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Inf-sup type differential games and Isaacs equations: dynamic programming approach

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1 Introduction

Let us consider the state dynamics influenced by two factors:

$$\begin{cases} \frac{dx}{ds}(s) = f(s, x(s), a(s), b(s)), \ t \leq s \leq T \\ x(t) = x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $a(\cdot)$, $b(\cdot)$ are measurable functions on $[t, T]$ taking its value in metric spaces A, B respectively. For given a and b, we introduce game payoff function:

$$J(t, x; a, b) \equiv \int_t^T l(s, x(s), a(s), b(s))ds + \Phi(x(T)), \tag{1.2}$$

where $l : [0, T] \times \mathbb{R}^N \times A \times B \to \mathbb{R}$, $\Phi : \mathbb{R}^N \to \mathbb{R}$. In this problem, $a(\cdot)$ is considered as control of minimizing player and $b(\cdot)$ as control of maximizing player. Under certain assumption on available information for each player, we can define two types of game value.

$$V(t, x) \equiv \inf_{\alpha} \sup_{b} J(t, x; \alpha[b], b) \tag{1.3}$$

$$W(t, x) \equiv \sup_{\beta} \inf_{a} J(t, x; a, \beta[a]). \tag{1.4}$$

In (1.3), supremum is taken over any $b(\cdot)$ and infimum is taken on a set of mappings from $b(\cdot)$ to $a(\cdot)$, which is considered as a class of strategies for minimizing player. Equation (1.4) is interpreted in the same way.

In conventional differential game problems, it is considered as a basic problem to find appropriate classes of strategies which enable us to characterize V, W and to identify V with W under min-max (Isaacs) condition. Evans and Souganidis [5] answered to these problems by using theory of viscosity solutions. In [5], by taking Elliott-Kalton strategy (see Definition 2.2 in the present paper), it is showed that Dynamic Programming Principles (DPPs) hold: For $0 \leq t < t + \delta \leq T$, $x \in \mathbb{R}^N$,

$$V(t, x) = \inf_{\alpha} \sup_{b} [\int_t^{t+\delta} l(s, x(s), \alpha[b](s), b(s))ds + V(t + \delta, x(t + \delta))] \tag{1.5}$$

$$W(t, x) = \sup_{\beta} \inf_{a} [\int_t^{t+\delta} l(s, x(s), a(s), \beta[a](s))ds + W(t + \delta, x(t + \delta))]. \tag{1.6}$$

Then, it is proved that V and W are characterized as the unique viscosity solutions of the following Isaacs equations respectively:

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \underline{H}(t,x,\nabla V(t,x)) = 0 \ \text{ in } (0,T)\times\mathbb{R}^N \\ V(T,x) = \Phi(x), \ x\in\mathbb{R}^N, \end{cases} \tag{1.7}$$

$$\begin{cases} \dfrac{\partial W}{\partial t}(t,x) + \overline{H}(t,x,\nabla W(t,x)) = 0 \ \text{ in } (0,T)\times\mathbb{R}^N \\ W(T,x) = \Phi(x), \ x\in\mathbb{R}^N, \end{cases} \tag{1.8}$$

where $\underline{H}$, $\overline{H}$ are defined as follows:

$$\underline{H}(s,x,p) = \sup_{b\in B}\inf_{a\in A}[f(s,x,a,b)\cdot p + l(s,x,a,b)]$$
$$\overline{H}(s,x,p) = \inf_{a\in A}\sup_{b\in B}[f(s,x,a,b)\cdot p + l(s,x,a,b)].$$

Here, we point out that the order of inf and sup in DPPs is flipped in the Isaacs equation. Since comparison theorems for Isaacs equations are proved under some condition and $\underline{H}\le\overline{H}$ holds, one can have

$$V(t,x)\le W(t,x).$$

Note that under min-max condition $\underline{H}=\overline{H}$, we can see that $V(t,x)=W(t,x)$ because of uniqueness of viscosity solutions (see Theorem 4.1, Corollary 4.2, [5]).

In certain problems, we often need to give a special role to each player. For instance, maximizing player is regarded as disturbance in H^∞-control (cf. [2]). In this interpretation, inf-sup type value (1.3) is preferable to sup-inf type value (1.4) because we want to control the worst case criterion caused by disturbance. Recently, Fleming [8] considers inf-sup type value in terms of max-plus stochastic control which gives a generalization of H^∞-theory. In [8], by using discretization method, sup-inf type value $W(t,x)$ in Elliott-Kalton sense is identified with inf-sup type value defined by a smaller class of strategies than Elliott-Kalton (see Theorem 4.1, [8]).

