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Stability of 1-dimensional stationary solution to the compressible Navier-Stokes equations on the half space

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1. Introduction

This article is concerned with the compressible Navier-Stokes equation on the half space $\mathbf{R}_+^n$ $(n \geq 2)$:

$$\begin{aligned} &\partial_t\rho + \mathrm{div}\,(\rho u) = 0, \\ (1.1) \qquad &\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) = \mu\Delta u + (\mu + \mu')\nabla \mathrm{div}\, u, \\ &p(\rho) = K\rho^\gamma. \end{aligned}$$

Here $\mathbf{R}_+^n = \{x = (x_1, x')\,;\, x' = (x_2, \cdots, x_n) \in \mathbf{R}^{n-1},\, x_1 > 0\}$; $\rho = \rho(x,t)$ and $u = (u^1(x,t), \cdots, u^n(x,t))$ denote the unknown density and velocity, respectively; μ, μ', K and γ are constants satisfying $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$, $K > 0$ and $\gamma > 1$. We consider (1.1) under the initial and boundary conditions

$$\begin{aligned} &u|_{x_1=0} = (u_b^1, 0, \cdots, 0), \\ (1.2) \qquad &\rho \to \rho_+, \quad u \to (u_+^1, 0, \cdots, 0) \quad (x_1 \to \infty), \\ &(\rho, u)|_{t=0} = (\rho_0, u_0), \end{aligned}$$

where ρ_+, u_+^1 and u_b^1 are given constants satisfying $\rho_+ > 0$ and $u_b^1 < 0$.

Kawashima, Nishibata and Zhu [4] investigated the conditions for ρ_+, u_+^1 and u_b^1 under which planar stationary motions occur. Namely, they showed that under suitable conditions for ρ_+, u_+^1 and u_b^1 there exists a stationary solution $(\widetilde{\rho}, \widetilde{u})$ of problem (1.1)–(1.2) in the form $\widetilde{\rho} = \widetilde{\rho}(x_1)$, $\widetilde{u} = (\widetilde{u}^1(x_1), 0, \cdots, 0)$. Furthermore, it was shown in [4] that $(\widetilde{\rho}, \widetilde{u})$ is asymptotically stable with respect to small one-dimensional perturbations, i.e., perturbations in the form $\rho - \widetilde{\rho} = \rho(x_1, t) - \widetilde{\rho}(x_1)$, $u - \widetilde{u} = (u^1(x_1, t) - \widetilde{u}^1(x_1), 0, \cdots, 0)$, provided that $|u_+^1 - u_b^1|$ is sufficiently small.

In this article we will give a summary of the results in [3], where $(\widetilde{\rho}, \widetilde{u})$ is shown to be asymptotically stable with respect to multi-dimensional perturbations small in $H^s(\mathbf{R}_+^n)$, provided that $|u_+^1 - u_b^1|$ is sufficiently small. Here s is an integer satisfying $s \geq [n/2] + 1$.

2. Stability Result

We first consider the one-dimensional stationary problem whose solutions represent planar stationary motions in $\mathbf{R}_+^n$. We look for a smooth stationary solution $(\widetilde{\rho}, \widetilde{u})$ of (1.1)–(1.2) of the form $\widetilde{\rho} = \widetilde{\rho}(x_1) > 0$ and $\widetilde{u} = (\widetilde{u}^1(x_1), 0, \cdots, 0)$. Then the problem for $(\widetilde{\rho}, \widetilde{u}^1)$ is written as

$$(2.1) \qquad \begin{aligned} &(\widetilde{\rho}\widetilde{u}^1)_{x_1} = 0 \quad (x_1 > 0), \\ &(\widetilde{\rho}(\widetilde{u}^1)^2)_{x_1} + p(\widetilde{\rho})_{x_1} = (2\mu + \mu')\widetilde{u}^1_{x_1 x_1} \quad (x_1 > 0), \\ &\widetilde{u}|_{x_1=0} = u_b^1, \\ &\widetilde{\rho} \to \rho_+, \quad \widetilde{u}^1 \to u_+^1 \quad (x_1 \to \infty), \end{aligned}$$

where subscript x_1 stands for differentiation in x_1.

