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Perfectness and Multicoloring of Unit Disk Graphs on Triangular Lattice Points

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概要

Given a pair of non-negative integers m and n, $P(m,n)$ denotes a subset of 2-dimensional triangular lattice points defined by $P(m,n) \stackrel{\text{def.}}{=} \{(x\boldsymbol{e}_1+y\boldsymbol{e}_2) \mid x \in \{0,1,\ldots,m-1\},\ y \in \{0,1,\ldots,n-1\}\}$ where $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$, $\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2,\sqrt{3}/2)$. Let $T_{m,n}(d)$ be an undirected graph defined on vertex set $P(m,n)$ satisfying that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d. In this paper, we discuss a necessary and sufficient condition that $T_{m,n}(d)$ is perfect. More precisely, we show that [$\forall m \in \mathrm{Z}_+$, $T_{m,n}(d)$ is perfect] if and only if $d \geq \sqrt{n^2-3n+3}$.

Given a non-negative vertex weight vector $\boldsymbol{w} \in \mathrm{Z}_+^{P(m,n)}$, a multicoloring of $(T_{m,n}(d),\boldsymbol{w})$ is an assignment of colors to $P(m,n)$ such that each vertex $v \in P(m,n)$ admits $w(v)$ colors and every adjacent pair of two vertices does not share a common color. We also give an efficient algorithm for multicoloring $(T_{m,n}(d),\boldsymbol{w})$ when $P(m,n)$ is perfect.

In general case, our results on the perfectness of $P(m,n)$ implies a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d),\boldsymbol{w})$. Our algorithm finds a multicoloring which uses at most $\alpha(d)\omega + \mathrm{O}(d^3)$ colors, where ω denotes the weighted clique number. When $d = 1, \sqrt{3}, 2, \sqrt{7}, 3$, the approximation ratio $\alpha(d) = (4/3), (5/3), (5/3), (7/4), (7/4)$, respectively. When $d > 1$, we showed that $\alpha(d) \leq \left(1 + \frac{2}{\sqrt{3}+\frac{2\sqrt{3}-3}{d}}\right)$.

We also showed the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d),\boldsymbol{w})$ with strictly less than $(4/3)\omega$ colors.

1 Introduction

Given a pair of non-negative integers m and n, $P(m,n)$ denotes the subset of 2-dimensional integer triangular lattice points defined by

$$P(m,n) \stackrel{\text{def.}}{=} \{(x\boldsymbol{e}_1+y\boldsymbol{e}_2) \mid x \in \{0,1,2,\ldots,m-1\},\ y \in \{0,1,2,\ldots,n-1\}\}$$

where $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$, $\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2,\sqrt{3}/2)$. Given a finite set of 2-dimensional points $P \subseteq \mathrm{R}^2$ and a positive real d, a *unit disk graph*, denoted by (P,d), is an undirected graph with vertex set

P such that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d. We denote the unit disk graph $(P(m,n),d)$ by $T_{m,n}(d)$.

Given an undirected graph H and a non-negative integer vertex weight $\boldsymbol{w}'$ of H, a *multicoloring* of $(H,\boldsymbol{w}')$ is an assignment of colors to vertices of H such that each vertex v admits $w'(v)$ colors and every adjacent pair of two vertices does not share a common color. A *multicoloring problem* on $(H,\boldsymbol{w}')$ finds a multicoloring of $(H,\boldsymbol{w}')$ which minimizes the required number of colors. The multicoloring problem is also known as weighted coloring [4], minimum integer weighted coloring [15] or w-coloring [12].

In this paper, we study weighted unit disk graphs on triangular lattice points $(T_{m,n}(d),\boldsymbol{w})$. First, we show a necessary and sufficient condition that $T_{m,n}(d)$ is a perfect graph. If the graph is perfect, we can solve the multicoloring problem easily. Next, we propose a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d),\boldsymbol{w})$. Our algorithm is based on the well-solvable case that the given graph is perfect. For any $d \geq 1$, our algorithm finds a multicoloring which uses at most

$$\left(1+\frac{\left\lfloor \frac{2}{\sqrt{3}}d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right)\omega + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right)\lfloor d+1 \rfloor^2$$

colors, where ω denotes the weighted clique number. Table 1 shows the values of the above approximation ratio in case that d is small.

