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Class A-f and A-f-paranormal operators

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1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

Furuta-Ito-Yamazaki [12] introduced the following class of non-normal operators.

Definition ([12]). $T \in$ class A $\Longleftrightarrow |T^2| \geq |T|^2$.

An operator T is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$ ([9][14]). Ando [3] showed that T is paranormal if and only if

$$T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0 \text{ for all } \lambda > 0,$$

and that if T is p-hyponormal (i.e., $(T^*T)^p \geq (TT^*)^p$) for some $p > 0$ or log-hyponormal (i.e., T is invertible and $\log T^*T \geq \log TT^*$), then T is paranormal. It was shown in [12] that class A includes the class of p-hyponormal and log-hyponormal operators, and is included in that of paranormal operators.

M. Fujii-D. Jung-S. H. Lee-M. Y. Lee-Nakamoto [8] introduced a generalization of class A. In fact, class A coincides with class A$(1, 1)$ ([24]).

Definition ([8]). For $s, t > 0$,

$$T \in \text{class A}(s,t) \Longleftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}.$$

On the other hand, Aluthge-Wang [1][2] introduced w-hyponormality. An operator T is said to be w-hyponormal if $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $T = U|T|$ is the polar decomposition and $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation), or equivalently,

$$(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \text{ and } |T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}}.$$

Ito-Yamazaki [17] showed that

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Longrightarrow A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$$

for $A, B \geq 0$ and $p, r \geq 0$, so that the class of w-hyponormal operators coincides with class $\mathrm{A}(\frac{1}{2}, \frac{1}{2})$.

As parallel concept to class $A(s,t)$, we introduced a generalization of paranormality in [26]. In fact, paranormality coincides with absolute-$(1,1)$-paranormality.

Definition ([26]). For $s, t > 0$,

$$\begin{aligned} &T \text{ is absolute-}(s,t)\text{-paranormal} \\ &\iff \||T|^s|T^*|^t x\|^t \geq \||T^*|^t x\|^{s+t} \text{ for every unit vector } x \in H \\ &\iff t|T^*|^t|T|^{2s}|T^*|^t - (s+t)\lambda^s|T^*|^{2t} + s\lambda^{s+t}I \geq 0 \text{ for all } \lambda > 0. \end{aligned}$$

We remark that class $\mathrm{A}(k)$ and absolute-k-paranormality introduced in [12] coincide with class $\mathrm{A}(k,1)$ and absolute-$(k,1)$-paranormality for each $k > 0$, respectively, and p-paranormality introduced in [7] coincides with absolute-(p,p)-paranormality for each $p > 0$.

2 Generalizations of class A and paranormality

We introduce further generalizations of class A and paranormality.

Definition 2.1. Let f be a non-negative continuous function on $[0, \infty)$.

(i) $T \in$ class A-$f \iff f(|T^*||T|^2|T^*|) \geq |T^*|^2$.

(ii) T is A-f-paranormal $\iff \lambda T \in$ class A-f for all $\lambda > 0$.

When f is a representing function of an operator connection σ (see [20]), we also call class A-f and A-f-paranormal class A-σ and A-σ-paranormal, respectively.

In fact, class A and paranormality coincide with class A-$\sharp$ and A-∇-paranormality, where ∇ and $\sharp$ are the arithmetic and geometric means, that is,

$$A \nabla B = \frac{1}{2}(A+B) \quad \text{and} \quad A \sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} \quad \text{for } A, B > 0,$$

and their representing functions are $f_\nabla(t) = \frac{1}{2}(1+t)$ and $f_\sharp(t) = t^{\frac{1}{2}}$, respectively. We remark that "$T \in$ class A $\Longrightarrow T$ is paranormal" can be shown as follows:

$$\begin{aligned} T \in \text{class A-}\sharp &\iff T \text{ is A-}\sharp\text{-paranormal} && \text{since } f_\sharp(\lambda^4 t) = \lambda^2 f_\sharp(t) \\ &\Longrightarrow T \text{ is A-}\nabla\text{-paranormal} && \text{since } f_\sharp(t) \leq f_\nabla(t). \end{aligned}$$

Moreover, we introduce further generalizations of class $\mathrm{A}(s,t)$ and absolute-(s,t)-paranormality.

Definition 2.2. Let f be a non-negative continuous function on $[0,\infty)$, and $s,t>0$.

(i) $T \in$ class $\mathrm{A}(s,t)$-$f \iff f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t}$.