Kawashima, Nishibata and Zhu [4] investigated problem (2.1) and gave a necessary and sufficient condition for the existence of solutions. Following [4], we introduce the Mach number at infinity defined by

$$M_+ \equiv \frac{|u_+|}{\sqrt{p'(\rho_+)}}.$$

We also set

$$\delta \equiv |u_+^1 - u_b^1|,$$

which measures the strength of the stationary solution.

Proposition 2.1.([4]) *Let $u_+^1 < 0$. Then problem (2.1) has a smooth solution $(\widetilde{\rho}, \widetilde{u}^1)$ if and only if $M_+ \geq 1$ and $w_c u_+ > u_b$, where w_c is a certain positive number. The solution $(\widetilde{\rho}, \widetilde{u}^1)$ is monotonic, in particular, $\widetilde{u}^1(x_1)$ is monotonically increasing when $M_+ = 1$. Furthermore, $(\widetilde{\rho}, \widetilde{u}^1)$ has the following decay properties as $x_1 \to \infty$.*

(i) *If $M_+ > 1$, then for any nonnegative integer k there exists a constant $C > 0$ such that*

$$|\partial_{x_1}^k (\widetilde{\rho} - \rho_+, \widetilde{u}^1 - u_+^1)| \leq C\delta e^{-\sigma x_1}$$

for some positive constant σ.

(ii) *If* $M_+ = 1$, *then for any nonnegative integer* k *there exists a constant* $C > 0$ *such that*

$$|\partial_{x_1}^k(\widetilde{\rho} - \rho_+, \widetilde{u}^1 - u_+^1)| \le C\frac{\delta^{k+1}}{(1+\delta x_1)^{k+1}}.$$

Our interest is the stability properties of $(\widetilde{\rho}, \widetilde{u})$, $\widetilde{u} = (\widetilde{u}^1, 0, \cdots, 0)$, with respect to multi-dimensional perturbations. To state our stability result we introduce function spaces. For $0 < T \le \infty$ and $\sigma \in \mathbf{Z}$, $\sigma \ge 0$, we define the Banach space

$$Z^\sigma(T) = X^\sigma(T) \times Y^\sigma(T)^n,$$

where

$$X^\sigma(T) = \bigcap_{j=0}^{[\frac{\sigma}{2}]} C^j([0,T]; H^{\sigma-2j})$$

and

$$Y^\sigma(T) = X^\sigma(T) \cap \bigcap_{j=0}^{[\frac{\sigma+1}{2}]} H^j(0,T; \widetilde{H}^{\sigma+1-2j}).$$

Here $\widetilde{H}^m = H^m \cap H_0^1$ when $m \ge 1$ and $\widetilde{H}^m = L^2$ when $m = 0$. The norm of $Z^\sigma(T)$ is defined by $\|U\|_{Z^\sigma(T)} = \|\phi\|_{X^\sigma(T)} + \|\psi\|_{Y^\sigma(T)}$ for $U = (\phi, \psi)$, where

$$\|\phi\|_{X^\sigma(T)} = \sup_{0\le t\le T} |[\phi(t)]|_\sigma, \quad \|\psi\|_{Y^\sigma(T)} = \left(\|\psi\|^2_{X^\sigma(T)} + \int_0^T |[\psi(t)]|^2_{\sigma+1}\,dt\right)^{1/2}$$

with

$$|[\phi(t)]|_{\sigma,k} = \left(\sum_{j=0}^k \|\partial_t^j \phi(t)\|^2_{H^{\sigma-2j}}\right)^{1/2}, \quad |[\phi(t)]|_\sigma = |[\phi(t)]|_{\sigma,[\frac{\sigma}{2}]}.$$

We simply denote by Z^σ, X^σ and Y^σ when $T = \infty$.

