

A MODIFIED HEINZ'S INEQUALITY

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Concerning the Heinz's inequality, we got the following result.

Theorem. The region of γ such that the operator inequality

$$(A^\gamma A^\alpha A^\gamma)^\beta \geq (A^\gamma B^\alpha A^\gamma)^\beta$$

holds for any operators A and B such as $A \geq B \geq bI$ (some $b > 0$) and for any given α and β such as $\alpha > 0$ and $\beta > 0$ is as follows ;

(1) $0 < \alpha \leq 1, 0 < \beta \leq 1, \quad -\infty < \gamma < +\infty$

(2) $0 < \alpha \leq 1, 1 < \beta \leq 2,$

$$\max \left\{ -\frac{1}{2}, \frac{-\alpha\beta}{2(\beta-1)} \right\} \leq \gamma \leq \min \left\{ 0, \frac{1-\alpha\beta}{2(\beta-1)} \right\}$$

(3) $0 < \alpha \leq 1, 2 < \beta \leq \frac{1}{\alpha}, \quad \gamma = 0$

(4) $0 < \alpha \leq 1, 2 < \beta,$

$$\max \left\{ \frac{2\alpha-1-\alpha\beta}{2(\beta-1)}, \frac{-\alpha\beta}{2(\beta-1)} \right\} \leq \gamma \leq \min \left\{ \frac{2\alpha-\alpha\beta}{2(\beta-1)}, \frac{1-\alpha\beta}{2(\beta-1)} \right\}$$

and (5) $1 < \alpha, 0 < \beta < 1,$

$$\max \left\{ 0, \frac{\alpha\beta-1}{2(1-\beta)} \right\} \leq \gamma.$$

In this lecture, we constituted counter examples of A and B for the cases where

$$0 < \alpha \leq 1, \quad 1 < \beta, \quad \min \left\{ \min \left(0, \frac{2\alpha-\alpha\beta}{2(\beta-1)} \right), \frac{1-\alpha\beta}{2(\beta-1)} \right\} < \gamma (\neq 0)$$

and

$$0 < \alpha \leq 1, \quad 1 < \beta, \quad \gamma < \max \left\{ \max \left(-\frac{1}{2}, \frac{2\alpha-1-\alpha\beta}{2(\beta-1)} \right), \frac{-\alpha\beta}{2(\beta-1)} \right\}$$

as follows.

For any a, b and μ such as $1 < a \leq \frac{17}{16}$, $0 < b \leq \frac{1}{2}$ and $0 < \mu \leq \min\{\alpha, 1 - \alpha\}$, let

$$x = a + \frac{(a-1)b^{1-\mu}}{a-1+b(1-b)} - \frac{(a-1)(b^{2-\mu} + b^{1-\mu})}{b(1-b)} - (1-b)b^{2-\alpha}. \quad (\#1)$$

Then

$$a - x = (1-b)b^{2-\alpha} + \frac{(a-1)[b^{2-\mu}\{a-1+b(1-b)\} + (a-1)b^{1-\mu}]}{b(1-b)\{a-1+b(1-b)\}} \quad (\#2)$$

and

$$1 - \frac{dx}{da} = \frac{(a-1)b^{1-\mu}\{a-1+2b(1-b)\}}{b(1-b)\{a-1+b(1-b)\}^2} + \frac{b^{2-\mu}}{b(1-b)}. \quad (\#3)$$

Since $\mu + 2 > 2 - \alpha$, $b^{\mu+2} < b^{2-\alpha}$ and, by taking $(1-b)b^{\mu+2} (< a - 1)$ sufficiently near by $a - 1$, we have $a - 1 < (1-b)b^{2-\alpha}$ and hence we can choose b such as

$$(1-b)b^{\mu+2} < a - 1 < (1-b)b^{2-\alpha}. \quad (\#4)$$

By (#2) and (#4), we have

$$\begin{aligned} a - 1 &< (1-b)b^{2-\alpha} < a - x \\ &< (1-b)b^{2-\alpha} + \frac{(a-1)b^{1-\mu}}{1-b} + \frac{(a-1)^2b^{-1-\mu}}{(1-b)^2} \\ &< (1-b)b^{2-\alpha} + b^{3-(\mu+\alpha)} + b^{3-(\mu+2\alpha)} \rightarrow 0 \quad (\text{as } a \rightarrow 1) \end{aligned} \quad (\#5)$$