(ii) T is $\mathrm{A}(s,t)$-f-paranormal $\iff \lambda T \in$ class $\mathrm{A}(s,t)$-f for all $\lambda > 0$.

When f is a representing function of an operator connection σ, we also call class $\mathrm{A}(s,t)$-f and $\mathrm{A}(s,t)$-f-paranormal class $\mathrm{A}(s,t)$-σ and $\mathrm{A}(s,t)$-σ-paranormal, respectively.

In fact, for each $s,t>0$, class $\mathrm{A}(s,t)$ and absolute-(s,t)-paranormality coincide with class $\mathrm{A}(s,t)$-$\sharp_{\frac{t}{s+t}}$ and $\mathrm{A}(s,t)$-$\nabla_{\frac{t}{s+t}}$-paranormality, where ∇_α and $\sharp_\alpha$ are generalized arithmetic and geometric means for $\alpha \in [0,1]$, that is,

$$A \nabla_\alpha B = (1-\alpha)A + \alpha B \quad \text{and} \quad A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } A,B>0,$$

and their representing functions are $f_{\nabla_\alpha}(t) = (1-\alpha)+\alpha t$ and $f_{\sharp_\alpha}(t) = t^\alpha$, respectively.

3 Properties of class A-f and A-f-paranormality

The following results have been shown on class $\mathrm{A}(s,t)$ and absolute-(s,t)-paranormal operators.

Theorem 3.A ([8][15][17][25][26]).

(i) *T is p-hyponormal for some $p>0$ or log-hyponormal $\Longrightarrow T \in$ class $A(s,t)$ for all $s,t>0$.*

(ii) *For each $s,t>0$, $T\in$ class $A(s,t) \Longrightarrow T$ is absolute-(s,t)-paranormal.*

(iii) *T is absolute-(s,t)-paranormal for some $s,t>0 \Longrightarrow T$ is normaloid (i.e., $\|T\| = r(T)$), where $r(T)$ is the spectral radius of T.*

(iv) *For each $0<s_1\leq s_2$ and $0<t_1\leq t_2$,*

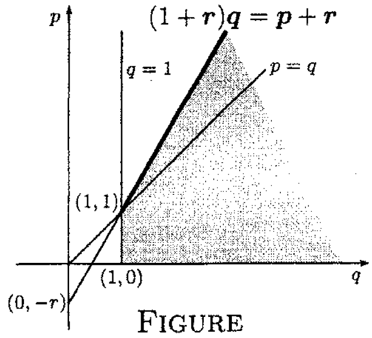
$$T \in \text{class } A(s_1,t_1) \Longrightarrow T \in \text{class } A(s_2,t_2),$$
$$T \text{ is absolute-}(s_1,t_1)\text{-paranormal} \Longrightarrow T \text{ is absolute-}(s_2,t_2)\text{-paranormal.}$$

(v) *T is invertible and absolute-(p,p)-paranormal for all $p>0 \Longrightarrow T$ is log-hyponormal.*

Theorem 3.B ([17][27]). *Let $s,t\in(0,1]$. Then*

$$T \in \text{class } A(s,t) \Longrightarrow T^n \in \text{class } A(\tfrac{s}{n},\tfrac{t}{n}) \text{ for every positive integer } n.$$

These were obtained as applications of the following result.



FIGURE

Theorem F (Furuta inequality [10]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

We remark that Theorem F yields Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [5][19] and also an elementary one-page proof in [11]. It is shown in [22] that the domain of p, q and r drawn in Figure is the best possible for Theorem F.

First, we show monotonicity of class A(s,t)-$f_{s,t}$ for s and t as a generalization of (iv) in Theorem 3.A.

Theorem 3.1. *Let $s_0, t_0 > 0$ and $\{f_{s,t} \mid s \geq s_0,\ t \geq t_0\}$ be a family of non-negative operator monotone functions on $[0,\infty)$ satisfying $f_{s,t}(x^t g(x)^s) = x^t$, where g is a continuous increasing function. Then*

$$T \text{ is invertible and } T \in \text{class } A(s_0,t_0)\text{-}f_{s_0,t_0}$$
$$\Longrightarrow T \in \text{class } A(s,t)\text{-}f_{s,t} \text{ for all } s \geq s_0 \text{ and } t \geq t_0.$$

We use the following result in order to give a proof of Theorem 3.1.