Theorem 2.2. *Let* s *be an integer satisfying* $s \ge [n/2]+1$ *and let* $(\widetilde{\rho}, \widetilde{u})$ *be the solution of* (2.1). *Then there exists a positive number* δ_0 *such that if* $|u_b^1 - u_+^1| < \delta_0$, *then* $(\widetilde{\rho}, \widetilde{u})$ *is stable with respect to perturbations small in* $H^s(\mathbf{R}^n_+)$ *in the following sense: there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that if the initial perturbation* $(\rho(0) - \widetilde{\rho}, u(0) - \widetilde{u}) \in H^s$ *and satisfies a suitable compatibility condition, then perturbation* $(\rho(t) - \widetilde{\rho}, u(t) - \widetilde{u})$ *exists in* Z^s, *and it satisfies*

$$\|(\rho(t) - \widetilde{\rho}, u(t) - \widetilde{u})\|_{H^s} \le C\|(\rho(0) - \widetilde{\rho}, u(0) - \widetilde{u})\|_{H^s}$$

for all $t \geq 0$ *and*

$$\lim_{t\to\infty} \|\partial_x(\rho(t)-\widetilde{\rho}, u(t)-\widetilde{u})\|_{H^{s-1}} = 0,$$

provided that $\|(\rho(0)-\widetilde{\rho}, u(0)-\widetilde{u})\|_{H^s} \leq \varepsilon_0$. *In particular,*

$$\lim_{t\to\infty} \|(\rho(t)-\widetilde{\rho}, u(t)-\widetilde{u})\|_{\infty} = 0.$$

Remarks. (i) The stability of $(\widetilde{\rho},\widetilde{u})$ was firstly investigated in [4] and they proved Theorem 2.1 for $n = 1$, i.e., $(\widetilde{\rho},\widetilde{u})$ is stable with respect to small perturbations in the form $\rho - \widetilde{\rho} = \rho(x_1,t) - \widetilde{\rho}(x_1)$, $u - \widetilde{u} = (u^1(x_1,t) - \widetilde{u}^1(x_1), 0, \cdots, 0)$.

(ii) We here consider large time behavior of solutions of (1.1)–(1.2) only under the conditions for ρ_+, u_b^1 and u_+^1 given in Proposition 2.1. As is easily imagined, if one of these conditions would be disturbed, then complicated phenomena might occur. In fact, Matsumura [5] proposed a classification of all possible time asymptotic states in terms of boundary data for one-dimensional problem. Some parts of this classification were already proved rigorously. See [5].

3. Outline of the Proof

Let us rewrite the problem into the one for perturbations. We set $(\phi,\psi) = (\rho-\widetilde{\rho}, u-\widetilde{u})$. Then problem (1.1)–(1.2) is transformed into

$$\begin{aligned}
&\partial_t\phi + u\cdot\nabla\phi + \rho\,\mathrm{div}\,\psi = F,\\
&\rho(\partial_t\psi + u\cdot\nabla\psi) + L\psi + p'(\rho)\nabla\phi = G,\\
&\psi|_{x_1=0} = 0; \quad (\phi,\psi)\to(0,0) \quad (x_1\to\infty),\\
&(\phi,\psi)|_{t=0} = (\phi_0,\psi_0)
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&L\psi = -\mu\Delta\psi - (\mu+\mu')\nabla\mathrm{div}\,\psi,\\
&F = -\psi\cdot\nabla\widetilde{\rho} - \phi\,\mathrm{div}\,\widetilde{u},\\
&G = -(\rho\psi + \phi\widetilde{u})\cdot\nabla\widetilde{u} - (p'(\rho) - p'(\widetilde{\rho}))\nabla\widetilde{\rho}.
\end{aligned}$$

The proof of Theorem 2.1 is thus reduced to showing the global existence of solution (ϕ,ψ) of (3.1) in the class Z^s, where s is an integer satisfying $s \geq [n/2]+1$.