表 1: Approximation ratio

d	$1\cdots$	$\sqrt{3}\cdots$	$\sqrt{7}\cdots$	$2\sqrt{3}\cdots$	$\sqrt{13}\cdots$	$\sqrt{19}\cdots$
ratio	4/3	5/3	7/4	2	9/5	2
d	$\sqrt{21}\cdots$	$3\sqrt{3}\cdots$	$\sqrt{31}\cdots$	$\cdots$		∞
ratio	11/6	2	13/7	$\cdots$		$1+2/\sqrt{3}$

We also show the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d),\boldsymbol{w})$ which uses strictly less than $(4/3)\omega$ colors.

The multicoloring problem has been studied in several context. When a given graph is the triangular lattice graph $T_{m,n}(1)$, the problem is related to the radio channel (frequency) assignment problem. McDiarmid and Reed [9] showed that the multicoloring problem on triangular lattice graphs is NP-hard. Some authors [9, 12] independently gave (4/3)-approximation algorithms for this problem. In case that a given graph H is a square lattice graph or a hexagonal lattice graph, the graph H becomes bipartite and so we can obtain an optimal multicoloring of $(H,\boldsymbol{w}')$ in polynomial time (see [9] for example). Halldórsson and Kortsarz [5] studied planar graphs and partial k-trees. For both classes, they gave a polynomial time approximation scheme (PTAS) for variations of multicoloring problem with min-sum objectives. These objectives appear in the context of multiprocessor task scheduling. For coloring (general) unit disk graphs, there exists a 3-approximation algorithm [6, 8, 14]. Here we note that the approximation ratio of our algorithm is less than $1+2/\sqrt{3} < 2.155$ for any $d \geq 1$.

2 Well-Solvable Cases and Perfectness

In this section, we discuss some well-solvable cases such that the multicoloring number is equivalent to the weighted clique number.

An undirected graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H, denoted by $\chi(H)$, is equal to its clique number $\omega(H)$. The following theorem is a main result of this paper.

Theorem 1 *When $n \geq 1$ and $d \geq 1$, we have the following;*
$[\forall m \in \mathrm{Z}_+$, $T_{m,n}(d)$ is perfect $]$ if and only if $d \geq \sqrt{n^2 - 3n + 3}$.

Table 2 shows the perfectness and imperfectness of $T_{m,n}(d)$ for small n and d.

表 2: Perfectness and imperfectness

n \ d	$1 \cdots \sqrt{3} \cdots 2 \cdots$	$\sqrt{7} \cdots 3 \cdots 2\sqrt{3} \cdots$	$\sqrt{13} \cdots 4 \cdots \sqrt{19} \cdots$	$\sqrt{21} \cdots$
1				
2				
3				
4				Perfect
5				
6	Imperfect			
$\vdots$				

To show the above theorem, we introduce some definitions. We say that an undirected graph has a transitive orientation property, if each edge can be assigned a one-way direction in such a way that the resulting directed graph (V, F) satisfies that $[(a, b) \in F$ and $(b, c) \in F$ imply $(a, c) \in F]$. An undirected graph which is transitively orientable is called *comparability graph.* The complement of a comparability graph is called *co-comparability graph.* It is well-known that every co-comparability graph is perfect.

Lemma 1 *For any integer $n \geq 1$, if $d \geq \sqrt{n^2 - 3n + 3}$, then $T_{m,n}(d)$ is a co-comparability graph.*

Proof: omitted.

The following lemma deals with the special case that $n = 3$, $d = 1$.

