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Singular sets for curvature equations of order k

広島大学・大学院理学研究科　滝本　和広 (Kazuhiro Takimoto)
Graduate School of Science
Hiroshima University

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ and K be a compact set contained in Ω. We consider the so-called *curvature equations* of the form

$$H_k[u]=S_k(\kappa_1,\ldots,\kappa_n)=\psi \quad \text{in } \Omega\setminus K, \tag{1.1}$$

where, for a function $u\in C^2(\Omega)$, $\kappa_1,\ldots,\kappa_n$ are the principal curvatures of the graph of the function u, namely, the eigenvalues of the matrix

$$\mathcal{C}=D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=\frac{1}{\sqrt{1+|Du|^2}}\left(I-\frac{Du\otimes Du}{1+|Du|^2}\right)D^2u, \tag{1.2}$$

and $S_k, k=1,\ldots,n$, is the k-th elementary symmetric function, that is,

$$S_k(\kappa)=\sum \kappa_{i_1}\cdots\kappa_{i_k}, \tag{1.3}$$

where the sum is taken over increasing k-tuples, $i_1,\ldots,i_k\subset\{1,\ldots,n\}$. The mean, scalar and Gauss curvatures correspond respectively to the special cases $k=1,2,n$ in (1.3).

Here we consider *generalized solutions* to curvature equations, which are solutions in a certain weak sense. In [23] the author introduced the notion of generalized solutions to

$$H_k[u]=\nu, \tag{1.4}$$

where ν is a non-negative Borel measure. Generalized solutions form a wider class than classical solutions or viscosity solutions under the convexity assumptions. In section 2, we give a definition of generalized solutions to curvature equations with some examples.

In the previous article [24], we discussed the removability of isolated singularities (i.e. $K = \{\text{one point}\}$) for solutions to homogeneous k-curvature equation (i.e. (1.1) with $\psi \equiv 0$), both in the viscosity sense and in the generalized sense. Among other things, we proved that for $1 \leq k \leq n-1$, isolated singularities are always removable under the continuity assumption on the solution. In this article, we study the removability of singular sets of generalized solutions to (1.4). We consider the following problem.

Problem: How large a singular set K can be allowed in the removable singularity theorem?

For the case of $k = 1$, which corresponds to the mean curvature equation in (1.1), such removability problems have been already studied. Bers [2], Nitsche [20] and De Giorgi-Stampacchia [12] proved the removability of isolated singularities for solutions to the equation of minimal surface ($\psi \equiv 0$) or constant mean curvature (ψ is a constant function). Serrin [21, 22] studied the same problem for a more general class of quasilinear equations of mean curvature type. He proved that any weak solution u to the mean curvature type equation in $\Omega \setminus K$ can be extended to a weak solution in Ω if the singular set K is a compact set of vanishing $(n-1)$-dimensional Hausdorff measure. For various semilinear and quasilinear equations, there are a number of papers concerning removability results.

Here we remark that (1.1) is a quasilinear equation for $k = 1$ while it is a *fully nonlinear* equation for $k \geq 2$. It is much harder to study the fully nonlinear equations' case. For Monge-Ampère equations' case, there are some results about the removability of isolated singularities (see, for example, [3, 14]). However, until recently, no results are known for other types of fully nonlinear elliptic PDEs except for the recent work of Labutin [16, 17, 18] who have studied the case of uniformly elliptic equations and Hessian equations.

We note that for the case $k = n$ which corresponds to Gauss curvature case, one has a solution to (1.1) with non-removable singularity at a single point. For example,

$$u(x) = \alpha|x|, \quad x \in \Omega = B_1(0) = \{|x| < 1\} \tag{1.5}$$

where $\alpha > 0$, satisfies the equation (1.1) with $\psi \equiv 0$ and $K = \{0\}$, in the classical sense as well as in the generalized sense. However, u does not satisfy $H_n[u] = 0$ in $\Omega = B_1(0)$ (see Example 2.1 (1)). Accordingly, it is sufficient to discuss our **Problem** for $1 \leq k \leq n-1$.

We state our main result in this article. We establish a removability result for a singular set of a generalized solution to the curvature equation. This is a Serrin type removability result for the curvature equation.