To make a motivation for the study of inf-sup type games clear, we shall give some observation through the small noise limit in risk-sensitive control. Let us consider the controlled stochastic differential equation:

$$\begin{cases} dX^\epsilon(s) = F(s,X^\epsilon(s),a(s))ds + \sqrt{\epsilon}\sigma(s,X^\epsilon(s),a(s))dW(s), \ t\le s\le T \\ X^\epsilon(t) = x\in\mathbb{R}^N, \end{cases}$$

where $\{W(s)\}$ is m-dimensional $\{\mathcal{F}_s\}$-standard Brownian motion on a filtered probability space $(\Omega,\mathcal{F},P,\{\mathcal{F}_s\})$, $a(s)$ is a control process which is usually considered to be A-valued $\{\mathcal{F}_s\}$-progressively measurable and ϵ is a positive parameter representing noise intensity. In risk-sensitive control, we try to minimize the logarithmic-exponential type criterion:

$$V^\epsilon(t,x)\equiv\inf_{a(\cdot)}\epsilon\log E\left[\exp\left\{\frac{1}{\epsilon}\left(\int_t^T L(s,X^\epsilon(s),a(s))ds + \Phi(X^\epsilon(T))\right)\right\}\right]$$

By DPP for V^ϵ, it is considered that $V^\epsilon(t,x)$ should be characterized as a solution of the Bellman equation:

$$\frac{\partial V^\epsilon}{\partial t} + \min_{a\in A}\Big[\frac{\epsilon}{2}\operatorname{tr}(A(t,x,a)D^2V^\epsilon) + \frac{1}{2}A(t,x,a)\nabla V^\epsilon\cdot\nabla V^\epsilon$$
$$+F(t,x,a)\cdot\nabla V^\epsilon + L(t,x,a)\Big] = 0 \text{ in } (0,T)\times\mathbb{R}^N \tag{1.9}$$
$$V^\epsilon(T,x) = \Phi(x),\ \ x\in\mathbb{R}^N, \tag{1.10}$$

where $A(t,x,a) = \sigma(t,x,a)\sigma(t,x,a)^*$ and $D^2V^\epsilon = [\partial^2V^\epsilon/\partial x_i\partial x_j]$. Indeed, (1.9) is equivalent to

$$\frac{\partial V^\epsilon}{\partial t} + \min_{a\in A}\max_{b\in\mathbb{R}^m}\Big[\frac{\epsilon}{2}(\operatorname{tr}A(t,x,a)D^2V^\epsilon)$$
$$+(F(t,x,a)+\sigma(t,x,a)b)\cdot\nabla V^\epsilon + L(t,x,a) - \frac{1}{2}|b|^2\Big] = 0. \tag{1.11}$$

By taking the limit as $\epsilon\to 0$, we expect that V^ϵ converges to some function V^0 and V^0 satisfies the limit equation of (1.11) in some sense:

$$\begin{cases} \dfrac{\partial V^0}{\partial t} + \min\limits_{a\in A}\max\limits_{b\in\mathbb{R}^m}\Big[(F(t,x,a)+\sigma(t,x,a)b)\cdot\nabla V^0 \\ \qquad\qquad +L(t,x,a) - \dfrac{1}{2}|b|^2\Big] = 0 \text{ in } (0,T)\times\mathbb{R}^N \\ \qquad V^0(T,x) = \Phi(x),\ \ x\in\mathbb{R}^N. \end{cases} \tag{1.12}$$

This equation is a special case of (1.8). For rigorous arguments in the case that $\sigma(t,x,a)$ does not depend on a, *i.e.*, $\sigma(t,x,a) = \hat\sigma(t,x)$, see [3], [9]. Note that if $\sigma(t,x,a) = \hat\sigma(t,x)$, we have min-max condition. But this is not true in general for (1.12). In terms of relationships to H^∞-control, it is preferable to associate V^0 with an inf-sup type game value V in (1.3). However, if we take Elliott-Kalton strategy class to define inf-sup type value, V corresponds to (1.7), not to (1.8) (cf. [5]). Thus, we need to seek a smaller class of strategies under which the inf-sup value is associated to (1.8).