Let us firstly consider the local existence of solutions. The local existence can be proved by applying the result in [2]. In fact, problem (3.1) is a hyperbolic-parabolic system satisfying the assumptions in [2] that guarantees the local solvability in H^s for s satisfying $s \geq [n/2]+1$. Therefore, we obtain the following

Proposition 3.1. *Let s be an integer satisfying $s \geq s_0 = [\frac{n}{2}]+1$. Assume that the initial value (ϕ_0, ψ_0) satisfies the following conditions.*

(a) $(\phi_0, \psi_0) \in H^s$ *and* (ϕ_0, ψ_0) *satisfies the* $\widehat{s}$*-th order compatibility condition, where* $\widehat{s} = [\frac{s-1}{2}]$.

(b) $\inf_x \rho_0(x) \geq -\frac{1}{4} \inf_{x_1} \widetilde{\rho}(x_1)$.

Then there exists a positive number T_0 depending on $\|(\phi_0, \psi_0)\|_{H^s}$ and $\inf_{x_1} \widetilde{\rho}(x_1)$ such that problem (3.1) has a unique solution $(\phi, \psi) \in Z^s(T_0)$ satisfying $\phi(x,t) \geq -\frac{1}{2}\inf_{x_1} \widetilde{\rho}(x_1)$ for all $(x,t) \in \mathbf{R}^n_+ \times [0, T_0]$. Furthermore, there exist constants $C > 0$ and $\gamma > 0$ depending on s, $\|(\phi_0, \psi_0)\|_{H^s}$ and $\inf_{x_1} \widetilde{\rho}(x_1)$ such that

$$\|(\phi, \psi)\|^2_{Z^s(T_0)} \leq C\{1 + \|(\phi_0, \psi_0)\|^2_{H^s}\}^\gamma \|(\phi_0, \psi_0)\|^2_{H^s}.$$

We next derive a priori estimates to show the global existence of solution. We define $E_\sigma(t)$ and $D_\sigma(t)$ by

$$E_\sigma(t) = \left(\sup_{0 \leq \tau \leq t} \{ |[\psi(\tau)]|^2_\sigma + \|\phi(\tau)\|^2_{H^\sigma} + |[\partial_\tau \phi(\tau)]|^2_{\sigma-1} \} \right)^{1/2}$$

and

$$D_\sigma(t) = \begin{cases} \left(\int_0^t \|\partial_x \psi\|^2_2 + \|\phi|_{x_1=0}\|^2_{L^2(\mathbf{R}^{n-1})} \, d\tau \right)^{1/2} & \text{for } \sigma = 0, \\ \left(\int_0^t \|\partial_x \psi\|^2_{H^\sigma} + \|\phi|_{x_1=0}\|^2_{L^2(\mathbf{R}^{n-1})} \right. \\ \quad \left. + \|\partial_x \phi\|^2_{H^{\sigma-1}} + |[\partial_\tau \phi]|^2_{\sigma-1} + |[\partial_\tau \psi]|^2_{\sigma-1} \, d\tau \right)^{1/2} & \text{for } \sigma \geq 1. \end{cases}$$

In what follows we will denote the solution (ϕ, ψ) and the initial value (ϕ_0, ψ_0) by

$$U = (\phi, \psi), \quad U_0 = (\phi_0, \psi_0).$$

Theorem 2.2 follows from Proposition 3.1 and the following a priori estimate.

Proposition 3.2. *Let $U = (\phi, \psi)$ be a solution of (3.1) on $[0,T]$. Assume that $E_s(t) < 1$ for all $t \in [0,T]$. Then there exist constants $\varepsilon_0 > 0$ and $C > 0$, which are independent of $T > 0$, such that*

$$E_s(t)^2 + D_s(t)^2 \le C\|U_0\|_{H^s}^2$$

for all $t \in [0,T]$, provided that $\|U_0\|_{H^s} < \varepsilon_0$.

Outline of the proof of Proposition 3.2

As in the one-dimensional problem studied in [4], the point in the proof of Proposition 3.2 is to derive a suitable bound for the L^2 norm of (ϕ, ψ). Due to the fact that the stationary solution has no shear components, one can obtain the L^2 bound in the same way as in the one-dimensional case in [4].