Theorem 1.1. *Let Ω be a convex domain in $\mathbb{R}^n$ and $K \Subset \Omega$ be a compact set whose $(n-k)$-dimensional Hausdorff measure is zero. Let $1 \leq k \leq n-1$, $\psi \in L^1(\Omega)$ be a non-negative function and u be a continuous function in $\Omega \setminus K$. We assume that u is a locally convex function in Ω and a generalized solution to $H_k[u] = \psi\, dx$ in $\Omega \setminus K$. Then u can be defined in the whole domain Ω as a generalized solution to $H_k[u] = \psi\, dx$ in Ω.*

2 Viscosity solutions and generalized solutions

In this section we give the definition of viscosity solutions and generalized solutions to curvature equations, both of which are solutions in a weak sense.

For a large class of elliptic PDEs, it is well known that one can consider a function which is not necessarily differentiable in a usual (classical) sense as a solution to the equation. Many mathematicians have investigated solutions in a generalized sense, such as *weak solutions* for quasilinear equations of divergence type and *distributional solutions* for semilinear equations. Moreover, in many nonlinear PDEs, the notion of *viscosity solutions* provides existence and uniqueness theorem under mild hypotheses. Crandall, Evans, Ishii, Lions and others have developed the theory of viscosity solutions since early 1980's (we refer to [9, 10, 11, 19]). First, we define the notion of viscosity solutions to the equation

$$H_k[u] = \psi(x) \quad \text{in } \Omega, \tag{2.1}$$

where Ω is an arbitrary open set in $\mathbb{R}^n$ and $\psi \in C^0(\Omega)$ is a non-negative function.

We define the admissible set of k-th elementary symmetric function S_k by

$$\begin{aligned} \Gamma_k &= \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid S_k(\lambda + \mu) \geq S_k(\lambda) \quad \text{for all } \mu_i \geq 0\} \qquad (2.3) \\ &= \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid S_j(\lambda) \geq 0, \quad j = 1, \ldots, k\}. \end{aligned}$$

Let Ω be an open set in $\mathbb{R}^n$. We say that a function $u \in C^2(\Omega)$ is *k-admissible* if $\kappa = (\kappa_1, \ldots, \kappa_n)$ belongs to Γ_k for every point $x \in \Omega$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of the graph of u at x.

Remark 2.1. Let $1 \leq k \leq n$ and $u \in C^2(\Omega)$.
(i) Γ_k is a cone in $\mathbb{R}^n$ with vertex at the origin, and

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \Gamma_+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i \geq 0, \quad i = 1, \ldots, n\}. \tag{2.4}$$

(ii) u is n-admissible if and only if u is (locally) convex in Ω.

Except for the case $k = 1$, equation (2.1) is not elliptic on all functions $u \in C^2(\Omega)$, but the following property is known.

Proposition 2.1. *The operator H_k is degenerate elliptic for k-admissible functions.*

This proposition is proved by Caffarelli, Nirenberg and Spruck [4, 5].

Now we define a viscosity solution to (2.1). A function $u \in C^0(\Omega)$ is said to be a *viscosity subsolution* (resp. *viscosity supersolution*) to (2.1) if for any k-admissible function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a maximum (resp. minimum) point of $u - \varphi$, we have

$$H_k[\varphi](x_0) \geq \psi(x_0) \quad (\text{resp. } \leq \psi(x_0)). \tag{2.5}$$

A function u is said to be a *viscosity solution* to (2.1) if it is both a viscosity subsolution and supersolution.

Remark 2.2. (i) The notion of viscosity subsolution does not change if all $C^2(\Omega)$ functions are allowed as comparison functions φ.

(ii) One can prove that a function $u \in C^2(\Omega)$ is a viscosity solution to (2.1) if and only if it is a k-admissible classical solution. Therefore, the notion of viscosity solutions is weaker than that of classical solutions.

The existence and uniqueness of Lipschitz solutions to the Dirichlet problem in the viscosity sense was established by Trudinger [25], under natural geometric restrictions and under relatively weak regularity hypotheses on ψ, for instance $\psi^{\frac{1}{k}} \in C^{0,1}(\overline{\Omega})$.