Lemma 2 *For any $m \in \mathrm{Z}_+$ and $1 \leq \forall d < \sqrt{3}$, the graph $T_{m,3}(d)$ is perfect.*

Proof: We only need to consider the case that $d = 1$, since $T_{m,n}(d) = T_{m,n}(1)$ when $1 \leq d < \sqrt{3}$. Let H be an induced subgraph of $T_{m,3}(1)$. When $\omega(H) \leq 2$, H has no 3-cycle. Then it is easy to show that H has no odd cycle and thus $\chi(H) = \omega(H)$, since H is bipartite. If $\omega(H) \geq 3$, then it is clear that $\omega(H) = 3$ and $\chi(H) \leq 3$, since $\omega(T_{m,3}(1)) = 3$ and $T_{m,3}(1)$ has a trivial 3-coloring. ∎

Note that though the graph $T_{m,3}(1)$ is perfect, the graph $T_{m,3}(1)$ is not co-comparability graph.

From the above, the perfectness of a graph satisfying the conditions of Theorem 1 is clear. In the following, we discuss the inverse implication. We say that an undirected graph G has an *odd-hole*, if G contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5. It is obvious that if a graph has an odd-hole, the graph is not perfect. In the following, we denote a point $(xe_1 + ye_2) \in P(m, n)$ by $\langle x, y\rangle$.

Lemma 3 *If* $1 \leq d < \sqrt{7}$, *then* $\forall m \geq 5$, $T_{m,4}(d)$ *has at least one odd-hole.*

Proof: If $1 \leq d < \sqrt{3}$, then a subgraph induced by $\{ \langle 2,0\rangle, \langle 1,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle \}$ is a 9-hole. If $\sqrt{3} \leq d < 2$, then a subgraph induced by $\{ \langle 3,0\rangle, \langle 1,1\rangle, \langle 0,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 4,2\rangle, \langle 4,1\rangle \}$ is a 7-hole. If $2 \leq d < \sqrt{7}$, then a subgraph induced by $\{ \langle 2,0\rangle, \langle 0,2\rangle, \langle 1,3\rangle, \langle 3,2\rangle, \langle 3,0\rangle \}$ is a 5-hole. When $1 \leq d < \sqrt{7}$, $T_{5,4}(d)$ has at least one odd-hole, and hence the proof is completed. ∎

Lemma 4 *If* $1 \leq d < \sqrt{13}$, *then* $\forall m \geq 6$, $T_{m,5}(d)$ *has at least one odd-hole.*

Proof: If $1 \leq d < \sqrt{7}$, then odd-holes in the proof of Lemma 3 are induced subgraph of $T_{6,5}(d)$. If $\sqrt{7} \leq d < 3$, then a subgraph induced by $\{ \langle 2,0\rangle, \langle 0,2\rangle, \langle 1,4\rangle, \langle 4,2\rangle, \langle 4,0\rangle \}$ is a 5-hole. If $3 \leq d < \sqrt{13}$, then a subgraph induced by $\{ \langle 3,0\rangle, \langle 0,3\rangle, \langle 2,4\rangle, \langle 5,3\rangle, \langle 5,0\rangle \}$ is a 5-hole. When $1 \leq d < \sqrt{13}$, $T_{6,5}(d)$ has at least one odd-hole, and hence the proof is completed. ∎

Lemma 5 *For any integer* $n \geq 4$, *if* $1 \leq d < \sqrt{n^2 - 3n + 3}$, *then* $\exists m \in \mathrm{Z}_+$, $T_{m,n}(d)$ *is imperfect.*

Proof: In the following, we show that $\forall n \geq 4$, if $1 \leq d < \sqrt{n^2 - 3n + 3}$, then $\exists m \in \mathrm{Z}_+$, $T_{m,n}(d)$ has at least one odd-hole, by induction on n. When $n = 4, 5$, it is clear from Lemmas 3 and 4, respectively.