Proposition 3.3. *There exists a constant $M > 0$ such that if*

$$E_s(t) \le M \tag{3.2}$$

for all $t \in [0,T]$, then

$$E_0(t)^2 + D_0(t)^2 \le C\{\|U_0\|_2^2 + R_0(t)^2\},$$

uniformly in $t \in [0,T]$, where $C > 0$ is independent of T and

$$R_0(t)^2 = -\int_0^t \{(\rho\psi\cdot\nabla\widetilde{u}, \psi) + ((p(\rho) - p(\widetilde{\rho}) - p'(\rho))\phi, \operatorname{div}\widetilde{u}) + (\frac{1}{\widetilde{\rho}}\phi L\widetilde{u}, \psi)\}\, d\tau.$$

Proof. As in [4], we introduce an energy functional based on the energy function defined by

$$\rho\mathcal{E} = \rho\{\frac{1}{2}|u|^2 + \Phi(\rho)\}, \quad \Phi(\rho) = \int^\rho \frac{p(\zeta)}{\zeta^2}\, d\zeta.$$

Note that $\Phi(\rho)$ is a strictly convex function of $\frac{1}{\rho}$. We then define

$$\rho\widetilde{\mathcal{E}} = \rho\{\frac{1}{2}|\psi|^2 + \Psi(\rho, \widetilde{\rho})\},$$

where

$$\begin{aligned}\Psi(\rho,\widetilde{\rho}) &= \Phi(\rho) - \Phi(\widetilde{\rho}) - \partial_{\frac{1}{\rho}}\Phi(\widetilde{\rho})\left(\frac{1}{\rho} - \frac{1}{\widetilde{\rho}}\right) \\ &= \int_{\widetilde{\rho}}^{\rho} \frac{p(\zeta) - p(\widetilde{\rho})}{\zeta^2}\, d\zeta.\end{aligned}$$

As shown in [4], $\rho\Psi(\rho,\widetilde{\rho})$ is equivalent to $|\rho-\widetilde{\rho}|^2$ for suitably small $|\rho-\widetilde{\rho}|$, and hence, there are positive constants c_0 and c_1 such that

$$c_0^{-1}|U| \le \rho\widetilde{\mathcal{E}} \le c_0|U|, \tag{3.3}$$

where $U=(\phi,\psi)$, $\phi=\rho-\widetilde{\rho}$ with $|\phi|\le c_1$.

Since $H^s \hookrightarrow L^\infty$ we can find a number $M>0$ such that if $E_s(t)\le M$, then $\|\phi(t)\|_\infty \le c_1$ and $\inf_x \phi(x,t) \ge -\frac{1}{4}\inf_{x_1}\widetilde{\rho}(x_1)$ for all $t\in[0,T]$.

A direct calculation shows

$$\begin{aligned} \partial_t(\rho\mathcal{E})+\operatorname{div}(\rho u\mathcal{E}+(p(\rho)-p(\widetilde{\rho}))\psi) &= \mu\operatorname{div}(\frac{1}{2}|\nabla\psi|^2)+(\mu+\mu')\operatorname{div}(\psi\operatorname{div}\psi) \\ &\quad -\mu|\nabla\psi|^2-(\mu+\mu')(\operatorname{div}\psi)^2+\mathcal{R}_0, \end{aligned}$$

where $\mathcal{R}_0=\mathcal{R}_0(x,t)$ is the function defined by

$$\mathcal{R}_0=-\rho(\psi\cdot\nabla\widetilde{u})\cdot\psi-(p(\rho)-p(\widetilde{\rho})-p'(\widetilde{\rho})\phi)\operatorname{div}\widetilde{u}-\frac{1}{\widetilde{\rho}}\phi\psi\cdot L\widetilde{u}.$$

Proposition 3.3 now follows from this identity and (3.3). This completes the proof.

