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About weak dissipations in Mathematicals models

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Abstract

In this paper we study models with weak dissipation, that is dissipations which are not able to produce exponential estability. Then our main task is to show the lack of exponential stability to fin a class of weak dissipations. Then after to show that the dissipation is weak we will find suitables norms and initial data, for which we show that the solution decays polynomially to zero.

1 Introduction

In this paper we study models with weak dissipation. For weak dissipations which mean dissipations that are not able to produce exponential estability. That is we study some dissipative models, then we show that the dissipation if weak by showing that there is no exponential decay of the energy. Then, when the lack of exponential decay is proved we look for a suitable norm and initial data, for which we get a polynomial stability. The main idea we use in this paper is to apply the following Theorem see [28].

Theorem 1. *Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(A) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - A)^{-1}\| < \infty$$

hold, where $\rho(A)$ is the resolvent set of A.

Alternatively we use also the energy method.

Therefore we will introduce some dissipative models, we will show that the dissipation is weak, then for an appropiate nor we will show that there exists a polynomial decay. The rest of the paper is organized as follows. In section 2 we consider system with frictional damping and we show that there no exist exponential stability but for apropriate norms in Hilbert spaces, there exists exponential stability. In section 3 we consider a model of the ionized atmosphere, which is a linear dissipative system of memory type. We prove as in section 2,that there is no exponential stability, but polynomial decay in appropiate Hilbert spaces. Finally, in section 4 and 5 we consider Models for acoustic waves and Magneto elasticity, in conum we have that we are not able to show wether exists or does not exist esponential stability. But using the energy methods we are able to show that there exists polynomial rate of decay in an appropriate norms.

2 Weak Dissipative system: Weak Frictional Damping

In colaboration with A. Pazoto

One simpler model of vibration with inertial term is given by

$$u_{tt} - u_{xxtt} - u_{xx} + u_t = 0,$$
$$u(0,t) = u(L,t) = 0$$
$$u(0) = u_0, u_t(0) = u_1$$

The above model was proposted by Love [9] (see page 429). Which can be rewriting in a general setting as

$$Cu_{tt} + Au + Bu_t = 0 \tag{2.1}$$
$$u(0) = u_0, u_t(0) = u_1 \tag{2.2}$$

where A, B and C are a self-adjoint positive definite operator with the domain $D(A) \subset D(C) \subset D(B)$ dense in a Hilbert space H. With this degree of generality we have

2.1 Asymptotic behaviour of the semigroup

We assume the existence of eigenfunctions λ_ν and eigenvectors w_ν, of the the operators $\tilde{A}$ and $\tilde{B}$ with unit norm in H satisfying

$$\begin{cases} Aw_\nu = \lambda_\nu w_\nu \\ Bw_\nu = f(\lambda_\nu)w_\nu, \quad \text{with} \quad f(\lambda) = o(\lambda^{\beta-\alpha}) \\ Cw_\nu = g(\lambda_\nu)w_\nu, \quad \text{with} \quad g(\lambda_\nu) = o(\lambda_\nu^\beta). \\ \lambda_\nu \to +\infty, \quad \text{and} \quad 0 \le \alpha, \beta < 1 \end{cases} \tag{2.3}$$

The following theorem descrives one of the main results of this paper

Theorem 2. *Let $S_B(t)$ be the C_0-semigroup of contractions generated by A_B, and*

$$E(t) = \frac{1}{2}||A^{1/2}u||_H^2 + \frac{1}{2}||C^{1/2}u_t||_H^2$$

the energy associated to (2.1). Then, if operators A and B satisfies (2.3), it follows that

(i) $S_B(t)$ is not exponentially stable; but

(ii) There exists a positive constant c; such that

$$E(t) \le \frac{c}{t}\, E_B(0), \quad \forall\, t > 0$$

where,

$$E_B(t) = \frac{1}{2}||L^{1/2}u_t||_H^2 + \frac{1}{2}||Q^{1/2}u||_H^2, \quad \forall\, t > 0$$

with $L = CB^{-1}C$ and $Q = CB^{-1}A$.

Proof: First we prove (i). To do that we make use of Theorem 1.

Let $F = (f,g) \in \mathcal{H}$, $U = (u,v)$ and consider the system

$$i\lambda U - A_B U = F$$

i.e.;

$$\begin{cases} i\lambda u - v = f \\ i\lambda v + C^{-1}Au + C^{-1}Bv = g \end{cases} \tag{2.4}$$

Solving for $f = 0$ and $g = w_\nu$

$$-\lambda^2 u + C^{-1}Au + i\lambda C^{-1}Bu = w_\nu.$$

Because of the boundary conditiond we can take

$$u = aw_\nu, \quad v = bw_\nu.$$

Then we have

$$\left(\underbrace{-\lambda^2 + \lambda_\nu^{-\beta+1}}_{:=0} + i\lambda\lambda_\nu^{-\alpha} \right) aw_\nu = w_\nu.$$

Therefore we get

$$\lambda = \lambda_\nu^{\frac{-\beta+1}{2}}, \quad u = -io(\lambda_\nu^{\alpha+\frac{\beta-1}{2}})w_\nu \quad \text{and} \quad v = o(\lambda_\nu^\alpha)w_\nu. \tag{2.5}$$