Now we consider the case that $n = n' \geq 6$ under the assumption that if $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3}$, then $\exists m' \in \mathrm{Z}_+$, $T_{m',n'-1}(d)$ has at least one odd-hole. If $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3} = \sqrt{n'^2 - 5n' + 7}$, then $T_{m',n'}(d)$ has at least one odd-hole, since $T_{m',n'-1}(d)$ is an induced subgraph of $T_{m',n'}(d)$. In the remained case that $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$, the set of points $\{ \langle n'-3, 0\rangle, \langle 0, n'-2\rangle, \langle n'-4, n'-1\rangle, \langle 2n'-7, n'-2\rangle, \langle 2n'-6, 0\rangle \}$ is contained in $P(m'', n')$, if $m'' = 2n' - 5$. It is easy to see that the above five vertices induces a 5-hole of $T_{m'',n'}(d)$, when $n' \geq 6$ and $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$ ∎

Lemma 5 shows the imperfectness of every graph which violates a condition of Theorem 1. Thus, we completed a proof of Theorem 1. From the above lemmas, the following is immediate.

Corollary 1 *Let* $d > 1$ *be a real number. Then,* $T_{m,n}(d)$ *is a co-comparability graph, if and only if* $n \leq \frac{3+\sqrt{4d^2-3}}{2}$.

Lastly, we discuss some algorithmic aspects. Assume that we have a co-comparability graph G and related digraph H which gives a transitive orientation of the complement of G. Then each independent set of G corresponds to a chain (directed path) of H. The multicoloring problem on G is essentially equivalent to the minimum size chain cover problem on H. Every clique of G corresponds to an anti-chain of H. Thus the equality $\omega(G) = \chi(G)$ is obtained from Dilworth's decomposition theorem [2]. It is well-known that the minimum size chain cover problem on an acyclic graph is solvable in polynomial time by using an algorithm for minimum-cost circulation flow problem (see [13] for example).

Though an weighted graph $(T_{m,3}(1), \boldsymbol{w})$ is not a co-comparability graph, we can construct exact multicoloring algorithm for the graph. Here we omit the detail.

3 Approximation Algorithm

In this section, we propose an approximation algorithm for multicoloring the graph $(T_{m,n}(d), \boldsymbol{w})$. When $d = 1$, McDiarmid and Reed [9] proposed an approximation algorithm for $(T_{m,n}(1), \boldsymbol{w})$, which finds a multicoloring with at most $(4/3)\omega(T_{m,n}(1), \boldsymbol{w}) + 1/3$ colors.

In the following, we propose an approximation algorithm for $(T_{m,n}(d), \boldsymbol{w})$ when $d > 1$. The basic idea of our algorithm is similar to the shifting strategy [7].

Theorem 2 *When $d > 1$, there exists a polynomial time algorithm for multicoloring $(T_{m,n}(d), \boldsymbol{w})$ such that the number of required colors is bounded by*

$$\left(1 + \frac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right) \omega(T_{m,n}(d), \boldsymbol{w}) + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right) \chi(T_{m,n}(d)).$$

Proof: We describe an outline of the algorithm. For simplicity, we define $K_1 = \left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor$ and $K_2 = \left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor + \left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor$.

First, we construct K_2 vertex weights $\boldsymbol{w}'_k$ for $k \in \{0, 1, \ldots, K_2 - 1\}$ by setting

$$w'_k(x, y) = \begin{cases} 0, & y \in \left\{k, k+1, \ldots, k + \left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor - 1\right\} \pmod{K_2}, \\ \left\lfloor \frac{w(x,y)}{K_1} \right\rfloor, & \text{otherwise.} \end{cases}$$