To estimate higher order derivatives, we rewrite (3.1) as

$$\begin{aligned} &\partial_t\phi+u\cdot\nabla\phi+\rho_+\operatorname{div}\psi=f, \\ &\partial_t\psi+\tfrac{1}{\rho_+}L\psi+\tfrac{p'(\rho_+)}{\rho_+}\nabla\phi=g, \\ &\psi|_{x_1=0}=0, \\ &(\phi,\psi)\to(0,0)\quad(x_1\to\infty), \\ &(\phi,\psi)|_{t=0}=(\phi_0,\psi_0) \end{aligned} \tag{3.4}$$

where $L\psi=-\mu\Delta\psi-(\mu+\mu')\nabla\operatorname{div}\psi$, $f=\widehat{f}+\widetilde{f}$ and $g=-\widetilde{u}\cdot\nabla\psi+\widehat{g}+\widetilde{g}$. Here $\widehat{f}=-\phi\operatorname{div}\psi$, $\widetilde{f}=-(\widetilde{\rho}-\rho_+)\operatorname{div}\psi-\psi\cdot\nabla\widetilde{\rho}-\phi\operatorname{div}\widetilde{u}$, and $\widehat{g}=\widehat{g}^{(1)}+\widehat{g}^{(2)}+\widehat{g}^{(3)}$, $\widetilde{g}=\widetilde{g}^{(1)}+\widetilde{g}^{(2)}+\widetilde{g}^{(3)}$ with

$$\begin{aligned} &\widehat{g}^{(1)}=\widehat{P}(\rho,\rho_+)\phi\nabla\phi,\quad \widehat{g}^{(2)}=\tfrac{1}{\rho\rho_+}\phi L\psi,\quad \widehat{g}^{(3)}=-\psi\cdot\nabla\psi, \\ &\widetilde{g}^{(1)}=\widehat{P}(\rho,\rho_+)(\widetilde{\rho}-\rho_+)\nabla\phi+\widehat{P}(\rho,\widetilde{\rho})\phi\nabla\widetilde{\rho}, \\ &\widetilde{g}^{(2)}=\frac{1}{\rho\widetilde{\rho}}(L\widetilde{u})\phi+\frac{1}{\rho\rho_+}(\widetilde{\rho}-\rho_+)L\psi, \\ &\widetilde{g}^{(3)}=-\psi\cdot\nabla\widetilde{u}, \\ &P(\rho_1,\rho_2)=\textstyle\int_0^1 p''(\rho_2+\theta(\rho_1-\rho_2))\,d\theta,\quad \widehat{P}(\rho_1,\rho_2)=\tfrac{p'(\rho_1)}{\rho_1\rho_2}-\tfrac{P(\rho_1,\rho_2)}{\rho_2}. \end{aligned}$$

Before proceeding further, we introduce some notations. We define $N_\sigma \geq 0$ by

$$
\begin{aligned}
N_\sigma(t)^2 &= \int_0^t |[\widehat{f}\,]|_\sigma^2 + |[\widehat{g}]|_{\sigma-1}^2 + |[\psi \cdot \nabla\phi]|_{\sigma-1}^2 \, d\tau \\
&\quad + \sum_{1 \leq 2j+|\alpha'| \leq \sigma} \int_0^t |(\partial_\tau^j \partial_{x'}^{\alpha'} \widehat{g}, \partial_\tau^j \partial_{x'}^{\alpha'} \psi)| \, d\tau \\
&\quad + \sum_{1 \leq 2j+|\alpha| \leq \sigma} \int_0^t |(\operatorname{div} \psi, |\partial_\tau^j \partial_x^\alpha \phi|^2)| \, d\tau \\
&\quad + \sum_{2j+|\alpha| \leq \sigma} \int_0^t \|[\partial_\tau^j \partial_x^\alpha, \psi \cdot \nabla]\phi\|_2^2, d\tau,
\end{aligned}
$$

