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Hajós Calculus on Planar Graphs

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Abstract: The Hajós calculus is a nondeterministic procedure which generates the class of non-3-colorable graphs [3]. If all non-3-colorable graphs can be constructed in polynomial steps by the calculus, NP = co-NP holds. Up to date, however, it remains open whether there exists a family of graphs that can be generated in polynomial steps. To attack this problem, we propose two graph calculi $\mathcal{PHC}$ and $\mathcal{PHC}^*$ that generate non-3-colorable planar graphs, where intermediate graphs in the calculi are also restricted to be planar. Then we prove that $\mathcal{PHC}$ and $\mathcal{PHC}^*$ are *sound* and *complete*. We also show that $\mathcal{PHC}^*$ can polynomially simulate $\mathcal{PHC}$.

Keywords: Hajós calculus, Planar graph, Coloring, Proof systems

1 Introduction

Graph k-coloring problem is the problem to decide whether we can assign one of k colors to each vertex so that adjacent pairs of vertices are assigned different colors [15]. This problem is one of the most fundamental NP-complete problems [5, 9]. Even when $k \geq 3$, it is NP-complete. When $k \leq 2$, we can solve the problem in polynomial time. If graphs are restricted to be planar, it is believed for a long time that every graph is 4-colorable [10]. Appel and Haken finally proved the Four-Color Theorem, i.e., every planar graph is 4-colorable [7, 8, 12, 19]. Therefore, when $k \geq 4$, we can decide whether given planar graph is k-colorable in polynomial time. When $k = 3$, the problem is still NP-complete.

In order to characterize k-colorable graphs, many approaches have been attempted. The most typical one is Hadwiger's conjecture to relate the non-k-colorability and the $(k+1)$-cliques [1]. Let k be the fewest number of colors necessary to color vertices in a given graph. Then, we can obtain a k-clique by contracting adjacent vertices. This conjecture is true for $k \leq 5$ [1, 2, 4].

Another approach is the Hajós calculus. The calculus is a nondeterministic procedure that constructs all non-k-colorable graphs from a $(k+1)$-clique [3]. A graph calculus is a collection of initial graphs, together with a finite set of rules which allows us to derive new graphs. A construction of a graph G is a sequence of graphs $(G_1, G_2, \ldots, G_\ell)$ such that the sequence ends with G (i.e., $G_\ell = G$) and every graph in the sequence is one of the initial graphs, or follows from its previous graphs by applying one of the rules.

The complexity of the Hajós calculus was first studied by Mansfield and Welsh [11]: If all non-3-colorable graphs have polynomial size Hajós constructions then, NP = co-NP holds, thus there may exist graphs that cannot be constructed in polynomial steps. A construction of a graph in the Hajós calculus gives the proof of the non-k-colorability of the graph.

Our Contribution: Our motivation is to give intermediate subsystems that are more powerful than bounded-depth Frege system and yet we can prove super-polynomial lower bounds. For this purpose, we consider the calculus on planar graphs, more precisely, the calculus that generates the class of non-3-colorable planar graphs, where intermediate graphs in the calculus are also restricted to be planar. Although the Hajós calculus can generate all non-3-colorable planar graphs, intermediate graphs are not guaranteed to be planar. When restricting the intermediate graphs to be planar, by adding only one new rule, we can obtain a sound and complete calculus $\mathcal{PHC}$. By modifying the second rule (edge elimination rule) in the Hajós calculus, we can obtain another sound and complete calculus $\mathcal{PHC}^*$. We compare the powers of the two calculi.

Previous work: It is known that the Hajós calculus is polynomially bounded if and only if Extended Frege proof systems are polynomially bounded [16]. This result links an open problem in graph theory to an important open problem in the complexity of propositional proof systems: Is there a strong system to produce a short proof of any tautology? As formalized by Cook and Reckhow [6], there exists a propositional proof system giving rise to short (polynomial-size) proofs of all tautologies if and only if NP equals co-NP. Since Extended Frege system is powerful enough that obtaining super-polynomial lower bounds is beyond our current technique, research interests shift into subsystems of the calculus. For example, Aijtai and others showed exponential lower bounds for bounded-depth Frege proofs [13, 14, 18], which lead exponential lower bounds on the subsystems of the Hajós calculus [16, 17].