Theorem 3.C ([23]). *Let A and B be positive operators, and let $\{\psi_r \mid r \geq a\}$ $(a > 0)$ be a family of non-negative operator monotone functions on $[0,\infty)$ satisfying $\psi_r(x^r g(x)) = x^r$, i.e., $x^{-r}\ \sigma_{\psi_r}\ g(x) = 1$, where g is a continuous increasing function. Then the following hold:*

(i) *If $A^a\ \sigma_{\psi_a}\ B \geq I$, then $A^r\ \sigma_{\psi_r}\ B$ is increasing for $r \geq a$.*

(ii) *If A and B are invertible and if $A^a\ \sigma_{\psi_a}\ B \leq I$, then $A^r\ \sigma_{\psi_r}\ B$ is decreasing for $r \geq a$.*

We also use the following result which is an extension of a result in [17].

Theorem 3.D ([16]). *Let A and B be positive operators, and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying $f(x)g(x) = x$. Then the following hold:*

(i) *$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ ensures $A - g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \geq A^{\frac{1}{2}}E_BA^{\frac{1}{2}} - g(0)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$.*

(ii) *$B \geq f(B^{\frac{1}{2}}AB^{\frac{1}{2}})$ ensures $g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) - A \geq g(0)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}} - A^{\frac{1}{2}}E_BA^{\frac{1}{2}}$.*

Here E_B and $E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$ are the orthoprojections to $\mathcal{N}(B)$ and $\mathcal{N}(A^{\frac{1}{2}}BA^{\frac{1}{2}})$, respectively.

Proof of Theorem 3.1. T belongs to class $\mathrm{A}(s_0, t_0)$-f_{s_0,t_0} if and only if

$$f_{s_0,t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0}) \geq |T^*|^{2t_0}.$$

Since $f_{s_0,t}(x^t g(x)^{s_0}) = x^t$ and $g(x)^{s_0}$ is a continuous increasing function,

$$|T^*|^{-t} f_{s_0,t}(|T^*|^t|T|^{2s_0}|T^*|^t)|T^*|^{-t} \geq |T^*|^{-t_0} f_{s_0,t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0})|T^*|^{-t_0}$$

holds for $t \geq t_0$ by (i) of Theorem 3.C. Hence

$$f_{s_0,t}(|T^*|^t|T|^{2s_0}|T^*|^t) \geq |T^*|^{2t}.$$

By (i) of Theorem 3.D, this implies

$$|T|^{2s_0} \geq f^{\perp}_{s_0,t}(|T|^{s_0}|T^*|^{2t}|T|^{s_0}),$$

where $f^{\perp}_{s,t}(x) = \frac{x}{f_{s,t}(x)}$. Since

$$f^{\perp}_{s,t}(x^s g^{-1}(x)^t) = \frac{x^s g^{-1}(x)^t}{f_{s,t}(x^s g^{-1}(x)^t)} = x^s$$

and $g^{-1}(x)^t$ is a continuous increasing function,

$$|T|^{-s} f^{\perp}_{s,t}(|T|^s|T^*|^{2t}|T|^s)|T|^{-s} \leq |T|^{-s_0} f^{\perp}_{s_0,t}(|T|^{s_0}|T^*|^{2t}|T|^{s_0})|T|^{-s_0}$$

holds for $s \geq s_0$ by (ii) of Theorem 3.C. Hence

$$|T|^{2s} \geq f^{\perp}_{s,t}(|T|^s|T^*|^{2t}|T|^s).$$

By (ii) of Theorem 3.D, this implies

$$f_{s,t}(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t},$$

that is, T belongs to class $\mathrm{A}(s,t)$-$f_{s,t}$. □

Secondly, we show a sufficient condition for log-hyponormality in terms of class $\mathrm{A}(s,t)$-f as a generalization of (v) in Theorem 3.A.

Theorem 3.2. *Let f be a non-negative, continuously differentiable and concave (or convex) function on $[0, \infty)$ satisfying $f(1) \leq 1$ and $0 < f'(1) < 1$, and $p_0 > 0$. Then*

$$T \text{ is invertible and } T \in \text{class } A(\theta' p, \theta p)\text{-}f \text{ for all } p \in (0, p_0) \Longrightarrow T \text{ is log-hyponormal},$$

where $\theta = f'(1)$ and $\theta + \theta' = 1$.