Now we claim that

$$||U||_{\mathcal{H}} \to +\infty.$$

Let us denote by $\tilde{A} = C^{-1}A$, and

$$A_B = \begin{pmatrix} 0 & I \\ C^{-1}A & C^{-1}B \end{pmatrix}$$

To prove the claim, note that

$$\begin{aligned} ||U||_{\mathcal{H}}^2 &= ||\tilde{A}^{1/2}u||_H^2 + ||v||_H^2 \\ &= ||o(\lambda_\nu^\alpha)w_\nu||_H^2 + ||C^{-1/2}A^{1/2}(-o(\lambda_\nu^{\alpha+\frac{\beta-1}{2}})w_\nu)||_H^2 \\ &= o(\lambda_\nu^{2\alpha})||w_\nu||_H^2 + ||-o(\lambda_\nu^{1/2-\frac{\beta}{2}})o(\lambda_\nu^{\alpha+\frac{\beta-1}{2}})w_\nu||_H^2 \\ &= 2\lambda_\nu^{2\alpha} \to +\infty. \end{aligned}$$

Recalling that

$$i\lambda U - A_B U = F \iff U = (i\lambda I - A_B)^{-1}F$$

it follows from Theorem 1, that $S_B(t)$ is not exponentially stable.

2.2 Polynomial decay

To prove (ii), first we observe that since

$$Cu_{tt} + Au + Bu_t = 0\ , \tag{2.6}$$

we have

$$CB^{-1}Cu_{tt} + CB^{-1}Au + Cu_t = 0. \tag{2.7}$$

Thus, taking the inner product of (2.7) with u_t we deduce that

$$\frac{dE_B}{dt} = -||C^{1/2}u_t||_H^2. \tag{2.8}$$

Now, if we take the inner product of (2.7) with u, and put

$$\phi(t) = (Cu, u_t)_H + \frac{1}{2}||B^{1/2}u||^2 \tag{2.9}$$

we get

$$\frac{d\phi}{dt} = ||C^{1/2}u_t||_H^2 - ||A^{1/2}u||_H^2. \tag{2.10}$$

Consequently, putting (2.7) and (2.9) together yields

$$\begin{aligned} \frac{d}{dt}\{E_B(t) + \varepsilon\phi(t)\} &= -||C^{1/2}u_t||_H^2 + \varepsilon||C^{1/2}u_t||_H^2 - \varepsilon||A^{1/2}u||_H^2 \\ &= -(1-\varepsilon)||C^{1/2}u_t||_H^2 - \varepsilon||A^{1/2}u||_H^2 \\ &\leq -\gamma_0 E(t) \end{aligned} \tag{2.11}$$

for some $\gamma_0 > 0$, since $0 < \varepsilon < 1$. Integrating (2.10) from 0 to t, we obtain

$$E_B(t) + \varepsilon\phi(t) + \gamma_0 \int_0^t E(s)ds \leq E_B(0) + \varepsilon\phi(0), \quad \forall t > 0 ,$$

and conclude that

$$\int_0^{+\infty} E(s)ds \leq cE_B(0) , \tag{2.12}$$

for some positive constant c. Finally, we have

$$\frac{d}{dt}\{tE(t)\} = E(t) + t\,\frac{dE}{dt}\,(t) \leq E(t) \tag{2.13}$$

and from (2.8), we obtain after integrating (2.13) that

$$tE(t) \leq \int_0^{+\infty} E(s)ds \leq cE_B(0) ,$$

i.e.,

$$E(t) \leq \frac{c}{t}\, E_B(0) .$$

This complestes the proof.

Some Examples

Plates

$$\begin{aligned} &\rho u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + \alpha u_t = 0, \quad \text{in} \quad \Omega\times]0,\infty[\\ &u = \Delta u = 0 \quad \text{in} \quad \partial\Omega\times]0,\infty[\\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \end{aligned}$$

where ρ, γ and α are positve constants

Abstract wave equation

Let H be a Hilbert space, A a positive selfadjint operator with $D(A) \subset H$ compact.

$$\begin{aligned} &\rho u_{tt} + Au + A^{-\alpha}u_t = 0, \quad \text{in} \quad L^2(0,\infty;H) \\ &u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in} \quad H. \end{aligned}$$

3 The model of the ionized atmosphere: polynomial decay

In colaboration with M.G Naso, E. Vuk (Brescia - Italy)
The simplified model is given by

$$\left\{\begin{array}{l} \boldsymbol{E}_{tt}(t) - \Delta \boldsymbol{E}(t) + \alpha_0 \boldsymbol{E}(t) - (\alpha * \boldsymbol{E})(t) = \mathbf{0} \\ \nabla \cdot \boldsymbol{E}(t) = 0 \,. \\ \boldsymbol{E} \times \boldsymbol{n} = \mathbf{0}, \quad \text{on} \quad \partial\Omega \end{array}\right. \tag{3.14}$$