2 Hajós Calculus

We describe the Hajós calculus for $k = 3$. The set of initial graphs in Hajós calculus contains all graphs isomorphic to complete graph K_4. There are three rules for generating new graphs:

1. **Vertex/edge introduction rule**: Add (any number of) vertices and edges.
2. **Join rule**: Let G_1 and G_2 be disjoint graphs, a_1 and b_1 adjacent vertices in G_1, and a_2 and b_2 adjacent vertices in G_2. Constrct a graph G_3 from $G_1 \cup G_2$ as follows. First, remove edges (a_1, b_1) and (a_2, b_2); then add an edge (b_1, b_2); lastly, contract vertices a_1 and a_2 into a single vertex, named a_1.
3. **Contraction rule:** Contract two nonadjacent vertices into a single vertex, and remove any resulting duplicated edges.

Vertex/edge introduction rule implies that if a subgraph of G has a construction, G also has a construction. Rules 1 and 2 increase vertices and/or edges, but Rule 3 reduces vertices and edges, thus the construction may not be polynomially bounded.

We consider a minor revision of the Hajós calculus, $\mathcal{HC}$. The system $\mathcal{HC}$ has the same set of initial graphs, as well as Rules 1 and 3 of the Hajós calculus, but now Rule 2 of the Hajós calculus is replaced by the following rule:

2. **Edge elimination rule**: Let G_1 and G_2 be two graphs with common vertex set $\{v_1, \dots, v_n\}$ which are identical except that G_1 contains edges (v_1, v_2) and (v_2, v_3) and not (v_1, v_3), whereas G_2 contains edges (v_1, v_2) and (v_1, v_3) and not (v_2, v_3). Then from G_1 and G_2, we can construct a graph G_3 that is identical to G_1 but does not contain (v_2, v_3).

To associate two calculi, Hajós calculus and $\mathcal{HC}$, we define a binary relation: Let $\mathcal{C}$ and $\mathcal{C}'$ be two graph calculus systems, then $\mathcal{C}$ *p-simulates* $\mathcal{C}'$ if there is a polynomial-time computable function f so that for all graphs G, if s is a graph construction of G in $\mathcal{C}'$, then $f(s)$ is a graph construction of G in $\mathcal{C}$. $\mathcal{C}$ and $\mathcal{C}'$ are *p-equivalent* if $\mathcal{C}$ p-simulates $\mathcal{C}'$ and $\mathcal{C}'$ p-simulates $\mathcal{C}$.

Fact 1 *$\mathcal{HC}$ is p-equivalent to the Hajós calculus.*

3 Planar Calculus $\mathcal{PHC}$

First, we propose planar calculus $\mathcal{PHC}$. The set of initial graphs in $\mathcal{PHC}$ contains all graphs isomorphic to K_4. There are four rules, where Rules 1 to 3 are same as the system $\mathcal{HC}$, but edge addition and vertex contraction are restricted so that the resulting graphs are planar. Rule 4 is as follows:

4. **Quadrilateral rule**: Let G_3 be a graph with vertex set $\{v_1, \dots, v_n\}$ that contains a face v_1, v_2, v_3, v_4. Let G_1 be a graph obtained by contracting vertices v_1 and v_3 of G_3. Let G_2 be a graph obtained by contracting vertices v_2 and v_4 of G_3. Then from G_1 and G_2, we can construct the graph G_3 (See Figure 1).

Rule 4 is important when given graph consists of only triangle and quadrilateral faces.

For example, we show that the graph G_3 of Figure 2 has a construction in $\mathcal{PHC}$. Let G_1 and G_2 be the graphs shown in Figure 2. G_1 contains K_4 as a subgraph induced by $\{v_1, v_2, v_3, v_5\}$. G_2 also contains K_4 as a subgraph induced by $\{v_1, v_3, v_4, v_5\}$. Therefore G_1 and G_2 can be constructed in $\mathcal{PHC}$. G_1 can be