where $[C, D]$ denotes the commutator of C and D

$$
[C, D] = CD - DC.
$$

We also define $R_\sigma \geq 0$ ($\sigma \geq 1$) by

$$
\begin{aligned}
R_\sigma(t)^2 &= R_{\sigma-1}(t)^2 + \int_0^t |[\widetilde{f}\,]|_\sigma^2 + |[\widetilde{g}]|_{\sigma-1}^2 + |[\widetilde{u} \cdot \nabla\phi]|_{\sigma-1}^2 \, d\tau \\
&\quad + \sum_{1 \leq 2j+|\alpha'| \leq \sigma} \int_0^t |(\partial_\tau^j \partial_{x'}^{\alpha'} \widetilde{g}, \partial_\tau^j \partial_{x'}^{\alpha'} \psi)| \, d\tau \\
&\quad + \sum_{1 \leq 2j+|\alpha| \leq \sigma} \int_0^t |(\operatorname{div} \widetilde{u}, |\partial_\tau^j \partial_x^\alpha \phi|^2)| \, d\tau \\
&\quad + \sum_{2j+|\alpha|+\ell \leq \sigma-1} \int_0^t \|[\partial_\tau^j \partial_x^\alpha \partial_{x_1}^{\ell+1}, \widetilde{u} \cdot \nabla]\phi\|_2^2, d\tau,
\end{aligned}
$$

Proposition 3.4. *Let $1 \leq \sigma \leq s$. Assume that (3.2) holds. Then there exists a constant $C > 0$ such that*

$$
E_\sigma(t)^2 + D_\sigma(t)^2 \leq C\{\|U_0\|_{H^s}^2 + R_\sigma(t)^2 + N_\sigma(t)^2\}.
$$

To prove Proposition 3.4 we introduce a notation

$$
|v|_k = \left(\sum_{|\alpha|=k} \|\partial_x^\alpha v\|_2^2 \right)^{1/2}.
$$

We also define $T_{j,\alpha'}$ by

$$
T_{j,\alpha'} v = \partial_t^j \partial_{x'}^{\alpha'} v.
$$

Proposition 3.4 follows from the following inequalities.

Proposition 3.5. *Let σ be a nonnegative integer satisfying $\sigma \le s$.*

(i) *Let j and α' satisfy $2j + |\alpha'| = \sigma$. Then*

$$\|T_{j,\alpha'}U(t)\|_2^2 + \int_0^t \|L^{1/2}T_{j,\alpha'}\psi\|_2^2\,d\tau \le C\{\|U_0\|_{H^s}^2 + R_\sigma(t)^2 + N_\sigma(t^2)\},$$

where $\|L^{1/2}\psi\|_2^2 = \mu\|\nabla\psi\|_2^2 + (\mu+\mu')\|\mathrm{div}\,\psi\|_2^2$.

(ii) *Let j and α' satisfy $2j + |\alpha'| = \sigma - 1$. Then*

$$\|L^{1/2}T_{j,\alpha'}\psi(t)\|_2^2 + \int_0^t \|T_{j+1,\alpha'}\psi\|_2^2\,d\tau \le \eta D_\sigma(t)^2 + C_\eta \mathcal{N}_\sigma(t)^2$$

for any $\eta > 0$. Here and in what follows $\mathcal{N}_\sigma(t)^2$ denotes

$$\mathcal{N}_\sigma(t)^2 = \|U_0\|_{H^s}^2 + E_{\sigma-1}(t)^2 + D_{\sigma-1}(t)^2 + R_\sigma(t)^2 + N_\sigma(t^2).$$

(iii) *Let j and α' satisfy $2j + |\alpha'| + \ell = \sigma - 1$. Then*

$$\begin{aligned}&\|T_{j,\alpha'}\partial_{x_1}^{\ell+1}\phi(t)\|_2^2 + \int_0^t \|T_{j,\alpha'}\partial_{x_1}^{\ell+1}\phi\|_2^2\,d\tau \\ \le\ & \eta D_\sigma(t)^2 + C_\eta\{\mathcal{N}_\sigma(t)^2 + \int_0^t \|T_{j+1,\alpha'}\partial_{x_1}^{\ell}\psi\|_2^2 + \|\partial_x\partial_{x'}T_{j,\alpha'}\partial_{x_1}^{\ell}\psi\|_2^2\,d\tau\}\end{aligned}$$

for any $\eta > 0$.