because $b \rightarrow 0$ (as $a \rightarrow 1$) by (#4) and $0 < (\mu + 2\alpha) \leq \begin{cases} 3\alpha & (0 < \alpha \leq \frac{1}{2}) \\ 1 + \alpha & (\frac{1}{2} \leq \alpha < 1) \end{cases} \leq 2$.

And, by (#3) and by (#4), we have

$$\begin{aligned} \frac{b^{1-\mu}}{1-b} < 1 - \frac{dx}{da} &< \frac{2(a-1)b^{1-\mu}}{b(1-b)\{a-1+b(1-b)\}} + \frac{b^{1-\mu}}{1-b} \\ &< \frac{2(a-1)b^{-1-\mu}}{(1-b)^2} + \frac{b^{1-\mu}}{1-b} \\ &< \frac{2b^{1-(\mu+\alpha)}}{1-b} + \frac{b^{1-\mu}}{1-b} \end{aligned} \quad (\#6)$$

For any ϵ such as $0 < \epsilon < \frac{(a-b)(a-1)}{b(1-b)}$, let $\delta = \frac{b(1-b)\epsilon}{a-1}$. Then $0 < \delta < a - b$ and $a - b - \delta > 0$. And let

$$A = \begin{pmatrix} a & \sqrt{\epsilon(a-b-\delta)} \\ \sqrt{\epsilon(a-b-\delta)} & b + \epsilon + \delta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

where $y = b - \frac{\epsilon(a-b-\delta)}{a-1}$. Then A and B are self-adjoint and clearly $B \geq \min(x, y) I$. Since $x \rightarrow 1$ (as $a \rightarrow 1$) by (#5), we may assume that $\min(x, y) > \frac{b}{2}$ because we shall

consider the case where $\epsilon \rightarrow 0$ and nextly $a \rightarrow 1$. Since the proper polynomial of

$$A - B = \left(\frac{a-x}{\sqrt{\epsilon(a-b-\delta)}} \quad \sqrt{\epsilon(a-b-\delta)} \right) \epsilon + \delta + \frac{\epsilon(a-b-\delta)}{a-1}$$

$$\lambda^2 - \left(a-x+\epsilon+\delta + \frac{\epsilon(a-b-\delta)}{a-1} \right) \lambda + (a-x)(\epsilon+\delta) + \left(\frac{a-x}{a-1} - 1 \right) \epsilon(a-b-\delta)$$

and since $a-x > a-1$ by (#5), $(a-x)(\epsilon+\delta) + \left(\frac{a-x}{a-1} - 1 \right) \epsilon(a-b-\delta) > 0$ and $A \geq B$.

If $(A^\gamma A^\alpha A^\gamma)^\beta \geq (A^\gamma B^\alpha A^\gamma)^\beta$, then we have