However, the requirement that ψ is a regular function is a serious limitation for curvature equations (for example, see Example 2.1 (1)). Weak solutions for quasilinear equations and distributional solutions for semilinear equations have an integral nature, while viscosity solutions do not have. It is difficult to define solutions with an integral nature for fully nonlinear PDEs. For some special types of fully nonlinear PDEs, one can introduce an appropriate notion of solutions that have such property, such as *generalized solutions* for the class of Monge-Ampère type equations (see [1, 6]) and for Hessian equations (see [8, 26, 27, 28]). We note that for $k = n$, (1.1) is a Monge-Ampère type equation. However, the concept of generalized solutions to curvature equations for $k = 1, \ldots, n-1$ has not been treated in the literature. Recently, the author [23] established a definition of generalized solutions for such cases as well as for $k = n$, which allows the inhomogeneous term ψ to be a Borel measure.

We give the definition of generalized solutions to curvature equations. We state some notations which we shall use. We assume that Ω is an open, convex and bounded subset of $\mathbb{R}^n$ and we look for solutions in the class of convex and (uniformly) Lipschitz functions defined in Ω. For a point $x \in \Omega$, let $\mathrm{Nor}(u;x)$ be the set of downward normal unit vectors to u at $(x, u(x))$. For a non-negative number ρ and a Borel subset η of Ω, we set

$$Q_\rho(u;\eta) = \{z \in \mathbb{R}^n \mid z = x + \rho v,\ x \in \eta,\ v \in \gamma_u(x)\}, \tag{2.6}$$

where $\gamma_u(x)$ is a subset of $\mathbb{R}^n$ defined by

$$\gamma_u(x) = \{(a_1, \ldots, a_n) \mid (a_1, \ldots, a_n, a_{n+1}) \in \mathrm{Nor}(u;x)\}. \tag{2.7}$$

The following theorem, which is an analogue of the so-called Steiner type formula, plays an important part in the definition of generalized solutions.

Theorem 2.2. *([23, Theorem 1.1]) Let Ω be an open convex bounded set in $\mathbb{R}^n$ and u be a convex and Lipschitz function defined in Ω. Then the following hold.*

(i) For every Borel subset η of Ω and for every $\rho \geq 0$, the set $Q_\rho(u;\eta)$ is Lebesgue measurable.

(ii) There exist $n+1$ non-negative, finite Borel measures $\sigma_0(u;\cdot), \ldots, \sigma_n(u;\cdot)$ such that

$$\mathcal{L}^n(Q_\rho(u;\eta)) = \sum_{m=0}^{n} \binom{n}{m} \sigma_m(u;\eta)\rho^m \tag{2.8}$$

for every $\rho \geq 0$ and for every Borel subset η of Ω, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

Remark 2.3. The measures $\sigma_k(u;\cdot)$ determined by u are characterized by the following two properties.

(i) If $u \in C^2(\Omega)$, then for every Borel subset η of Ω,

$$\binom{n}{k}\sigma_k(u;\eta) = \int_\eta H_k[u](x)\,dx. \tag{2.9}$$

(The proof is given in [23, Proposition 2.1].)

(ii) If u_i converges uniformly to u on every compact subset of Ω, then

$$\sigma_k(u_i;\cdot) \rightharpoonup \sigma_k(u;\cdot) \quad \text{(weakly)} \tag{2.10}$$

Therefore we can say that for $k = 1, \ldots, n$, the measure $\binom{n}{k}\sigma_k(u;\cdot)$ generalizes the integral of the function $H_k[u]$. Moreover, if the curvature equation (1.1) has a convex solution, then ψ must be a Borel measure.

Now we state the definition of a generalized solution to (1.4).

Definition 2.3. Let Ω be an open convex bounded set in $\mathbb{R}^n$ and ν be a non-negative finite Borel measure on Ω. A convex and Lipschitz function $u \in C^{0,1}(\Omega)$ is said to be a *generalized solution* to

$$H_k[u] = \nu \quad \text{in } \Omega, \tag{2.11}$$

if it holds that

$$\binom{n}{k}\sigma_k(u;\eta) = \nu(\eta) \tag{2.12}$$

for every Borel subset η of Ω.

We note that one can also define the notion of a generalized solution stated above when Ω is merely an open set which is not necessarily convex and u is a locally convex function in Ω. Indeed, we shall say that u is a generalized solution to (2.11) if for any point $x \in \Omega$ and for any ball $B = B_R(x) \subset \Omega$, (2.12) holds for every Borel subset η of $B_R(x)$.

