B}^0_{1,1}} \leq Ct^{-\frac{n}{2}-\alpha}t^{\frac{n}{2}}||(-\Delta)^\alpha G_1||_{\dot{B}^0_{1,1}} \leq C_\alpha t^{-\alpha}.$$

For any fixed $j \in \mathbb{Z}$ one sees that

$$\phi_j * (-\Delta)^\alpha G_1 = \phi_j * (F^{-1}(|\xi|^{2\alpha}\widehat{G_1})) = F^{-1}(\widehat{\phi_j}|\xi|^{2\alpha}\widehat{G_1}).$$

Since $\widehat{\phi_j}(\xi) = \widehat{\phi_0}(2^{-j}\xi)$ we continue

$$\begin{aligned}
\phi_j * (-\Delta)^\alpha G_1 &= \int e^{ix\cdot\xi}\widehat{\phi_0}(2^{-j}\xi)|\xi|^{2\alpha}\widehat{G_1}(\xi)d\xi \\
&= \int e^{ix\cdot 2^j\xi}\widehat{\phi_0}(\xi)|2^j\xi|^{2\alpha}\widehat{G_1}(2^j\xi)2^{jn}d\xi \\
&= 2^{jn+j2\alpha}\int e^{i2^jx\cdot\xi}|\xi|^{2\alpha}\widehat{\phi_0}(\xi)\widehat{G_1}(2^j\xi)d\xi \\
&= 2^{jn+j2\alpha}[F^{-1}(|\xi|^{2\alpha}\widehat{\phi_0}(\xi)\widehat{G_1}(2^j\xi))](2^jx) \\
&= 2^{jn+j2\alpha}[F^{-1}(|\xi|^{2\alpha}\widehat{\phi_0}(\xi)) * F^{-1}(\widehat{G_1}(2^j\xi))](2^jx).
\end{aligned}$$

It follows from

$$F^{-1}(\widehat{G_1}(2^j\xi)) = F^{-1}(\frac{1}{2^{jn}}[F(G_1(\frac{x}{2^j}))](\xi)) = \frac{1}{2^{jn}}G_1(\frac{x}{2^j})$$

and $F^{-1}(|\xi|^{2\alpha}\widehat{\phi_0}(\xi)) = (-\Delta)^\alpha\phi_0$ that

$$\phi_j * (-\Delta)^\alpha G_1 = 2^{j2\alpha}[(-\Delta)^\alpha\phi_0 * G_1(\frac{\cdot}{2^j})](2^jx). \tag{A.2}$$

Hence Young's inequality yields

$$\begin{aligned}
||\phi_j * (-\Delta)^\alpha G_1||_1 &= 2^{j2\alpha}||[(-\Delta)^\alpha\phi_0 * G_1(\frac{\cdot}{2^j})](2^jx)||_1 \\
&= 2^{j2\alpha-jn}||[(-\Delta)^\alpha\phi_0 * G_1(\frac{\cdot}{2^j})](x)||_1 \\
&\le 2^{j2\alpha-jn}||(-\Delta)^\alpha\phi_0||_1||G_1(\frac{\cdot}{2^j})||_1 \\
&= 2^{j2\alpha}||(-\Delta)^\alpha\phi_0||_1||G_1||_1 \\
&\le C_\alpha 2^{j2\alpha}.
\end{aligned} \tag{A.3}$$

Here we used $||G_1||_1 = 1$ and $||(-\Delta)^\alpha\phi_0||_1 = ||F^{-1}(|\xi|^{2\alpha}\widehat{\phi_0})||_1 \le C_\alpha$ because $|\xi|^{2\alpha}\widehat{\phi_0} \in \mathcal{S}$. On the other hand we shift $(-\Delta)^\alpha$ to $G_1(\frac{\cdot}{2^j})$ in RHS of (A.2) to get

$$\begin{aligned}
\phi_j * (-\Delta)^\alpha G_1 &= 2^{j2\alpha}[\phi_0 * (-\Delta)^\alpha G_1(\frac{\cdot}{2^j})](2^jx) \\
&= 2^{j2\alpha}[((-\Delta)^{-\beta}\phi_0) * (-\Delta)^{\alpha+\beta}G_1(\frac{\cdot}{2^j})](2^jx).
\end{aligned}$$

Here we put $(-\Delta)^{\beta-\beta} = \mathrm{I}$ with some $\beta > 0$. Then we get by Young's inequality for any fixed $j \in \mathbb{N}$ that

$$\begin{aligned}
||\phi_j * (-\Delta)^\alpha G_1||_1 &= 2^{j2\alpha}||[((-\Delta)^{-\beta}\phi_0) * (-\Delta)^{\alpha+\beta}G_1(\frac{\cdot}{2^j})](2^jx)||_1 \\
&= 2^{j2\alpha-jn}||[((-\Delta)^{-\beta}\phi_0) * (-\Delta)^{\alpha+\beta}G_1(\frac{\cdot}{2^j})](x)||_1 \\
&\le 2^{j2\alpha-jn}||(-\Delta)^{-\beta}\phi_0||_1||(-\Delta)^{\alpha+\beta}(G_1(\frac{\cdot}{2^j}))||_1.
\end{aligned}$$

Noting that $(-\Delta)^\gamma(G_1(\frac{\cdot}{a})) = a^{-2\gamma}((-\Delta)^\gamma G_1)(\frac{\cdot}{a})$ for $a > 0, \gamma > 0$, and $||(-\Delta)^{-\beta}\phi_0||_1 = ||F^{-1}(|\xi|^{-2\beta}\widehat{\phi_0})||_1 \leq C_\beta$ for some $C_\beta > 0$ because $|\xi|^{-2\beta}\widehat{\phi_0} \in \mathcal{S}$ we continue

$$\begin{aligned} ||\phi_j * (-\Delta)^\alpha G_1||_1 &\leq C_\beta 2^{j2\alpha-jn}||2^{-2(\alpha+\beta)j}((-\Delta)^{\alpha+\beta}G_1)(\frac{x}{2^j})||_1 \\ &\leq C_\beta 2^{j2\alpha-jn}2^{-2(\alpha+\beta)j}2^{jn}||((-\Delta)^{\alpha+\beta}G_1)(\frac{x}{2^j})||_1 \\ &= C_\beta 2^{-2\beta j}||((-\Delta)^{\alpha+\beta}G_1)(x)||_1. \end{aligned}$$

Because

$$||(-\Delta)^\gamma G_1||_1 \leq C_\gamma \quad \text{for} \quad \gamma > 0$$

we get

$$||\phi_j * (-\Delta)^\alpha G_1||_1 \leq C_{\alpha,\beta} 2^{-2\beta j}. \tag{A.4}$$

Fix $\beta > 0$ to get from (A.3) and (A.4) that

$$||(-\Delta)^\alpha G_t(x)||_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \leq \sum_{j\leq 0} C_\alpha 2^{j2\alpha} + \sum_{j>0} C_\alpha 2^{-j2\beta} \leq C_\alpha.$$

□

Acknowledgements

The author would like to thank Professor Yoshikazu Giga, Professor Alex Mahalov and Professor Shin'ya Matsui for their encouragement. The author also would like to thank Dr. Juergen Saal, Dr. Kazuyuki Yamauchi, Dr. Jun Kato, Dr. Okihiro Sawada and Mr. Yutaka Terasawa for the useful discussions.

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