$$\begin{aligned} & a^{2\gamma} b^{2\gamma} (x^\alpha - y^\alpha)^2 (a-b) \epsilon \left\{ 1 + \frac{o(\epsilon)}{\epsilon} \right\} \\ & \times \left[\left\{ a^{2\gamma\beta} x^{\alpha\beta} - a^{(\alpha+2\gamma)\beta} \right\} + a^{2\gamma\beta} \left\{ \frac{2\gamma(x^{\alpha\beta} - a^{\alpha\beta})}{a} + \frac{x^{\alpha\beta}(y^\alpha - x^\alpha)}{x^\alpha(a-b)} \right. \right. \\ & \quad \left. \left. + \frac{x^{\alpha\beta} b^{2\gamma} (x^\alpha - y^\alpha)^2}{(a-b)x^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})} - \alpha a^{\alpha\beta-1} \right\} \beta \epsilon + o(\epsilon) \right] \\ & \times b^{(\alpha+2\gamma)\beta} \left[\left\{ \frac{\alpha(1-b)}{a-1} - \frac{x^\alpha - y^\alpha}{(a-b)y^\alpha} + \frac{\alpha(a-b)}{b(a-1)} \right. \right. \\ & \quad \left. \left. + \frac{a^{2\gamma}(x^\alpha - y^\alpha)^2}{(a-b)y^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})} \right\} \beta \epsilon + o(\epsilon) \right] \\ & \leq \frac{a^{4\gamma} b^{4\gamma} (x^\alpha - y^\alpha)^4 \epsilon^2}{(a^{2\gamma} x^\alpha - y^\alpha b^{2\gamma})^2} \left\{ 1 + \frac{o(\epsilon)}{\epsilon} \right\} \\ & \times \left[\left\{ a^{(\alpha+2\gamma)\beta} - y^{\alpha\beta} b^{2\gamma\beta} \right\} + \left\{ (\alpha+2\gamma)a^{(\alpha+2\gamma)\beta-1} - \left(\frac{2\gamma(1-b)}{a-1} \right. \right. \right. \\ & \quad \left. \left. + \frac{x^\alpha - y^\alpha}{(a-b)y^\alpha} - \frac{a^{2\gamma}(x^\alpha - y^\alpha)^2}{(a-b)y^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})} \right) y^{\alpha\beta} b^{2\gamma\beta} \right\} \beta \epsilon + o(\epsilon) \right] \\ & \times \left[\left\{ b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta} x^{\alpha\beta} \right\} + \left\{ \frac{(\alpha+2\gamma)(1-b)b^{(\alpha+2\gamma)\beta}}{a-1} - \left(\frac{2\gamma}{a} \right. \right. \right. \\ & \quad \left. \left. + \frac{y^\alpha - x^\alpha}{x^\alpha(a-b)} + \frac{b^{2\gamma}(x^\alpha - y^\alpha)^2}{(a-b)x^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})} \right) a^{2\gamma\beta} x^{\alpha\beta} \right\} \beta \epsilon + o(\epsilon) \right] \quad (i) \end{aligned}$$

where $o(\epsilon)$ is a function of ϵ such that $\lim_{\epsilon \rightarrow 0} \frac{o(\epsilon)}{\epsilon} = 0$.

By multiplying $\{a^{2\gamma(1+\beta)} b^{4\gamma} (x^\alpha - y^\alpha)^4 \epsilon^2\}^{-1}$ to the both side of the inequality (i) and by putting $\epsilon \rightarrow 0$, we have

$$\begin{aligned} & \frac{\beta(\alpha^\beta - x^{\alpha\beta}) b^{(\alpha+2\gamma)\beta-2\gamma}}{(x^\alpha - b^\alpha)^2} \left\{ \frac{x^\alpha - b^\alpha}{b^\alpha} - \frac{\alpha(a-b^2)(a-b)}{b(a-1)} - \frac{a^{2\gamma}(x^\alpha - b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha - b^{\alpha+2\gamma})} \right\} \\ & \leq \frac{a^{2\gamma(1-\beta)}}{(b^{\alpha+2\gamma} - a^{2\gamma}x^\alpha)^2} \left\{ a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta} \right\} \left\{ b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta} x^{\alpha\beta} \right\} \quad (ii) \end{aligned}$$

and, by multiplying $b^{-2(\alpha+2\gamma)(\beta-1)}$ to the both side of (ii), we have

$$\begin{aligned} & \frac{\beta(a^{\alpha\beta} - x^{\alpha\beta})b^{2\gamma(1-\beta)-\alpha\beta+2\alpha}}{(x^\alpha - b^\alpha)^2} \left\{ \frac{x^\alpha - b^\alpha}{b^\alpha} - \frac{\alpha(a-b^2)(a-b)}{b(a-1)} \right. \\ & \qquad \qquad \qquad \left. - \frac{a^{2\gamma}(x^\alpha - b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha - b^{\alpha+2\gamma})} \right\} \\ & \leq \frac{a^{2\gamma(1-\beta)}}{\{1 - a^{2\gamma}x^\alpha b^{-(\alpha+2\gamma)}\}^2} \left\{ a^{(\alpha+2\gamma)\beta} b^{-(\alpha+2\gamma)\beta} - 1 \right\} \left\{ 1 - a^{2\gamma\beta} x^{\alpha\beta} b^{-(\alpha+2\gamma)\beta} \right\}. \quad (\text{iii}) \end{aligned}$$