Together with the thermodynamic restrictions, we assume that the kernel $\alpha \in C^2(\mathbb{R}^+) \cap W^{2,1}(\mathbb{R}^+)$ satisfies the following set of hypotheses

$$-c_0\,\alpha(s) \leq \alpha'(s) \leq -c_1\,\alpha(s)\,, \quad \forall s \in \mathbb{R}^+, \quad \alpha(s) > 0 \tag{3.15}$$

$$\breve{\alpha}(t) := \alpha_0 - \int_0^t \alpha(\tau)\,d\tau > 0\,, \quad \forall t \in \mathbb{R}^+ \tag{3.16}$$

$$|\alpha''(s)| \leq c_2\,\alpha(s)\,, \quad \forall s \in \mathbb{R}^+ \tag{3.17}$$

with c_i, $i = 0, 1, 2$ positive constants.
The set of auxiliar functionals

$$\begin{aligned} \mathcal{E}_1(t) &= \frac{1}{2}\int_\Omega \left[|\boldsymbol{E}_t|^2 + |\nabla \times \boldsymbol{E}|^2 + \breve{\alpha}\,|\boldsymbol{E}|^2 + (\alpha \Box \boldsymbol{E})\right] dx \\ \mathcal{E}_2(t) &= \frac{1}{2}\int_\Omega |\nabla \times \boldsymbol{E}_t|^2 + |\Delta \boldsymbol{E}|^2 + \breve{\alpha}|\nabla \times \boldsymbol{E}|^2 + (\alpha \Box \nabla \times \boldsymbol{E})dx \\ \mathcal{F}(t) &= \frac{1}{2}\int_\Omega |(\alpha * \boldsymbol{E})(t)|^2\,dx - \int_\Omega \boldsymbol{E}_t(t) \cdot (\alpha * \boldsymbol{E})_t(t)\,dx \\ \mathcal{K}(t) &= \int_\Omega \boldsymbol{E}_t(t) \cdot \boldsymbol{E}(t)\,dx \\ \mathcal{L}(t) &= N\,\mathcal{E}_1(t) + N\,\mathcal{E}_2(t) + \mathcal{F}(t) + \frac{\alpha(0)}{4}\mathcal{K}(t)\,, \end{aligned}$$

with $N > 0$. The main result of this section is given by

Theorem 3. *Let us suppose that* $\alpha \in C^2(\mathbb{R}^+) \cap W^{2,1}(\mathbb{R}^+)$. *Then there exists a positive constant* C *such that*

$$\mathcal{E}_1(t) \leq \frac{C\,[\mathcal{E}_1(0) + \mathcal{E}_2(0)]}{t}\,.$$

Proof.- there exists a positive constant γ_0 such that

$$\frac{d}{dt}\mathcal{L}(t) \;\; \leq \;\; -\gamma_0 \mathcal{E}_1(t)\,.$$

By an integration with respect to t, we get

$$\mathcal{L}(t) - \mathcal{L}(0) + \gamma_0 \int_0^t \mathcal{E}_1(\tau)\,d\tau \leq 0\,.$$

Because of $\frac{d}{dt}\mathcal{L}(t) \leq 0$, we observe that $\mathcal{L}(t) \leq \mathcal{L}(0)$, for any $t > 0$. Then, we have

$$\int_0^t \mathcal{E}_1(\tau)\,d\tau \leq \mathcal{L}(0)\,, \quad \forall t > 0\,.$$

Since $\dfrac{d}{dt}\mathcal{E}_1(t) \le 0$.

$$\frac{d}{dt}\left[t\,\mathcal{E}_1(t)\right] = \mathcal{E}_1(t) + t\,\frac{d}{dt}\mathcal{E}_1(t) \le \mathcal{E}_1(t)\,, \quad \forall t > 0\,.$$

By an integration with respect to t, we find

$$t\,\mathcal{E}_1(t) \le \int_0^t \mathcal{E}_1(\tau)\,d\tau \le \mathcal{L}(0)\,, \quad \forall t > 0\,,$$

and it follows that there exists a positive constant C such that

$$\mathcal{E}_1(t) \le \frac{C\left[\mathcal{E}_1(0) + \mathcal{E}_2(0)\right]}{t}\,.$$

Remark 1. *Note that the polynomial decay obtained in Theorem 3 is not in the same norm for the solution and the initial data. Then, it is not possible to get exponential decay using the semigroup property.*

3.1 Non-exponential stability

We use that a C_0-semigroup $e^{\mathcal{A}t}$ in a Hilbert space X is exponentially stable if and only if $i\,\mathbb{R} \subset \rho(\mathcal{A})$ and there exists $M \ge 1$ such that $\|(i\,\lambda\,I - \mathcal{A})^{-1}\| < M$, $\forall \lambda \in \mathbb{R}$, where $\mathcal{A}$ is its generator.