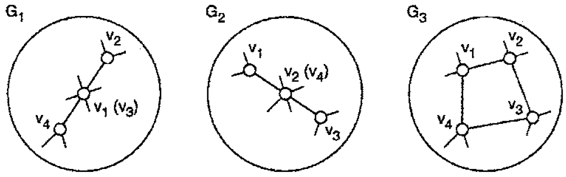

Figure 1: G_1, G_2 and G_3 of Rule 4 in $\mathcal{PHC}$

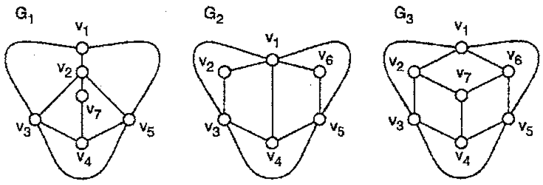

Figure 2: Example of the system $\mathcal{PHC}$

constructed from G_1 and G_2 by Rule 4, since v_1, v_2, v_7, v_6 is a quadrilateral face and G_1 is identical to G_3 with v_2 and v_6 contracted and G_2 is identical to G_3 with v_1 and v_7 contracted.

Since G_3 is edge minimal with respect to the 3-colorability, G_3 cannot be constructed directly by Rule 1. Each face of G_3 is triangle or quadrilateral, thus there is not a triplet of vertices v, v', v'' of satisfying the condition of Rule 2. This means that G_3 cannot be constructed directly by Rule 2. Contraction rule cannot break the structure of non-3-colorablility. Therefore, probably G_3 is an example of graphs that essentially need Rule 4 in $\mathcal{PHC}$.

In the rest of this section, we prove the soundness and the completeness of $\mathcal{PHC}$.

Theorem 2 *$\mathcal{PHC}$ is sound.*

Proof: We only need to show that Rule 4 is sound since other rules also appear in $\mathcal{HC}$ and are shown to be sound [3]. Assume that there exists a 3-colorable graph G_3 generated by Rule 4. Then, its face v_1, v_2, v_3, v_4 has a coloring satisfying one of the following cases:

Case 1: color(v_1) = color(v_3).

Case 2: color(v_2) = color(v_4).

Note that, if neither of the cases are satisfied, we have color(v_1) $\neq$ color(v_3) and color(v_2) $\neq$ color(v_4). In this case, we need more than four colors to the face, which contradicts the 3-colorability of G_3. Cases 1 (Case 2, respectively) implies that G_1 (G_2, respectively) is 3-colorable. Therefore, only non-3-clorable graphs are generated. □

Theorem 3 *$\mathcal{PHC}$ is complete.*

Proof: We prove this theorem by induction on the size n of the graph. In case $n < 4$, all graphs are colorable by at most 3 colors. In case $n = 4$, K_4 is the initial graph of $\mathcal{PHC}$. Other graphs of size 4 are all 3-colorable, so we do not care them.

Assume that all non-3-colorable graphs of size $n-1$ can be constructed in $\mathcal{PHC}$. We assume that there exists a nonempty set $\mathcal{G}$ of non-3-colorable graphs of size n that cannot be constructed in $\mathcal{PHC}$. Then we lead a contradiction that edge maximal graph $G \in \mathcal{G}$ can be constructed. By considering the size of the faces in G, we have the following three cases.

Case 1: All faces are triangle. According to Theorem 5 (we prove this theorem later), G can be constructed in $\mathcal{PHC}$.

Case 2: There is a face f of size $k \geq 5$. Let $v_1, v_2, v_3, v_4, \ldots, v_k$ be the vertices of face f. $G' = G+(v_1, v_3)$ and $G'' = G+(v_1, v_4)$ can be constructed, since G is a edge maximal graph in $\mathcal{G}$. Therefore we can construct G from G' and G'' applying by Rule 2.

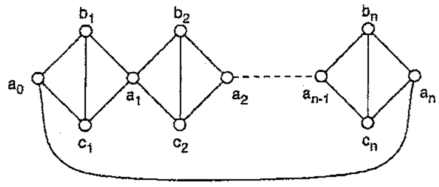

Figure 3: G_n

Case 3: G is composed of triangle or quadrilateral faces. Let $f = v_1, v_2, v_3, v_4$ be a quadrilateral face of G. Let G' be a graph obtained by contracting vertices v_1 and v_3 of G. Let G'' be a graph obtained by contracting vertices v_2 and v_4 of G. G' and G'' can be constructed in $\mathcal{PHC}$ because of the assumption. Therefore we can construct G from G' and G'' applying the Rule 4.