Here are some examples of generalized solutions.

Example 2.1. Let $B_1(0)$ be a unit ball in $\mathbb{R}^n$ and α be a positive constant.

(1) Let $u_1(x) = \alpha|x|$, which is a function we have already seen in (1.5), is a generalized solution to

$$H_n[u_1] = \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^n \omega_n \delta_0 \quad \text{in } B_1(0), \tag{2.13}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and δ_0 is the Dirac measure at 0.

(2) Let $u_2(x) = \alpha\sqrt{x_1^2 + \cdots + x_k^2}$, where $x = (x_1, \ldots, x_n)$. One can see that u_2 cannot be a viscosity solution to $H_k[u_2] = \psi$ in $B_1(0)$ for any $\psi \in C^0(B_1(0))$. However, u_2 is a generalized solution to

$$H_k[u_2] = \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^k \omega_k \mathcal{L}^{n-k}\lfloor T \quad \text{in } B_1(0), \tag{2.14}$$

where ω_k denotes the k-dimensional measure of the unit ball in $\mathbb{R}^k$ and $T = \{(x_1, \ldots, x_n) \in B_1(0) \mid x_1 = \cdots = x_k = 0\}$.

We state some properties of generalized solutions to (2.11) defined above.

Remark 2.4. (i) If $u \in C^2(\Omega)$ is a generalized solution to (2.11), then u is a classical solution to $H_k[u] = \psi$ for some $\psi \in C^0(\Omega)$ and $\nu = \psi(x)\,dx$.

(ii) For $k = n$ which corresponds to Gauss curvature equation, there is a notion of generalized solutions, since they are in a class of Monge-Ampère type. As far as the Gauss curvature equation is concerned, the definition of generalized solutions for Monge-Ampère type equations coincides with the one introduced in Definition 2.3. (The proof is given in [23, Theorem 3.3].)

In the last part of this section, we prove that the notion of generalized solutions is weaker than that of viscosity solutions in some sense.

Proposition 2.4. *Let $1 \le k \le n$ and Ω be an open convex bounded set in $\mathbb{R}^n$. Let ψ be a positive function with $\psi^{1/k} \in C^{0,1}(\overline{\Omega})$ and u be a locally convex function in Ω. If u is a viscosity solution to $H_k[u] = \psi$ in Ω, then u is a generalized solution to $H_k[u] = \nu$ in Ω, where $\nu = \psi(x)\,dx$.*

Proof. Let x_0 be any point in Ω. We wish to show that u is a generalized solution to $H_k[u] = \nu\,dx$ in some ball centered at x_0. We fix a sufficiently small constant $r > 0$ such that

$$\|\psi\|_{L^{\frac{n}{k}}(B_r(x_0))} < \frac{1}{2}\binom{n}{k}\omega_n^{\frac{k}{n}}, \tag{2.15}$$

which assures C^0-a priori bound for a solution to $H_k[u] = \psi$ (see [25]). We may assume that $\Omega = B_r(x_0)$.

First we extend the function u to a convex function defined in $\mathbb{R}^n$, which is proved in [7]. Let φ be a non-negative function in $C_0^\infty(\mathbb{R}^n)$ vanishing outside $B_1(0)$ and satisfying $\int_{B_1(0)} \varphi\,dx = 1$. We define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right), \tag{2.16}$$

and set $u_i = \varphi_{\frac{1}{i}} * u$, the regularization of u. It turns out that u_i converges uniformly to u in Ω as $i \to \infty$.

Next, let $\{\Omega_i\}_{i=1}^\infty$ be a sequence of convex domains such that $\Omega_1 \Subset \Omega_2 \Subset \cdots$ and that $\Omega = \bigcup_{i=1}^\infty \Omega_i$. In the case of $1 \le k \le n-1$, we take $\{\psi_i\}_{i=1}^\infty \subset C^\infty(\overline{\Omega})$ which satisfies that

$$\psi_i \to \psi \text{ in } L^1(\Omega) \text{ and uniformly in } C^0(\overline{\Omega_j}) \text{ for every } j \in \mathbb{N}, \tag{2.17}$$

$$\text{for every } j \in \mathbb{N},\ \sup_{i=1,2,\ldots} |D\psi_i| \text{ is bounded in } \Omega_j, \tag{2.18}$$