We consider the spectrum of the "$\nabla\times\nabla\times$" operator with homogeneous Dirichlet boundary conditions, namely

$$\begin{cases} \nabla\times\nabla\times \boldsymbol{e}_\nu = \lambda_\nu\,\boldsymbol{e}_\nu & \text{in } \Omega \\ \nabla\cdot\boldsymbol{e}_\nu = 0 & \text{in } \Omega \\ \boldsymbol{e}_\nu \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega \end{cases} \qquad \text{and} \quad \|e_\nu\|_{L^2(\Omega)} = 1\,,\ \nu \ge 1$$

where $(\lambda_\nu)_{\nu\ge 1} \to \infty$. To use the semigroup approach we put

$$\eta^t(s) = \boldsymbol{E}(t) - \boldsymbol{E}(t-s) \tag{3.18}$$

where η^t represents the *relative electric field history*.

$$\begin{cases} \boldsymbol{E}_{tt} + \nabla\times\nabla\times\boldsymbol{E} + \widehat{\alpha}\,\boldsymbol{E} + \displaystyle\int_0^\infty \alpha(s)\,\eta(s)\,ds = \boldsymbol{0} \\ \eta_t - \boldsymbol{E}_t + \eta_s = \boldsymbol{0}\,, \end{cases} \tag{3.19}$$

where $\widehat{\alpha} := \alpha_0 - \displaystyle\int_0^\infty \alpha(\tau)\,d\tau$. Let $L^2_\alpha(\mathbb{R}^+, V(\Omega))$ be the α-weighted L^2 spaces of functions on $\mathbb{R}^+$ with values in $V(\Omega)$ endowed with the inner product

$$\langle \eta_1, \eta_2\rangle_\alpha := \int_0^\infty \alpha(s)\langle \eta_1(s), \eta_2(s)\rangle\,ds\,.$$

Finally, let us introduce the Hilbert space

$$Z := H^1_{t0}(\Omega) \times V(\Omega) \times L^2_\alpha(\mathbb{R}^+, V(\Omega))\,.$$

Setting $\boldsymbol{v} := \boldsymbol{E}_t$ and

$$U(t) := [\boldsymbol{E}(t), \boldsymbol{v}(t), \eta(t)]^\top, \; U_0 := [\boldsymbol{E}_0, \boldsymbol{v}_0, \eta_0]^\top \in Z$$

problem (3.19 can be rewritten as an abstract linear evolution equation in the Hilbert space Z of the form

$$\begin{cases} U_t(t) = \mathcal{A}\, U(t) \\ U(0) = U_0 \,. \end{cases} \tag{3.20}$$

The operator $\mathcal{A}$ is defined as

$$\mathcal{A}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{v} \\ \eta \end{pmatrix} = \begin{pmatrix} \boldsymbol{v} \\ -\nabla \times \nabla \times \boldsymbol{E} - \widehat{\alpha}\,\boldsymbol{E} - \displaystyle\int_0^\infty \alpha(s)\,\eta(s)\,ds \\ \boldsymbol{v} - \eta_s \end{pmatrix},$$

with domain

$$D(\mathcal{A}) := \Big\{ U \in Z : \; \mathcal{A}U \in Z, \quad \int_0^\infty \alpha(s)\,\eta(s)\,ds \in V(\Omega), \\ \eta_s \in L^2_\alpha(\mathbb{R}^+, V(\Omega)), \quad \eta(0) = \boldsymbol{0} \Big\}.$$

Theorem 4. *Under the above notation, the semigroup associated with the above system is not exponentially stable.*

Proof. Let $F = [\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3]^\top \in Z$ and consider the following equation

$$(i\,\beta\, I - \mathcal{A})U = F, \quad \beta \in \mathbb{R}$$

which in components reads

$$\begin{aligned} & i\,\beta\,\boldsymbol{E} - \boldsymbol{v} = \boldsymbol{F}_1 \\ & i\,\beta\,\boldsymbol{v} + \nabla \times \nabla \times \boldsymbol{E} + \widehat{\alpha}\,\boldsymbol{E} + \int_0^\infty \alpha(s)\,\eta(s)\,ds = \boldsymbol{F}_2 \\ & i\,\beta\,\eta - \boldsymbol{v} + \eta_s = \boldsymbol{F}_3 \,. \end{aligned} \tag{3.21}$$

Setting

$$\boldsymbol{F}_1 = \boldsymbol{0}, \quad \boldsymbol{F}_2 = \boldsymbol{0}, \quad \boldsymbol{F}_3 = \beta_\nu^{\frac{1}{2}}\, e^{-\beta_\nu s}\, \boldsymbol{e}_\nu$$

we look for solutions of the type

$$\boldsymbol{E} = A\,\boldsymbol{e}_\nu, \quad \boldsymbol{v} = B\,\boldsymbol{e}_\nu, \quad \eta(s) = \varphi(s)\,\boldsymbol{e}_\nu,$$

solving the above system for $\beta^2 = \lambda_\nu + a$ we find

$$\begin{aligned} & B = i\beta A, \\ & (\widehat{\alpha} - a)\,A = -\int_0^\infty \alpha(s)\,\varphi(s)\,ds \,. \end{aligned} \tag{3.22}$$

Denoting $\beta = \sqrt{\lambda_\nu + a} =: \beta_\nu$. So we have that

$$\varphi(s) = C\,e^{-i\beta_\nu s} + A + \frac{1}{i-1}\,\beta_\nu^{-\frac{1}{2}} e^{-\beta_\nu s}. \tag{3.23}$$

By the initial data we have $\eta(0) = \mathbf{0}$. Then,

$$C = -A - \frac{1}{i-1}\beta_\nu^{-\frac{1}{2}}$$

and (3.23 becomes

$$\varphi(s) = \left(-A - \frac{1}{i-1}\beta_\nu^{-\frac{1}{2}}\right) e^{-i\beta_\nu s} + A + \frac{1}{i-1}\beta_\nu^{-\frac{1}{2}} e^{-\beta_\nu s}. \tag{3.24}$$