In any case, G ($\in \mathcal{G}$) can be constructed in $\mathcal{PHC}$, which contradict to the definition of $\mathcal{G}$. Thus, any non-3-colorable graph can be constructed in $\mathcal{PHC}$. □

Now, we prove that any triangulate non-3-colorable planar graph can be constructed in polynomial number of steps. First we prove the following lemma, which construct an essential component of triangulate planar graphs.

Lemma 4 *Let $G_n = (V, E)$ be a graph of $3n + 1$ vertices, where $n \geq 1$,*

$$V = \{a_0\} \cup \{a_i,\ b_i,\ c_i \mid i \in \{1, \ldots, n\}\},$$

$$E = \{(a_0, a_n)\} \cup \{(a_{i-1}, b_i), (a_{i-1}, c_i), (a_i, b_i), (a_i, c_i), (b_i, c_i) \mid i \in \{1, \ldots, n\}\}$$

then, G has a linear size construction in $\mathcal{PHC}$.

PROOF: We prove this lemma by induction on n. In case $n = 1$, the lemma obviously holds since G_1 is isomorphic to an initial graph K_4.

We prove that G_n can be constructed by the assumption that G_{n-1} can be constructed. $G' = G_n + (a_0, a_{n-1})$ can be constructed in $\mathcal{PHC}$ since G_{n-1} is subgraph of G'. $G'' = G_n + (a_{n-1}, a_n)$ can be constructed in $\mathcal{PHC}$ since subgraph of G'' induced by $\{a_{n-1}, a_n, b_n, c_n\}$ is isomorphic to K_4. Therefore we can construct G_n by applying Rule 2 to G' and G''. Since we apply Rule 1 twice and Rule 2 once at each induction step, the whole construction is linearly bounded. □

Theorem 5 *Triangulate non-3-colorable planar graphs have a polynomial size construction in $\mathcal{PHC}$.*

PROOF: Our goal is to find a structure G_n of Lemma 4 as a subgraph of a given graph G. We try to assign colors to vertices of G. Initially, we choose a triangle face v_1, v_2, v_3 and assign different color to each vertex. color(v_1) $= R$, color(v_3) $= G$ and color(v_3) $= B$. We introduce three trees T_R, T_G, T_B. The root node of each tree is one of the vertices v_1, v_2, v_3. The face that its vertices are already assigned a color is called a colored face. Next, we repeat the following procedure until all vertices are assigned a color or adjacent vertices are assigned the same color. Choose a non-colored triangle face f' adjacent to colored face f. We need not to think about the case that non-colored faces exist but are not adjacent to colored face because the given graph is connected and trianglate. Let v be a vertex that belongs to f and does not belong to f'. Let v' be a vertex that belongs to f' and does not belong to f. Vertices v and v' are uniquely determined. Then we assign $c =$ color(v) to v' and add the vertex v' to the tree T_c as a child node of v. This replication stops before all vertices assigned a color because G is non-3-colorable. When the repetition stops, we find adjacent vertices v' and v'' on G that are assigned the same color c. The tree T_c includes v' and v'' so that there is a path p from v' to v'' in T_c. An Edge $(v_i, v_j) \in T_c$ corresponds to a subgraph of G as the Figure 4. Let G' be a subgraph of G that corresponds to path p (G' of Figure 5 is an example) that corresponds to path p (dotted line of the figure). Let G'' be a graph $G_{|p|}$ of Lemma 4. G'' can be constructed because of Lemma 4. G' can be constructed from G'' with some vertices contracted. G can be constructed from G' by Rule 1. Therefore G has a construction in $\mathcal{PHC}$. □

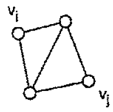

Figure 4: Relation between v_i and v_j in G

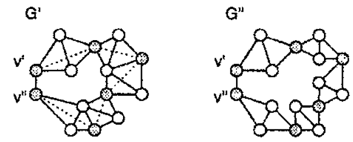

Figure 5: Subgraph of G and its structure

4 Planar Calculus $\mathcal{PHC}^*$

In this section, we propose another planar calculus $\mathcal{PHC}^*$. The set of initial graphs in $\mathcal{PHC}^*$ contains all graphs isomorphic to K_4. There are three rules for generating new graphs. Rule 1 and Rule 3 are same as the system $\mathcal{PHC}$. Our new Rule 2 is as follows:

2. **Vertex division/edge elimination rule**: Let G_1 be a graph with n vertices $\{v_1, \ldots, v_n\}$ that contains an edge (v_1, v_2), and G_2 be the graph obtained by contracting v_1 and v_2 of G_1.Then from G_1 and G_2, we can construct a graph G_3 graph that are identical to G_1 but does not contain (v_1, v_2).