(iv) *Let j and α' satisfy $2j + |\alpha'| + \ell = \sigma - 1$ and set $\dfrac{D\phi}{Dt} = \partial_t\phi + u\cdot\nabla\phi$. Then*

$$\int_0^t \Big|T_{j,\alpha'}\frac{D\phi}{Dt}\Big|_{\ell+1}^2 d\tau \le \eta D_\sigma(t)^2 + C_\eta\{\mathcal{N}_\sigma(t)^2 + \int_0^t \|T_{j+1,\alpha'}\partial_{x_1}^{\ell}\psi\|_2^2 + \|\partial_x\partial_{x'}T_{j,\alpha'}\partial_{x_1}^{\ell}\psi\|_2^2\,d\tau\}$$

for any $\eta > 0$.

(v) *Let j and α' satisfy $2j + |\alpha'| + \ell = \sigma - 1$. Then*

$$\begin{aligned}\int_0^t |T_{j,\alpha'}\psi|_{\ell+2}^2 + |T_{j,\alpha'}\phi|_{\ell+1}^2\,d\tau \ \le\ & C\int_0^t \{|T_{j+1,\alpha'}\psi|_\ell^2 + |T_{j,\alpha'}f|_{\ell+1}^2 + \Big|T_{j,\alpha'}\frac{D\phi}{Dt}\Big|_{\ell+1}^2 \\ & + |T_{j,\alpha'}(\widetilde{u}\cdot\nabla\psi)|_\ell^2 + |T_{j,\alpha'}\widehat{g}|_\ell^2 + |T_{j,\alpha'}\widetilde{g}|_\ell^2\}\,d\tau.\end{aligned}$$

(vi) *Let j and α' satisfy $2j + 1 \le \sigma$. Then*

$$\|\partial_t^{j+1}\phi(t)\|_2^2 + \int_0^t \|\partial_\tau^{j+1}\phi\|_2^2\,d\tau \le \eta D_\sigma(t)^2 + C_\eta\mathcal{N}_\sigma(t)^2$$

for any $\eta > 0$.

Proof. Proposition 3.5 can be proved by the energy method as in [1, 6]. The details can be found in [3].

It remains to estimate R_σ and N_σ. To estimate R_0 we will use a special case of Hardy's inequality

$$\left\| \frac{1}{x_1} \int_0^{x_1} v(y)\, dy \right\|_{L^2(0,\infty)} \le C\|v\|_{L^2(0,\infty)}. \tag{3.5}$$

In a similar manner as in [1, 4], applying (3.5) and the decay estimates in Proposition 2.1 together with the Gagliardo-Nirenberg inequality, one can show that

$$R_0(t)^2 \le C\{\delta D_0(t)^2 + E_s(t)D_s(t)^2\}.$$

Here we note that we also use the monotonicity of $\widetilde{u}^1(x_1)$ when $M_+ = 1$.

For $\sigma \ge 1$, one can show, as in [1], that

$$R_\sigma(t)^2 + N_\sigma(t)^2 \le C\{D_{\sigma-1}(t)^2 + \delta D_\sigma(t)^2 + E_s(t)D_s(t)^2\},$$

provided that $E_s(t) < \min\{M, 1\}$. Therefore, it follows that if δ is sufficiently small and $E_s(t) < \min\{M, 1\}$ then

$$E_s(t)^2 + D_s(t)^2 \le C\{\|U_0\|_{H^s}^2 + E_s(t)D_s(t)^2\},$$

and hence, we conclude that

$$E_s(t)^2 + D_s(t)^2 \le C\|U_0\|_{H^s}^2,$$

provided that $\|U_0\|_{H^s}$ is sufficiently small. This completes the proof of Proposition 3.2.

References

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