Taking

$$\alpha(s) = e^{-\gamma s}, \quad \gamma \in \mathbb{R}^+,$$

we find

$$A \approx C\,\beta_\nu^{-\frac{1}{2}} \approx C\,\lambda_\nu^{-\frac{1}{4}}, \ \textit{as}\ \lambda_\nu \to \infty.$$

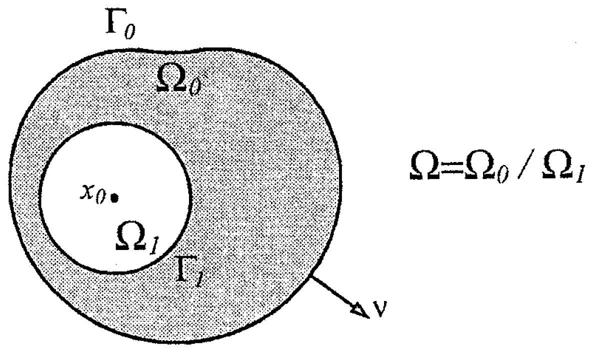
Finally, recalling that $\boldsymbol{E} = A\,e_\nu$,

$$\|\boldsymbol{E}\|_{H^1_{t0}(\Omega)} \approx \lambda_\nu^{\frac{1}{4}} \to \infty\,, \text{ as } \lambda_\nu \to \infty\,.$$

Then, the solution of system can not decay exponentially. □

4 Model with acustic boundary condition

In colaboration with Yuming Qin (Shangai - China)



We consider the model

$$\phi_{tt} = c^2\Delta\phi \quad \text{in} \quad \Omega$$

where c is the speed of sound in the medium. We assume that the boundary $\partial\Omega = \Gamma$ is divided into two parts,

$$\Gamma = \Gamma_0 \cup \Gamma_1$$

such that

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset, \Gamma_0 \neq \emptyset.$$

We assume that

- Each point reacts to excess pressure of the acoustic wave like a resistive harmonic oscillator.

- Different parts of the boundary Γ do not influence each other, that is, the surface is *locally reacting* but subject to small oscillations.

Then the normal displacement δ of Γ_0 into the domain satisfies an equation of the form

$$m(x)\delta_{tt}(x,t) + d(x)\delta_t(x,t) + k(x)\delta(x,t) = -\rho\phi_t(x,t) \quad \text{in} \quad \Gamma_0$$

where ρ is the density of the fluid, m, d and k are mass per unit area, resistivity and spring constant on Γ_0, respectively. If we also assume that Γ_0 is impenetrable, we obtain a third equation from the continuity of the velocity at the boundary Γ_0

$$\delta_t(x,t) = \frac{\partial\phi(x,t)}{\partial\nu} \quad \text{in} \quad \Gamma_0$$

where $\frac{\partial\phi(x,t)}{\partial\nu} = \nabla\phi(x,t)\cdot\nu$ denotes the outward normal velocity at $x \in \Gamma_0$ and $\nu = \nu(x)$ stands for the outward normal vector at $x \in \Gamma$.

We assume that Γ_1 is rigid and on it ϕ satisfies Dirichlet boundary condition, that is,

$$\phi(x,t) = 0 \quad \text{in} \quad \Gamma_1.$$

For more details on the model we refer to [1-3] and [11]. Moreover, we assume that there is a point $x_0 \in \mathbb{R}^3$ such that

$$\begin{aligned}
\Gamma_1 &= \{x \in \Gamma | (x - x_0)\cdot\nu(x) \leq 0\}, \\
\Gamma_0 &= \{x \in \Gamma | (x - x_0)\cdot\nu(x) \geq a > 0\}
\end{aligned}$$

for some constant $a > 0$.

Additionally, we prescribe the initial conditions

$$\begin{aligned}
\phi(x,0) &= \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) \qquad \forall x \in \Omega, && (4.25) \\
\delta(x,0) &= \delta_0(x), \quad \delta_t(x,0) = \delta_1(x) \qquad \forall x \in \Gamma_0. && (4.26)
\end{aligned}$$

We assume that $m(x), d(x)$ and $k(x)$ are positive smooth and bounded functions on Γ_0

4.1 The semigroup approach

Let us define the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_0)$$

with

$$H^1_{\Gamma_1}(\Omega) = \{u : u \in H^1(\Omega), u|_{\Gamma_1} = 0\}.$$

with the inner product

$$< \mathbf{u}, \mathbf{w} >= \int_\Omega (\rho \nabla u_1 \cdot \nabla w_1 + \rho c^{-2} u_2 w_2) dx + \int_{\Gamma_0} (k u_3 w_3 + m u_4 w_4) ds$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)^T, \mathbf{w} = (w_1, w_2, w_3, w_4)^T \in \mathcal{H}$.
The induced norm on $\mathcal{H}$ is

$$|u|^2_{\mathcal{H}} = \int_\Omega (\rho|\nabla u_1|^2 + \rho c^{-2}|u_2|^2) dx + \int_{\Gamma_0} (k|u_3|^2 + m|u_4|^2) ds$$

for any $\mathbf{u} = (u_1, u_2, u_3, u_4)^\tau \in \mathcal{H}$.