Rule 2 is simple but powerful to generate non-3-colorable graphs. This rule means that none adjacent vertices v_1 and v_2 can be assigned the same color or different colors.

In the rest of this section, we prove the soundness and the completeness of $\mathcal{PHC}^*$.

Theorem 6 *$\mathcal{PHC}^*$ is sound.*

PROOF: We only need to show the soundness of Rule 2 since other rules also appear in $\mathcal{HC}$ and are shown to be sound [3]. Assume that there exists a 3-colorable graph G_3 generated by Rule 2. Then, its vertices v_1 and v_2 has a coloring satisfying one of the following two cases:

Case 1: color(v_1) $\neq$ color(v_2).

Case 2: color(v_1) $=$ color(v_2).

In Case 1, the coloring is also valid for G_1, i.e., G_1 is 3-colorable. In Case 2, we can contract vertices v_1 and v_2 in G_3, i.e., G_2 is also 3-colorable. Therfore, in $\mathcal{PHC}^*$, all graphs generated by Rule 2 are non-3-colorable. □

Theorem 7 *$\mathcal{PHC}^*$ is complete.*

PROOF: All non-3-colorable planar graphs can be constructed in $\mathcal{PHC}$. We can simulate $\mathcal{PHC}$ by $\mathcal{PHC}^*$, so that $\mathcal{PHC}^*$ is complete. □

5 Polynomial-Time Simulation

We show the relationship between $\mathcal{PHC}$ and $\mathcal{PHC}^*$. First direction is that we simulate $\mathcal{PHC}$ by $\mathcal{PHC}^*$.

Theorem 8 *$\mathcal{PHC}^*$ p-simulates $\mathcal{PHC}$.*

PROOF: Rules 1 and 3 are common in $\mathcal{PHC}^*$ and $\mathcal{PHC}$. We only need to simulate Rule 2 and Rule 4 in $\mathcal{PHC}$ by $\mathcal{PHC}^*$. According to Lemma 9, Rule 2 can be simulated. According to Lemma 10, Rule 4 can be simulated. In each case the series of simulating steps can be constructed in polynomial time. Therefore, $\mathcal{PHC}^*$ p-simulates $\mathcal{PHC}$. □

Lemma 9 *$\mathcal{PHC}^*$ p-simulates Rule 2 of $\mathcal{PHC}$.*

PROOF: We prove that a graph G_3 can be constructed from G_1 and G_2 in $\mathcal{PHC}^*$. Let G_1 and G_2 be two graphs with common vertex set $\{v_1, \ldots, v_n\}$ which are identical except that G_1 contains edges (v_1, v_2) and (v_2, v_3) and not (v_1, v_3), whereas G_2 contains edges (v_1, v_2) and (v_1, v_3) and not (v_2, v_3). G_3 is identical to G_1 but does not contain (v_2, v_3). Let G_1' be a graph identical to G_1 with vertices v_1 and v_3 are contracted. (G_1, G_1', G_2, G_3) is a subsequence of a construction in $\mathcal{PHC}^*$ since G_1' can be constructed from G_1 by Rule 3 and G_3 can be constructed from G_1' and G_2 by Rule 2 with paticular vertices v_1 and v_3. □