Proof. There exists a continuous function g on $[0,\infty)$ such that $f'(g(x)) = \frac{f(x)-f(1)}{x-1}$ for $x \neq 1$ by the mean value theorem and concavity (or convexity) of f. Then we have

$$\begin{aligned}\frac{|T^*|^{2\theta p}-I}{p} &\leq \frac{f(|T^*|^{\theta p}|T|^{2\theta' p}|T^*|^{\theta p})-f(1)I}{p}\\ &= f'(g(|T^*|^{\theta p}|T|^{2\theta' p}|T^*|^{\theta p}))\frac{|T^*|^{\theta p}|T|^{2\theta' p}|T^*|^{\theta p}-I}{p}\\ &= f'(g(|T^*|^{\theta p}|T|^{2\theta' p}|T^*|^{\theta p}))\left(|T^*|^{\theta p}\frac{|T|^{2\theta' p}-I}{p}|T^*|^{\theta p}+\frac{|T^*|^{2\theta p}-I}{p}\right)\end{aligned}$$

for $0<p<p_0$. By tending $p \to +0$, we have

$$\log |T^*|^{2\theta} \leq \theta\left(\log |T|^{2\theta'} + \log |T^*|^{2\theta}\right),$$

hence T is log-hyponormal. □

Thirdly, we show a result on powers of class A(s,t)-f operators as a generalization of Theorem 3.B.

Theorem 3.3. *Let f be a non-negative operator monotone function on $[0,\infty)$ satisfying $f(0)=0$, and $s,t \in (0,1]$. Then*

$$\begin{aligned}&T \in \text{class } A(s,t)\text{-}f \text{ and } T \in \text{class } A\\ &\qquad\Longrightarrow T^n \in \text{class } A\left(\tfrac{s}{n},\tfrac{t}{n}\right)\text{-}f \text{ for every positive integer } n.\end{aligned}$$

We use the following result in order to give a proof of Theorem 3.3.

Theorem 3.E ([17][27]). *If T belongs to class A, then*

$$|T^n|^{\frac{2}{n}} \geq |T|^2 \quad \text{and} \quad |T^*|^2 \geq |T^{n*}|^{\frac{2}{n}}$$

for every positive integer n.

We also use the following which is an extension of results in [18][27].

Lemma 3.4. *Let A, B and C be positive operators, and f be an operator monotone function on $[0,\infty)$ satisfying $f(0)\leq 0$. Then*

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \text{ and } B \geq C \Longrightarrow f(C^{\frac{1}{2}}AC^{\frac{1}{2}}) \geq C.$$

Proof. There exists an operator X such that

$$B^{\frac{1}{2}}X = X^*B^{\frac{1}{2}} = C^{\frac{1}{2}} \quad \text{and} \quad \|X\| \leq 1$$

by Douglas' theorem [4]. Then we have

$$f(C^{\frac{1}{2}}AC^{\frac{1}{2}}) = f(X^*B^{\frac{1}{2}}AB^{\frac{1}{2}}X) \geq X^*f(B^{\frac{1}{2}}AB^{\frac{1}{2}})X \geq X^*BX = C$$

by Hansen's inequality [13]. □

Proof of Theorem 3.3. By Löwner-Heinz theorem and Theorem 3.E,

$$|T^n|^{\frac{2s}{n}} \geq |T|^{2s} \quad \text{and} \quad |T^*|^{2t} \geq |T^{n*}|^{\frac{2t}{n}}.$$

Hence $f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t}$ implies

$$f(|T^{n*}|^{\frac{t}{n}}|T|^{2s}|T^{n*}|^{\frac{t}{n}}) \geq |T^{n*}|^{\frac{2t}{n}}$$

by Lemma 3.4, and

$$f(|T^{n*}|^{\frac{t}{n}}|T^n|^{\frac{2s}{n}}|T^{n*}|^{\frac{t}{n}}) \geq |T^{n*}|^{\frac{2t}{n}}$$

since f is operator monotone. □

The following can be obtained similarly by using Lemma 3.4, which is an extension of a result on class A(s,t) operators in [21].

Proposition 3.5. *Let f be a non-negative operator monotone function on $[0,\infty)$ satisfying $f(0) = 0$, and $s,t \in (0,1]$. Then*

$$T \in \text{class } A(s,t)\text{-}f \Longrightarrow T|_{\mathcal{M}} \in \text{class } A(s,t)\text{-}f,$$

where $\mathcal{M}$ is an invariant subspace of T and $T|_{\mathcal{M}}$ is the restriction of T onto $\mathcal{M}$.