Next we define an operator $\mathcal{A}$ on $\mathcal{H}$ so that for smooth $\mathbf{u} = (\phi, \phi_t, \delta, \delta_t)^\tau$, (1.1)-(1.4) are equivalent to $\mathbf{u}(t) \in D(\mathcal{A})$ and $\mathbf{u}_t = \mathcal{A}\mathbf{u}$. We define

$$\mathcal{A}\mathbf{u} = [u_2, c^2 \triangle u_1, u_4, -m^{-1}(\rho u_2 + ku_3 + du_4)]^\tau$$

for $\mathbf{u} = (u_1, u_2, u_3, u_4)^\tau \in D(\mathcal{A})$, where

$$D(\mathcal{A}) = \{\mathbf{u} : \Delta u_1 \in L^2(\Omega), u_2 \in H^1_{\Gamma_1}(\Omega), \frac{\partial u_1}{\partial \nu} = u_4\}.$$

Here u_2 in the last component of $\mathcal{A}\mathbf{u}$ is understood as the trace in $H^{1/2}(\Gamma_0)$ and $\frac{\partial u_1}{\partial \nu} = u_4$ is meant in the weak sense

$$\int_\Omega [(\Delta u_1)\psi + \nabla u_1 \cdot \nabla \psi] dx = \int_{\Gamma_0} u_4 \psi ds, \qquad \forall \psi \in H^1_{\Gamma_1}(\Omega).$$

The relation $u_1 \in H^2(\Omega)$ is equivalent to the condition that u_4 is the normal derivative of u_1 as a trace. Thus, similar to the proofs in [1-3], we readily obtain the following results on the global existence and regularity of solutions.

Theorem 5. *$\mathcal{A}$ is closed, densely defined, and dissipative. It is the generator of a C_0-semigroup. If $d \equiv 0, \mathcal{A}$ is skew-adjoint and generates a unitary group.*

Theorem 6. *Assume that $\mathbf{u}_0 \in \mathcal{H}$ is C^∞ and vanishes near $\partial\Omega$; let $\mathbf{u}(t)$ be the solution of $\mathbf{u}'(t) = \mathcal{A}\mathbf{u}(t), t \geq 0$, with $\mathbf{u}(0) = \mathbf{u}_0$. Then $u_1(t), u_2(t) \in C^\infty(\bar{\Omega})$ and $u_3(t), u_4(t) \in C^\infty(\Gamma_0)$ for any $t \geq 0$.*

Let us define energy functions

$$\begin{aligned} E_0(t; \phi, \delta) &= \frac{1}{2}\int_\Omega (\rho |\nabla \phi|^2 + \rho c^{-2} \phi_t^2) dx + \frac{1}{2}\int_{\Gamma_0} (k(x)\delta^2 + m(x)\delta_t^2) ds, \\ E_j(t) &\equiv E_j(t; \phi, \delta) = E_0(t; \partial_t^j \phi, \partial_t^j \delta), \quad j = 1, 2, \cdots \end{aligned}$$

The novelty of this paper is the following results on the asymptotic behaviour of solutions.

Theorem 7. *Under the above assumptions and with smooth initial data $(\phi_0, \phi_1, \delta_0, \delta_1)$ such that*

$$\sum_{j=0}^{k+1} E_j(0) < \infty \tag{4.27}$$

for $k \geq 0$. Then there is a positive constant $\hat{C}$ such that

$$\sum_{j=0}^{k} E_j(t) \leq \frac{\hat{C}}{t} \sum_{j=0}^{k+1} E_j(0), \qquad \forall t > 0. \tag{4.28}$$

4.2 Energy Estimates

We use multiplicative techniques to establish some energy estimates.

By (1.1)-(1.6) and Green's formula, it is not hard to verify

$$\frac{d}{dt}E_0(t;\phi,\delta) = -\int_{\Gamma_0} d(x)\ \delta_t^2 ds. \tag{4.1}$$

Similarly, noting Eq. (1.1) and boundary conditions (1.2)-(1.3) are all linear, we have that for $j = 0, 1, \cdots, k+1$,

$$\frac{d}{dt}E_j(t;\phi,\delta) = -\int_{\Gamma_0} d(x)\ |\partial_t^{j+1}\delta|^2 ds. \tag{4.2}$$

Define

$$\begin{aligned} q(x) &= x - x_0, \\ F_0(t;\phi,\delta) &= \int_\Omega (\phi_t\ q\cdot\nabla\phi + \phi\ \phi_t)dx, &(4.3)\\ F_j(t) &\equiv F_j(t;\phi,\delta) \equiv F_0(t;\partial_t^j\phi,\partial_t^j\delta), \quad j = 1, 2, \cdots, k. &(4.4)\end{aligned}$$