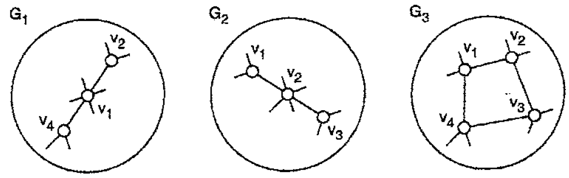

Figure 6: G_1, G_2 and G_3 of Rule 4 in $\mathcal{PHC}$

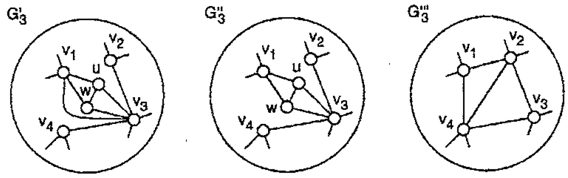

Figure 7: Intermediate graphs of simulation in $\mathcal{PHC}^*$

Lemma 10 *$\mathcal{PHC}^*$ p-simulates Rule 4 of $\mathcal{PHC}$.*

PROOF: Let G_1, G_2 and G_3 be graphs as Figure 6. G_3 is a graph with vertex set $\{v_1, \ldots, v_n\}$ that contains a face v_1, v_2, v_3, v_4. G_1 is a graph obtained by contracting vertices v_1 and v_3 of G_3. G_2 is a graph obtained by contracting vertices v_2 and v_4 of G_3. We prove that a graph G_3 can be constructed from G_1 and G_2 in $\mathcal{PHC}^*$. Let G'_1 be the graph as Figure 7. G'_1 is identical to G_1, but has two additional vertices u and w and three additional edges $(v_1, u), (v_1, w), (u, w)$. G'_1 can be constructed from G_1 by Rule 1, since G'_1 is a subgraph of G'_3. Let G'_3, G''_3 and G'''_3 be graphs as Figure 7 that are identical to G_3 but some vertices and edges in the figure is different from G_3. G'_3 can be constructed from K_4 by Rule 1, since a subgraph induced by $\{v_1, v_3, u, w\}$ is isomorphic to K_4. G''_3 can be constructed from G'_1 and G'_3 by Rule 2 in $\mathcal{PHC}^*$ with paticular vertices v_1 and v_3 (G'_3 has an edge (v_1, v_3) and G'_1 is identical to G''_3 with v_1 and v_3 contracted). G'''_3 can be constructed from G''_3 by contracting two pairs of vertices (v_2, u) and (v_4, w). This construction needs twice of applying Rule 3 in $\mathcal{PHC}^*$. Then G_3 can be constructed from G'''_3 and G_2 by Rule 2 in $\mathcal{PHC}^*$ with paticular vertices v_2 and v_4. Thus Rule 4 in $\mathcal{PHC}$ can be p-simulated by $\mathcal{PHC}^*$. □

Theorem 8 implies that the modified rule (Rule 2) is at least as powerful as the original one in $\mathcal{HC}$.

Corollary 11 *$\mathcal{HC}$ can be p-simulated by $\mathcal{PHC}^*$ without planarity ristiction on the intermediate graphs.*

It is difficult to show that $\mathcal{PHC}$ p-simulates $\mathcal{PHC}^*$. For example, as shown in Figure 8, Rule 2 in $\mathcal{PHC}^*$ can generate a new quadrilateral of G_3 from G_1 and G_2. To simulate this construction, we must remove an edge (v_2, v_4) of G_1 by rules in $\mathcal{PHC}$, but edge elimination rule cannot be applied since (v_2, v_4) are sandwiched between triangle faces and the other rules cannot eliminate edges.

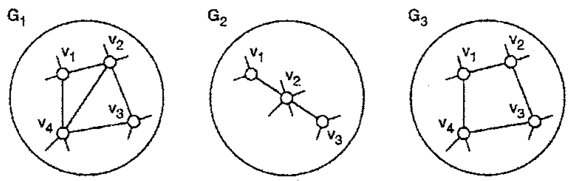

Figure 8: Rule 2 in $\mathcal{PHC}^*$

6 Concluding Remarks

We show that there exist a system of generating non-3-colorable planar graphs, where intermediate graphs in the system are restricted to be planar. Two calculi $\mathcal{PHC}$ and $\mathcal{PHC}^*$ are sound and complete graph construction system that generates the class of non-3-colorable planar graphs. $\mathcal{PHC}^*$ is simple but powerful calculus, since $\mathcal{PHC}^*$ p-simulates $\mathcal{PHC}$.

Relationship between construction in planar graph calculus and general graph calculus may be interesting. There is a structure that can replace crossing edges keeping the colorability condition, so that non-planar graphs can be mapped to planar graphs. Thus a class of graphs that have super-polynomial lower bound in $\mathcal{HC}$ may be associated with a class of graphs in a planar calculus. For future discussion, we would like to consider polynomial-time simulation of $\mathcal{PHC}^*$ by $\mathcal{PHC}$. Also lower bound of planar graph calculus is an interesting work.

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