$$S_k(\kappa_1', \ldots, \kappa_{n-1}', 0) \ge \psi_i \text{ on } \partial\Omega, \tag{2.19}$$

where $\kappa' = (\kappa'_1, \dots, \kappa'_{n-1})$ denotes the principal curvatures of the boundary $\partial\Omega$ and that

$$\psi_i > 0 \text{ in } \overline{\Omega}. \tag{2.20}$$

For $k = n$, the condition (2.20) is replaced by

$$\psi_i > 0 \text{ in } \Omega \quad\text{and}\quad \psi_i = 0 \text{ on } \partial\Omega. \tag{2.21}$$

One can get $\{\psi_i\}_{i=1}^{\infty}$ by using the regularizations of ψ.

Now we consider the following Dirichlet problem:

$$\begin{cases} H_k[v_i] = \psi_i & \text{in } \Omega, \\ v_i = u_i & \text{on } \partial\Omega. \end{cases} \tag{2.22}$$

By virtue of the results in [13, 25], there exists a unique classical solution $v_i \in C^\infty(\overline{\Omega})$ to (2.22), for sufficiently large i. From the maximum principle [25], the sequence $\{v_i\}$ is uniformly bounded. We also see that for any open set $\Omega' \Subset \Omega$, the interior gradient bound by Korevaar [15] implies that $\{v_i\}$ is equicontinuous in Ω'. Therefore, using the diagonal argument, we deduce from Ascoli-Arzelà's theorem that there exists a subsequence of $\{v_i\}$ (we relabel it as $\{v_i\}$ again) converging uniformly to some function $v \in C^0(\Omega)$ on every compact subset of Ω. By the stability property of viscosity solutions, it follows that v is a viscosity solution to

$$\begin{cases} H_k[v] = \psi & \text{in } \Omega, \\ v = u & \text{on } \partial\Omega. \end{cases} \tag{2.23}$$

The uniqueness of solutions to the Dirichlet problem (2.23) implies that $u \equiv v$ in Ω.

We set

$$\mu_i(\eta) = \int_\eta \psi_i(x)\, dx \tag{2.24}$$

for Borel subset η of Ω. From (2.17), we obtain

$$\mu_i \to \nu \quad \text{(strongly)}. \tag{2.25}$$

On the other hand, from the uniform convergence of $\{v_i\}$ on every compact subset of Ω and Remark 2.3 (ii) (see also [23, Proposition 3.2]), we see that

$$\mu_i \rightharpoonup \binom{n}{k} \sigma_k(u;\cdot) \quad \text{(weakly)}. \tag{2.26}$$

Then, the uniqueness of the weak limit yields

$$\binom{n}{k}\sigma_k(u;\eta) = \int_\eta \psi(x)\,dx \tag{2.27}$$

for every Borel subset η of Ω. Hence the proposition is proved. $\square$

3 Proof of Theorem 1.1

Before giving a proof of Theorem 1.1, we introduce some notations. We write $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$. $B_r^{n-1}(x') \subset \mathbb{R}^{n-1}$ denotes the $(n-1)$-dimensional open ball of radius r centered at x'.

Proof. The proof is split into two steps.

Step 1. (Extension of u to a convex function in Ω)

Here we prove that u can be extended to a convex function in the whole domain Ω. The idea of the proof is adapted from that of Yan [29].

Let y, z be any two distinct points in $\Omega \setminus K$. Without loss of generality we may assume that y is the origin and $z = (0, \ldots, 0, 1)$. First we prove the following lemma.

Lemma 3.1. *There exist sequences $\{y_j\}_{j=1}^\infty, \{z_j\}_{j=1}^\infty \subset \Omega \setminus K$ such that $y_j \to y, z_j \to z$ as $j \to \infty$ and*

$$[y_j, z_j] = \{ty_j + (1-t)z_j \,|\, 0 \le t \le 1\} \subset \Omega \setminus K. \tag{3.1}$$