Under the above notations, we have

Lemma 2. *For $j = 0, 1, \cdots, k$, we obtain the following identity*

$$\begin{aligned} \frac{d}{dt}F_j(t;\phi,\delta) &= -\frac{1}{2}\int_\Omega(|\partial_t^{j+1}\phi|^2 + c^2|\nabla\partial_t^j\phi|^2)dx \\ &\quad -\frac{c^2}{2}\int_{\Gamma_0} q\cdot\nu\,|\nabla\partial_t^j\phi|^2 ds \\ &\quad +c^2\int_{\Gamma_0}\partial_t^{j+1}\delta\ q\cdot\nabla\partial_t^j\phi ds + \frac{c^2}{2}\int_{\Gamma_1} q\cdot\nu\,|\nabla\partial_t^j\phi|^2 ds \\ &\quad +\frac{1}{2}\int_{\Gamma_0} q\cdot\nu\,|\partial_t^{j+1}\phi|^2 ds + c^2\int_{\Gamma_0}\partial_t^{j+1}\delta\ \partial_t^j\phi ds. \end{aligned}$$

Lemma 3. *For $j = 0, 1, 2, \cdots, k$, we obtain*

$$\frac{d}{dt}G_j(t;\phi,\delta) = -\int_{\Gamma_0} k(x)\ |\partial_t^j\delta|^2 ds + \int_{\Gamma_0}[m(x)|\partial_t^{j+1}\delta|^2 + \rho\partial_t^j\phi\ \partial_t^{j+1}\delta]ds. \tag{4.5}$$

Now we define the following Liapunov functional

$$\mathcal{L}_k(t) = N^2\sum_{j=0}^{k+1}E_j(t) + N^{1/2}\sum_{j=0}^{k}G_j(t) + \sum_{j=0}^{k}F_j(t)$$

where N is a large positive number specified later on.

Lemma 4. *For N large enough, there are positive constants C_0, C_1, C_2 and C_3 such that*

$$0 \le C_0\sum_{j=0}^{k+1}E_j(t) \le \mathcal{L}_k(t) \le C_1\sum_{j=0}^{k+1}E_j(t), \quad \forall t \ge 0 \tag{4.6}$$

and

$$\frac{d}{dt}\mathcal{L}_k(t) \le -C_3\sum_{j=0}^{k}E_j(t), \quad \forall t > 0.$$

Based on the estimates obtained above, we are able to finish the proof of Theorem 1.3.
Proof of Theorem 1.3: By (1.12) and Lemma 2.3, we get

$$\int_0^t \sum_{j=0}^{k} E_j(\tau)d\tau \leq C_3^{-1}(\mathcal{L}_k(0) - \mathcal{L}_k(t)) \leq C_3^{-1}C_1\sum_{j=0}^{k+1} E_j(0) < \infty. \tag{4.7}$$

Clearly, we get for any $t > 0$

$$\frac{d}{dt}[t\sum_{j=0}^{k} E_j(t)] = \sum_{j=0}^{k} E_j(t) + t\sum_{j=0}^{k}\frac{d}{dt}E_j(t) \leq \sum_{j=0}^{k} E_j(t)$$

whence,

$$\sum_{j=0}^{k} E_j(t) \leq \frac{\hat{C}}{t}\sum_{j=0}^{k+1} E_j(0) \tag{4.8}$$

with $\hat{C} = C_3^{-1}C_1$. The proof of Theorem 1.3 is now complete.

5 Magneto Elasticity

In colaboration with R. Racke (Konstanz-Germany)
and
M. Santos (Belem-Brazil)

Let $\Omega \subset \mathbb{R}^2$, the the displacement vector $u = (u^1, u^2, 0)' = u(t,x)$ depending on the time variable $t \geq 0$ and on the space variable $x \in \Omega$, and for the magnetic field $h = (h^1, h^2, 0)' = h(t,x)$ are:

$$\begin{aligned} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u - \alpha[\nabla \times h]\times \vec{H} &= 0, \\ h_t - \Delta h - \beta\nabla \times [u_t \times \vec{H}] &= 0, \end{aligned}$$

Here λ, μ and κ are positive constants. The coupling constants α, β satisfy $\alpha\beta > 0$. $\vec{H} = (H, 0, 0)'$ is a constant vector with $H \neq 0$.

Additionally, one has initial conditions

$$u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; h(0,x) = h_0(x) \tag{5.9}$$

and the following classical Dirichlet-type boundary conditions

$$u = 0, \quad \nu \times (\nabla \times h) = 0, \;\; \nu \cdot h = 0 \quad \text{on } \Gamma := \partial\Omega; \tag{5.10}$$

moreover,

$$\operatorname{div} h = 0, \tag{5.11}$$

which follows from (5.9) if div $h_0 = 0$. The vector $\nu = (\nu_1, \nu_2, 0)' = \nu(x)$ denotes the exterior normal vector in $x \in \Gamma$ the boundary of Ω.