Our aim is to provide a general framework how to relate inf-sup type games with the corresponding Isaacs equations without proof (see [11] for details). To utilize viscosity solution methods, we have to work with two steps: one is DPP and the other is identification of infinitesimal generator (cf. Chapter II, [10]). When we give a class of strategy, we can define an inf-sup type value by (1.3). Then, it would be natural to expect that DPP holds in the form of (1.5). If DPP holds, we formally have the following equation:

$$\lim_{\delta\to 0+}\frac{1}{\delta}\inf_{\alpha}\sup_{b}[\int_t^{t+\delta} l(s,x(s),\alpha[b](s),b(s))ds + V(t+\delta,x(t+\delta)) - V(t,x)] = 0.$$

Indeed, this can be rewritten as follows:

$$\lim_{\delta\to 0+}\frac{1}{\delta}(F_{t,t+\delta}V(t+\delta,\cdot)(x) - V(t,x)) = 0 \tag{1.13}$$

where $F_{t,s}$ is defined for $\phi : \mathbb{R}^N \to \mathbb{R}$ as follows

$$F_{t,s}\phi(x) = \inf_{\alpha} \sup_{b} [\int_t^s l(r, x(r), \alpha[b](r), b(r))dr + \phi(x(s))], \ x \in \mathbb{R}^N.$$

In general, it is known that V is not smooth. So, we consider the left hand side of (1.13) for smooth function $\varphi(t,x)$ instead of V:

$$\lim_{\delta \to 0} \frac{1}{\delta}(F_{t,t+\delta}\varphi(t+\delta, \cdot)(x) - \varphi(t,x)). \tag{1.14}$$

In Section 2, we shall study the infinitesimal generators (1.14). We give upper and lower bounds of (1.14) for quite general classes of strategies. In fact, lower bound is given by $\underline{H}$ and upper bound by $\overline{H}$. In general, it is not easy to find a class which has infinitesimal generator. However, we can show that the lower bound of generators corresponds to Elliott-Kalton strategy (which we call *progressive* in this paper) and the upper bound corresponds to *strictly progressive* strategy.

Relationships between DPP and infinitesimal generator are discussed in Section 3. Once the infinitesimal generator is obtained, we can consider the corresponding Isaacs equation. Following the general ideas of viscosity theory, we shall prove that if we have infinitesimal generator and DPP, the inf-sup value is a viscosity solution of the corresponding Isaacs equation. In this procedure, we also have to see if DPP holds. DPP for Elliott-Kalton case is proved in [5]. We shall see that DPP holds for strictly progressive strategy.

We assumed that the control spaces for both of players are compact, but this is not the case in (1.12) because $B = \mathbb{R}^m$. In Section 4, we shall make some remarks on the possibility of an extension to (1.12). Indeed, if we assume that $F(t,x,a)$ and $L(t,x,a)$ are linearly growing, $\sigma(t,x,a)$ is bounded and A is compact, the same results could be obtained. This case can be applied to differential game problems obtained by totally risk-averse limit in classical optimal investment problems (cf. [6], [7]).

2 Infinitesimal generators for game values

As we mentioned in Introduction, it is an important step to study infinitesimal generators associated to game values. In this section, we shall give bounds for infinitesimal generators of inf-sup type value for general class of strategies. In particular, we shall identify infinitesimal generators for progressive and strictly progressive strategies.

In Sections 2, 3, we suppose the followings:

$$A, \ B \text{ are compact subsets of Euclidean spaces.} \tag{A.1}$$

$$f, \ l \text{ and } \Phi \text{ are bounded and uniformly continuous.} \tag{A.2}$$

There exists $L > 0$ such that for each $s \in [0,T], \ x, y \in \mathbb{R}^N, a \in A, \ b \in B$

$$\begin{aligned} |f(s,x,a,b) - f(s,y,a,b)| &\le L|x-y| \\ |l(s,x,a,b) - l(s,y,a,b)| &\le L|x-y| \\ |\Phi(x) - \Phi(y)| &\le L|x-y|. \end{aligned} \tag{A.3}$$

$\mathcal{A}_{t,s}$ (*resp.* $\mathcal{B}_{t,s}$) is the set of A-valued (*resp.* B-valued) measurable functions on $[t,s]$, which is considered as all of the controls for minimizing player (*resp.* maximizing player). We denote the set of mappings from $\mathcal{B}_{t,s}$ into $\mathcal{A}_{t,s}$ (*resp.* $\mathcal{A}_{t,s}$ into $\mathcal{B}_{t,s}$) as $\Gamma^0_{t,s}$ (*resp.* $\Delta^0_{t,s}$). $\Gamma^0_{t,s}$ (*resp.* $\Delta^0_{t,s}$) is considered as all the possible strategies of minimizing player (*resp.* maximizing player).