Proof. Let P be the orthogonal projection onto $\mathcal{M}$, and $T_0 = TP$. Then

$$|T_0|^{2s} = (P|T|^2P)^s \geq P|T|^{2s}P$$

by Hansen's inequality [13], so that $|T_0^*|^t|T_0|^{2s}|T_0^*|^t \geq |T_0^*|^t|T|^{2s}|T_0^*|^t$. And also,

$$|T_0^*|^{2t} = (TPT^*)^t \leq (TT^*)^t = |T^*|^{2t}$$

by Löwner-Heinz theorem. Hence $f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t}$ implies

$$f(|T_0^*|^t|T|^{2s}|T_0^*|^t) \geq |T_0^*|^{2t}$$

by Lemma 3.4, and

$$f(|T_0^*|^t|T_0|^{2s}|T_0^*|^t) \geq |T_0^*|^{2t}$$

since f is operator monotone. □

Lastly, we show a result on Aluthge transformation of class A(s,t)-f operators. We remark that for each $s,t > 0$, if T belongs to class $A(s,t)$, then $\tilde{T}_{s,t}$ is $\frac{\min\{s,t\}}{s+t}$-hyponormal ([15][17]). An operator T is said to be f-hyponormal if $f(T^*T) \geq f(TT^*)$ for a continuous function f ([6]).

Theorem 3.6. *Let f and g be non-negative continuous increasing functions on $[0,\infty)$ satisfying $f(x)g(x) = x$ and $g(0) = 0$, and $s,t > 0$. If $T \in$ class $A(s,t)$-f, then the following hold, where $T = U|T|$ is the polar decomposition and $\tilde{T}_{s,t} = |T|^sU|T|^t$:*

(i) $\tilde{T}_{s,t}$ *is* f*-hyponormal if* $f \circ g^{-1}$ *is operator monotone and* $x^t \geq (f \circ g^{-1})(x^s)$.

(ii) $\tilde{T}_{s,t}$ *is* g*-hyponormal if* $g \circ f^{-1}$ *is operator monotone and* $(g \circ f^{-1})(x^t) \geq x^s$.

We use the following result in order to give a proof of Theorem 3.6.

Lemma 3.F ([16]). *Let* A *be a positive operator and* U *be a partial isometry such that* $N(U) \subseteq N(A)$, *and let* f *be a continuous function on* $[0, \infty)$. *Then*

$$Uf(A)U^* = f(UAU^*) - f(0)(I - UU^*).$$

Proof of Theorem 3.6. Noting that $N(U^*) = N(|T^*|) \subseteq N(|T^*|^t|T|^{2s}|T^*|^t)$,

$$\begin{aligned} f\left((\tilde{T}_{s,t})^*\tilde{T}_{s,t}\right) &= f(|T|^tU^*|T|^{2s}U|T|^t) \\ &= f(U^*|T^*|^t|T|^{2s}|T^*|^tU) \\ &= U^*f(|T^*|^t|T|^{2s}|T^*|^t)U + f(0)(I - U^*U) \quad \text{by Lemma 3.F} \\ &\geq U^*|T^*|^{2t}U \\ &= |T|^{2t}. \end{aligned}$$

On the other hand, $f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t}$ implies

$$|T|^{2s} \geq g(|T|^s|T^*|^{2t}|T|^s) = g(|T|^sU|T|^{2t}U^*|T|^s) = g\left(\tilde{T}_{s,t}(\tilde{T}_{s,t})^*\right)$$

by (i) of Theorem 3.D. If $f \circ g^{-1}$ is operator monotone and $x^t \geq (f \circ g^{-1})(x^s)$,

$$f\left((\tilde{T}_{s,t})^*\tilde{T}_{s,t}\right) \geq |T|^{2t} \geq (f \circ g^{-1})(|T|^{2s}) \geq f\left(\tilde{T}_{s,t}(\tilde{T}_{s,t})^*\right),$$

hence $\tilde{T}_{s,t}$ is f-hyponormal. If $g \circ f^{-1}$ is operator monotone and $(g \circ f^{-1})(x^t) \geq x^s$,

$$g\left((\tilde{T}_{s,t})^*\tilde{T}_{s,t}\right) \geq (g \circ f^{-1})(|T|^{2t}) \geq |T|^{2s} \geq g\left(\tilde{T}_{s,t}(\tilde{T}_{s,t})^*\right),$$

hence $\tilde{T}_{s,t}$ is g-hyponormal. □

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