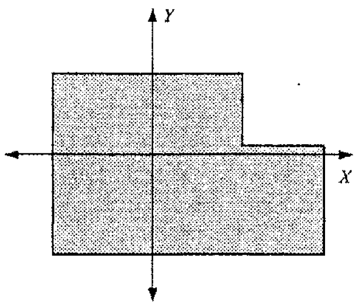


Figure Type I

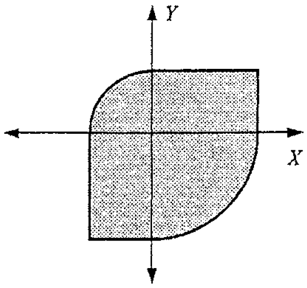


Figure Type II

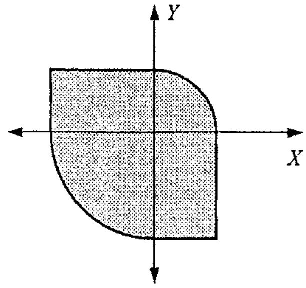


Figure Type III

We show that the solution of the magnetoelastic system decays polynomially as time goes to infinity, provided Ω is of one of the following type

I: Ω is the union of finitely many rectangles with axes paralles to the x_1- and x_2-axes, respectively, see Figure 1.1.

II: Ω satisfies $\nu_1\nu_2 = 0$ in the first quadrant (where $x_1 \geq 0$ and $x_2 \geq 0$) and in the third quadrant (where $x_1 \leq 0$ and $x_2 \leq 0$). In the second and fourth quadrant Ω satisfies $x\nu \geq \alpha_0 > 0$, for some α_0.

III: Ω satisfies $\nu_1\nu_2 = 0$ in the second and fourth quadrant. In the first and third quadrant Ω satisfies $x\nu \geq \alpha_0 > 0$, for some α_0, see Figure 1.3.

By domains of partial rectangular type *I*, all sufficiently smoothly bounded, connected domains can be exhausted, also all connected Jordan measurable sets.

The energy $E = E(t)$(of first order) associated to the equations (5.9), (5.9) is given by

$$E(t; u, h) := \frac{1}{2}\int\limits_{\Omega}\left(|u_t|^2 + \mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2 + \frac{\alpha}{\beta}|h|^2\right)(t,x)dx. \tag{5.12}$$

We shall also use energy terms of higher order given for $j \in \mathbb{N}$ by

$$E_j(t) := E(t; \partial_t^j u, \partial_t^j h). \tag{5.13}$$

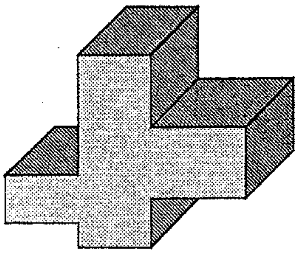
Then it will be proved that the energy $E(t)$ decays like t^{-1}. More precisely, the main theorem is the following

Theorem 8. *Let (u, h) be the solution to the initial boundary value problem (5.9)-(5.11). Then the energy E defined in (5.12) decays polynomially,*

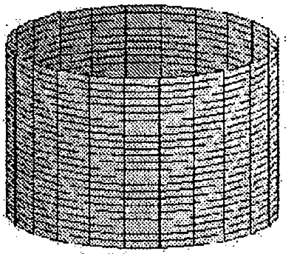
$$\exists d > 0 \quad \forall t \geq 0: \quad E(t) \leq \frac{d}{t}\sum_{j=0}^{7} E_j(0).$$

This result presents a polynomial decay that is uniform with respect to initial data but involves derivatives at time $t = 0$ higher than those estimated for $t > 0$. Indeed, it is open whether there is a uniform exponential decay of the associated semigroup, and our calculations do not assist this possibility.
The method we use is an energy method, looking for appropriate multipliers and Lyapunov functionals.

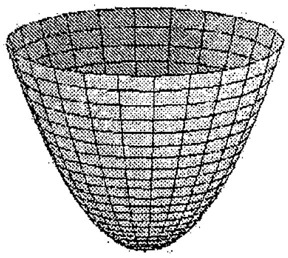
3-Dimensinal Magneto Elasticity



Solid of type I



Solid of type II



Solid of type III

Remark 5. *For solid of paraboloid type we have that the lateral surface* Γ *is defined by the equation* $x_3 = x_1^2 + x_2^2$, *and the top surface* Γ_0 *is given by the plane* $x_3 = 1$. *In this case we have that*

$$\nu = (0,0,1) \quad on \quad \Gamma_0, \qquad \nu = \frac{1}{R}(2x_1, 2x_2, -1) \quad on \quad \Gamma,$$

where $R = \sqrt{4x_1^2 + 4x_2^2 + 1}$. *Let us take* $x_1^0 = 0$ *and* $x_3^0 > 1/2$. *Under this conditions we have that the function* I_1 *defined in section 1, satisfies*

$$I_1 = -\frac{R^2-1}{R^3}\left\{-\frac{x_1^2}{2x_1^2+2x_2^2} + x_3 + x_3^0\right\} \leq 0,$$

which is an important inequality to get the decay.

Remark 6. *In general we have*

$$\frac{d}{dt}E_j(t) = -\frac{\alpha}{\beta}\int_\Omega |\nabla \times \partial_t^j h|^2 dx.$$

For $j \geq 0$. *Using the fact that* Ω *is simply connected, we conclude that*

$$\int_\Omega |h|^2 dx \leq c \int_\Omega |\nabla \times h|^2 dx$$

hence

$$\frac{d}{dt}E_j(t) \leq -c\{\int_\Omega |\partial_t^j h|^2 dx + \int_\Omega |\nabla \times h|^2 dx\}. \tag{5.14}$$

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