Since $0 < a - x < 3b$ by (#5), we have

$$|a^{\alpha\beta-1} - x^{\alpha\beta-1}| \leq Kb \text{ for some constant } K > 0 \quad (\text{b}_1)$$

(case 1) Let $0 < \alpha \leq 1$, $1 < \beta$, $\min\left\{\min\left(0, \frac{2\alpha-\alpha\beta}{2(\beta-1)}\right), \frac{1-\alpha\beta}{2(\beta-1)}\right\} < \gamma (\neq 0)$.

In this case

$$\begin{cases} (\alpha + 2\gamma)\beta - 2\gamma - \alpha - \min\{\min(\alpha(\beta - 1), \alpha), 1 - \alpha\} \\ \quad = 2\gamma(\beta - 1) + \alpha\beta - \alpha - \min\{\min(\alpha(\beta - 1), \alpha), 1 - \alpha\} > 0 \\ \text{and } (\alpha + 2\gamma)(\beta - 1) = (\alpha + 2\gamma)\beta - 2\gamma - \alpha \\ \quad > \min\{\min(\alpha(\beta - 1), \alpha), 1 - \alpha\} \geq 0 \\ \text{and hence } \alpha + 2\gamma > 0 \text{ because } \beta > 1 \end{cases} \quad (\text{b}_1)$$

Let $a \rightarrow 1$. Then $b \rightarrow 0$ by (#4) and $x \rightarrow 1$ by (#5) and hence we have

$$\lim_{a \rightarrow 1} \frac{a^{2\gamma(1-\beta)}}{(b^{\alpha+2\gamma} - a^{2\gamma}x^\alpha)^2} \left\{ a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta} \right\} \left\{ b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta} x^{\alpha\beta} \right\} = -1$$

because $\alpha + 2\gamma > 0$ by (b₁).

Since $0 < \min\{\min(\alpha(\beta - 1), \alpha), 1 - \alpha\} \leq \min\{\alpha, 1 - \alpha\}$, let

$$\mu = \min\{\min(\alpha(\beta - 1), \alpha), 1 - \alpha\}.$$

Then $(\alpha + 2\gamma)\beta - 2\gamma - \alpha - \mu > 0$ by (b₁) and since $b \rightarrow 0$ (as $a \rightarrow 1$) by (#4) and $x \rightarrow 1$ (as $a \rightarrow 1$) by (#5), we have

$$\begin{aligned} \lim_{a \rightarrow 1} \frac{a^{\alpha\beta} - x^{\alpha\beta}}{a - 1} b^{(\alpha+2\gamma)\beta-2\gamma-1} &= \lim_{a \rightarrow 1} \alpha\beta \left(a^{\alpha\beta-1} - x^{\alpha\beta-1} \frac{dx}{da} \right) b^{(\alpha+2\gamma)\beta-2\gamma-1} \\ &= \alpha\beta \lim_{b \rightarrow 0} \left(1 - \frac{dx}{da} \right) b^{(\alpha+2\gamma)\beta-2\gamma-1} = 0 \text{ by (b}_1) \end{aligned}$$

because $\frac{b^{1-\mu}}{1-b} < 1 - \frac{dx}{da} < \frac{2b^{1-(\mu+\alpha)}}{1-b} + \frac{b^{1-\mu}}{1-b}$ by (#6) and hence we have

$$\lim_{a \rightarrow 1} \frac{\beta(a^{\alpha\beta} - x^{\alpha\beta})b^{(\alpha+2\gamma)\beta-2\gamma}}{(x^\alpha - b^\alpha)^2} \left\{ \frac{x^\alpha - b^\alpha}{b^\alpha} - \frac{\alpha(a-b^2)(a-b)}{b(a-1)} - \frac{a^{2\gamma}(x^\alpha - b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha - b^{\alpha+2\gamma})} \right\} = 0.$$

This contradicts (ii).