Next, we exactly solve K_2 multicoloring problems defined by K_2 pairs $(T_{m,n}(d), \boldsymbol{w}'_k)$, $k \in \{0, 1, \ldots, K_2 - 1\}$ and obtain K_2 multicolorings. We can solve each problem exactly in polynomial time, since every connected component of the graph induced by the set of vertices with positive weight is a perfect graph discussed in the previous section. Thus $\chi(T_{m,n}(d), \boldsymbol{w}'_k) = \omega(T_{m,n}(d), \boldsymbol{w}'_k)$ for any $k \in \{0, 1, \ldots, K_2 - 1\}$. Put $\boldsymbol{w}'' = \boldsymbol{w} - \sum_{k=0}^{K_2-1} \boldsymbol{w}'_k$. Then each element of $\boldsymbol{w}''$ is less than or equal to $K_1 - 1$. Thus we can find a multicoloring of $(T_{m,n}(d), \boldsymbol{w}'')$ from the direct sum of $K_1 - 1$ trivial colorings of $T_{m,n}(d)$. The obtained multicoloring uses at most $(K_1 - 1)\chi(T_{m,n}(d))$colors. Lastly, we output the direct sum of $K_2 + 1$ multicolorings obtained above. The definition of the weight vector $\boldsymbol{w}'_k$ implies that $\forall k \in \{0, 1, \ldots, K_2 - 1\}$, $K_1 \omega(T_{m,n}(d), \boldsymbol{w}'_k) \leq \omega(T_{m,n}(d), \boldsymbol{w})$. Thus, the obtained multicoloring uses at most $(K_2/K_1)\omega(T_{m,n}(d), \boldsymbol{w}) + (K_1 - 1)\chi(T_{m,n}(d))$ colors. ∎

The following lemma gives the chromatic number of $T_{m,n}(d)$.

表 3: Unit disk graphs on square lattice points

n \ d	$1\cdots$	$\sqrt{2}\cdots$	$2\cdots$	$\sqrt{5}\cdots 2\sqrt{2}\cdots$
1				
2				Perfect
3				
4			Imperfect	

Lemma 6 *If m, n are sufficiently large, then $\chi(T_{m,n}(d)) = \widehat{d}^2$ where $\widehat{d}$ is the minimum Euclidean distance between two points in $P(m,n)$ subject to that distance being greater than d. Clearly, $d < \widehat{d} \leq \lfloor d+1 \rfloor$.*

Proof: See McDiarmid [9] for example. ∎

When d is small, Table 1 shows the approximation ratio. The following corollary gives a simple upper bound of the approximation ratio.

Corollary 2 *For any $d \geq 1$, we have* $1 + \dfrac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lceil \frac{3+\sqrt{4d^2-3}}{2} \right\rceil} \leq 1 + \dfrac{2}{\sqrt{3} + \frac{2\sqrt{3}-3}{d}}$.

Here we note that if we apply our algorithm in the case that $d = 1$, then the algorithm finds a multicoloring which uses at most $(4/3)\omega(T_{m,n}(1), \boldsymbol{w}) + 6$ colors.

4 Discussion

In this paper, we dealt with the triangular lattice. In the following, we discuss the square lattice. Given a pair of non-negative integers m and n, $Q(m,n) \stackrel{\text{def.}}{=} \{0,1,2,\ldots,m-1\} \times \{0,1,2,\ldots,n-1\}$ denotes the subset of 2-dimensional integer square lattice points. We denote the unit disk graph $(Q(m,n), d)$ by $S_{m,n}(d)$. In case that $d < \sqrt{2}$, it is clear that $S_{m,n}(d) = S_{m,n}(1)$ and the graph is bipartite for any m and n. If $d = \sqrt{2}$, we proposed a (4/3)-approximation algorithm for multicoloring $(S_{m,n}(\sqrt{2}), \boldsymbol{w})$ in our previous paper [11]. We also showed the NP-hardness of the problem.

Unfortunately, Theorem 1 is not extensible to the square lattice case. Table 3 shows the perfectness and imperfectness of unit disk graphs on the square lattice for small n and d. The perfectness of $T_{m,3}(\sqrt{2})$ was shown in [11]. The graph $S_{m,3}(2)$ contains a 5-hole: $\{(0,0),(2,0),(2,1),(1,2),(0.2)\}$.

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