Proof. To the contrary, we suppose that there exist $\delta > 0$ such that for every $\tilde{y} \in B_\delta(y)$ and for every $\tilde{z} \in B_\delta(z)$, there exists $\tilde{t} \in (0,1)$ such that $\tilde{t}\tilde{y} + (1-\tilde{t})\tilde{z} \in K$. Here we note that $\tilde{t}\tilde{y} + (1-\tilde{t})\tilde{z}$ must be in Ω since Ω is assumed to be convex. In particular, if we set $\tilde{y} = (a_1, \ldots, a_{n-1}, 0), \tilde{z} = (a_1, \ldots, a_{n-1}, 1)$ with $a' = (a_1, \ldots, a_{n-1}) \in B_\delta^{n-1}(0)$, one sees that there exists $t_{a'} \in (0,1)$ such that $(a', t_{a'}) \in K$. We define the set V by

$$V = \{(a', t_{a'}) \,|\, a' \in B_\delta^{n-1}(0)\}. \tag{3.2}$$

Clearly $V \subset K$.

The assumption on K implies that the $(n-1)$-dimensional Hausdorff measure of K is zero. Hence there exist countable balls $\{B_{r_i}(x_i)\}_{i=1}^\infty$ such that

$$K \subset \bigcup_{i=1}^\infty B_{r_i}(x_i) \quad \text{and} \quad \sum_{i=1}^\infty r_i^{n-1} < \delta^{n-1}. \tag{3.3}$$

It follows that V is also covered by $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$. By projecting both V and $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ onto $\mathbb{R}^{n-1} \times \{0\}$, we have that

$$B_{\delta}^{n-1}(0) \subset \bigcup_{i=1}^{\infty} B_{r_i}^{n-1}(x_i'). \tag{3.4}$$

Taking $(n-1)$-dimensional measure of each side of (3.4), we obtain that

$$\omega_{n-1}\delta^{n-1} \le \sum_{i=1}^{\infty} \omega_{n-1} r_i^{n-1} < \omega_{n-1}\delta^{n-1}, \tag{3.5}$$

which is a contradiction. Lemma 3.1 is thus proved. □

Let $\lambda \in [0,1]$ and set $x = \lambda y + (1-\lambda)z \in \Omega \setminus K$. From the above lemma and the local convexity of u, it follows that

$$u(x) \le \lambda u(y_j) + (1-\lambda)u(z_j) \tag{3.6}$$

for all $j \in \mathbb{N}$, where $\{y_j\}_{j=1}^{\infty}$ and $\{z_j\}_{j=1}^{\infty}$ are sequences which we obtained in Lemma 3.1. Since u is locally convex in $\Omega \setminus K$, u is continuous in $\Omega \setminus K$. Taking $j \to \infty$,

$$u(x) \le \lambda u(y) + (1-\lambda)u(z). \tag{3.7}$$

Next let U be the supergraph of u, that is,

$$U = \{(x,w) \mid x \in \Omega \setminus K, w \ge u(x)\} \subset \mathbb{R}^{n+1}, \tag{3.8}$$

and for every set $X \subset \mathbb{R}^{n+1}$, $\operatorname{co} X$ denotes the convex hull of X. Now we define the function $\tilde{u}$ by

$$\tilde{u}(x) = \inf\{w \in \mathbb{R} \mid (x,w) \in \operatorname{co} U\}. \tag{3.9}$$

One can easily show that the convex hull of $\Omega \setminus K$ (in $\mathbb{R}^n$) is Ω, so that $\tilde{u}$ is defined in the whole Ω. Moreover, $\tilde{u}$ is a convex function due to the convexity of $\operatorname{co} U$. Finally, we show that $\tilde{u}$ is an extension of u defined in $\Omega \setminus K$. To see this, fix a point $x \in \Omega \setminus K$. The definition of $\tilde{u}$ follows that $\tilde{u}(x) \le u(x)$. Taking the infimum of the right-hand side of (3.7) over all $y, z \in \Omega \setminus K$, we have that $u(x) \le \tilde{u}(x)$. Consequently, it holds that $u \equiv \tilde{u}$ in $\Omega \setminus K$. $\tilde{u}$ is the desired function.