(case 2) Let $0 < \alpha \leq 1$, $1 < \beta$, $\gamma < \max\left\{\max\left(-\frac{1}{2}, \frac{2\alpha-1-\alpha\beta}{2(\beta-1)}\right), \frac{-\alpha\beta}{2(\beta-1)}\right\}$.

In this case

$$\begin{cases} 2\gamma(\beta-1) + \alpha\beta + 1 - \alpha - \max\{\max(\alpha\beta + 2 - \beta - \alpha, \alpha), 1 - \alpha\} < 0 \\ \text{and } (\alpha + 2\gamma)(\beta - 1) < -1 + \max\{\max(\alpha\beta + 2 - \beta - \alpha, \alpha), 1 - \alpha\} \\ \qquad \qquad \qquad = -\min\{(1 - \alpha)\min(\beta - 1, 1), \alpha\} \leq 0 \\ \text{and hence } \alpha + 2\gamma < 0 \text{ because } \beta > 1 \end{cases} \quad (b_2)$$

Let $a \rightarrow 1$. Then $b \rightarrow 0$ by (#4) and $x \rightarrow 1$ by (#5) and hence we have

$$\begin{aligned} & \lim_{a \rightarrow 1} \frac{a^{2\gamma(1-\beta)}}{\{1 - a^{2\gamma}x^\alpha b^{-(\alpha+2\gamma)}\}^2} \left\{ a^{(\alpha+2\gamma)\beta} b^{-(\alpha+2\gamma)\beta} - 1 \right\} \left\{ 1 - a^{2\gamma\beta} x^{\alpha\beta} b^{-(\alpha+2\gamma)\beta} \right\} \\ & = -1 \end{aligned}$$

because $\alpha + 2\gamma < 0$ by (b₂).

Since $0 < \min\{(1 - \alpha)\min(\beta - 1, 1), \alpha\} \leq \min\{\alpha, 1 - \alpha\}$, let

$$\mu = \min\{(1 - \alpha)\min(\beta - 1, 1), \alpha\}.$$

Then $2\gamma(1 - \beta) - \alpha\beta + \alpha - \mu > 0$ by (b₂) and since $b \rightarrow 0$ (as $a \rightarrow 1$) by (#4) and $x \rightarrow 1$ (as $a \rightarrow 1$) by (#5), we have

$$\begin{aligned} & \lim_{a \rightarrow 1} \frac{a^{\alpha\beta} - x^{\alpha\beta}}{a - 1} b^{2\gamma(1-\beta) - \alpha\beta + 2\alpha - 1} \\ & = \lim_{a \rightarrow 1} \alpha\beta \left(a^{\alpha\beta-1} - x^{\alpha\beta-1} \frac{dx}{da} \right) b^{2\gamma(1-\beta) - \alpha\beta + 2\alpha - 1} \\ & = \alpha\beta \lim_{b \rightarrow 0} \left(1 - \frac{dx}{da} \right) b^{2\gamma(1-\beta) - \alpha\beta + 2\alpha - 1} = 0 \text{ by (i)} \end{aligned}$$

because $\frac{b^{1-\mu}}{1-b} < 1 - \frac{dx}{da} < \frac{2b^{1-(\mu+\alpha)}}{1-b} + \frac{b^{1-\mu}}{1-b}$ by (#6) and hence we have

$$\lim_{a \rightarrow 1} \frac{\beta(a^{\alpha\beta} - x^{\alpha\beta})b^{2\gamma(1-\beta) - \alpha\beta + 2\alpha}}{(x^\alpha - b^\alpha)^2} \left\{ \frac{x^\alpha - b^\alpha}{b^\alpha} - \frac{\alpha(a-b^2)(a-b)}{b(a-1)} - \frac{a^{2\gamma}(x^\alpha - b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha - b^{\alpha+2\gamma})} \right\} = 0.$$

This contradicts (iii).