Step 2. (Removability of the singular set K)

We denote the extended function constructed in Step 1 by the same symbol u. Theorem 2.2 implies that there exists a non-negative Borel measure ν whose support is contained in K such that u is a generalized solution to

$$H_k[u] = \psi\, dx + \nu \quad \text{in } \Omega. \tag{3.10}$$

We fix arbitrary $\varepsilon > 0$. By the assumption we can cover K by countable open balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} r_i^{n-k} < \varepsilon. \tag{3.11}$$

For any $\rho \geq 0$, it holds that

$$\begin{aligned} \omega_n(r_i+\rho)^n &\geq \mathcal{L}^n(Q_\rho(u; B_{r_i}(x_i))) \\ &= \sum_{m=0}^{n} \binom{n}{m} \sigma_m(u; B_{r_i}(x_i))\rho^m \\ &\geq \binom{n}{k} \sigma_k(u; B_{r_i}(x_i))\rho^k \\ &= \left(\int_{B_{r_i}(x_i)} \psi\, dx + \nu(B_{r_i}(x_i)) \right) \rho^k \geq \nu(B_{r_i}(x_i))\rho^k. \end{aligned} \tag{3.12}$$

The first inequality in (3.12) is due to the fact that $Q_\rho\left(u; B_{r_i}(x_i)\right) \subset B_{r_i+\rho}(x_i)$, since taking any $z \in Q_\rho\left(u; B_{r_i}(x_i)\right)$ we obtain

$$|z - x_i| = |y + \rho v - x_i| \leq |y - x_i| + \rho|v| < r_i + \rho, \tag{3.13}$$

for some $y \in B_{r_i}(x_i)$, $v \in \gamma_u(y)$. Inserting $\rho = r_i$ in (3.12), we obtain that

$$\omega_n 2^n r_i^n \geq \nu\left(B_{r_i}(x_i)\right) r_i^k. \tag{3.14}$$

Consequently, it holds that

$$\nu\left(B_{r_i}(x_i)\right) \leq \omega_n 2^n r_i^{n-k}. \tag{3.15}$$

Now taking the summation for $i \geq 1$, we have that

$$\begin{aligned} \nu(K) &\leq \nu\left(\bigcup_{i=1}^{\infty} B_{r_i}(x_i) \right) \\ &\leq \sum_{i=1}^{\infty} \nu(B_{r_i}(x_i)) \\ &\leq \sum_{i=1}^{\infty} \omega_n 2^n r_i^{n-k} \\ &< \omega_n 2^n \varepsilon. \end{aligned} \tag{3.16}$$

Since we can take $\varepsilon > 0$ arbitrarily, we see that $\nu(K) = 0$. Therefore, $\nu \equiv 0$. We conclude that K is a removable set. $\square$

We see from Example 2.1 (2) that the number $(n-k)$ in Theorem 1.1 is optimal, since the Hausdorff dimension of T is $n-k$.

4 Future work

There are a number of results concerning the Dirichlet problem for curvature equations (1.1) in the literature, for general $k = 1, 2, \ldots, n$. Such problems were investigated by Caffarelli, Nirenberg and Spruck [5] and Ivochkina [13] in the classical sense, and by Trudinger [25] in the viscosity sense.

Therefore, it seems an interesting problem to study the solvability of the Dirichlet problem

$$\begin{cases} H_k[u] = \nu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

in the class of generalized solutions, where ν is a non-negative Borel measure. For $k = n$ (Gauss curvature case) which is an equation of Monge-Ampère type, the existence and uniqueness of generalized solutions to the Dirichlet problem (4.1) in a bounded convex domain have been studied. We refer the reader to [1], for example. We would like to seek appropriate conditions on ν which guarantee the solvability of generalized solutions to (4.1) for the case of $1 \leq k \leq n-1$. However, we obtain few results about that so far. Theorem 1.1 in this article implies that, for example, there exist no generalized solutions to (4.1) when $1 \leq k \leq n-1$ and $\nu = C\delta_{x_0}$ where C is a positive constant and δ_{x_0} is a Dirac delta measure at $x_0 \in \Omega$. In fact, if we write $\nu = \psi\, dx + \mu$ where ψ is a non-negative $L^1(\Omega)$ function and μ is the singular part of ν with respect to the Lebesgue measure, then either of the two alternatives must hold:

(i) the $(n-k)$-dimensional Hausdorff measure of the support of μ is non-zero; or

(ii) $\mu = 0$.

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Department of Mathematics, Faculty of Science, Hiroshima University
1-3-1 Kagamiyama, Higashi-Hiroshima city, Hiroshima 739-8526, Japan

E-mail address: `takimoto@math.sci.hiroshima-u.ac.jp`