For given $\Gamma_{t,T} \subset \Gamma^0_{t,T}$, we define inf-sup type value:

$$\begin{aligned} V(t,x) &\equiv \inf_{\alpha\in\Gamma_{t,T}} \sup_{b\in\mathcal{B}_{t,T}} J(t,x;\alpha[b],b) \\ &\equiv \inf_{\alpha\in\Gamma_{t,T}} \sup_{b\in\mathcal{B}_{t,T}} [\int_t^T l(s,x(s),\alpha[b](s),b(s))ds + \Phi(x(T))], \end{aligned} \tag{2.1}$$

where $x(\cdot)$ is a solution of (1.1) with initial condition $x(t) = x$ and controls $a = \alpha[b]$, b. We also introduce the operators on $C(\mathbb{R}^N)$ associated to (2.1):

$$\begin{aligned} F^{a,b}_{t,s}\phi(x) &= \int_t^s l(r,x(r),a(r),b(r))dr + \phi(x(s)), \\ F_{t,s}\phi(x) &\equiv \inf_{\alpha\in\Gamma_{t,s}} \sup_{b\in\mathcal{B}_{t,s}} F^{\alpha[b],b}_{t,s}\phi(x), \ \phi \in C(\mathbb{R}^N), \end{aligned} \tag{2.2}$$

where $x(\cdot)$ is a solution of (1.1) with $x(t) = x$

Proposition 2.1. *Suppose $\Gamma_{t,s}$ includes open loop strategies, i.e., $\alpha : \mathcal{B}_{t,s} \to \mathcal{A}_{t,s}$ defined in the following belongs to $\Gamma_{t,s}$: for arbitrarily given $a \in \mathcal{A}_{t,s}$*

$$\alpha[b](r) \equiv a(r), t \le r \le s, \ b \in \mathcal{B}_{t,s}. \tag{2.3}$$

Then, for $\varphi \in C^1((0,T)\times\mathbb{R}^N)$, we have

$$\overline{\lim_{\delta\to 0+}} \frac{1}{\delta}(F_{t_\delta,t_\delta+\delta}\varphi(t_\delta+\delta,\cdot)(x_\delta) - \varphi(t_\delta,x_\delta)) \le \frac{\partial\varphi}{\partial t}(t,x) + \overline{H}(t,x,\nabla_x\varphi(t,x)), \tag{2.4}$$

$$\frac{\partial\varphi}{\partial t}(t,x) + \underline{H}(t,x,\nabla_x\varphi(t,x)) \le \underline{\lim_{\delta\to 0+}} \frac{1}{\delta}(F_{t_\delta,t_\delta+\delta}\varphi(t_\delta+\delta,\cdot)(x_\delta) - \varphi(t_\delta,x_\delta)), \tag{2.5}$$

where (t_δ, x_δ) is any sequence such that $(t_\delta,x_\delta) \to (t,x) \in (0,T)\times\mathbb{R}^N$ as $\delta \to 0+$.

In general, it is not obvious to find an explicit form of infinitesimal generators. However, we can identify the limit for progressive and strictly progressive strategies. We introduce these two notions of strategies.

Definition 2.2. (cf. [4], [13], [14]) $\alpha \in \Gamma^0_{t,s}$ is called *progressive strategy* for minimizing player if the following condition is satisfied:

$$\text{For each } r \in [t,s], \text{ if } b \equiv \tilde{b} \ a.e. \text{ on } [t,r], \text{ then } \alpha[b] \equiv \alpha[\tilde{b}] \ a.e. \text{ on } [t,r].$$

We denote by $\Gamma^P_{t,s}$ the set of progressive strategies for minimizing player. Progressive strategy for maximizing player $\beta \in \Delta^0_{t,s}$ is defined in a similar way and the set of progressive strategies for maximizing player is denoted by $\Delta^P_{t,s}$.

Definition 2.3. (cf. [8]) $\alpha \in \Gamma^P_{t,s}$ is *strictly progressive strategy* if for any $\beta \in \Delta^P_{t,s}$, there exist $a \in \mathcal{A}_{t,s}$ and $b \in \mathcal{B}_{t,s}$ such that

$$\alpha[b] = a, \ \beta[a] = b \ a.e. \text{ on } [t, s].$$

We denote the set of strictly progressive strategies as $\Gamma^{SP}_{t,s}$.

Remark 2.4. $\Gamma^P_{t,s}$ and $\Gamma^{SP}_{t,s}$ include open loop strategies.

We define the operators associated with these two classes of strategies:

$$F^P_{t,s}\phi(x) \equiv \inf_{\alpha \in \Gamma^P_{t,s}} \sup_{b \in \mathcal{B}_{t,s}} F^{\alpha[b],b}_{t,s}\phi(x),$$
$$F^{SP}_{t,s}\phi(x) \equiv \inf_{\alpha \in \Gamma^{SP}_{t,s}} \sup_{b \in \mathcal{B}_{t,s}} F^{\alpha[b],b}_{t,s}\phi(x)$$

We also introduce the corresponding inf-sup type game values:

$$V^P(t,x) \equiv \inf_{\alpha \in \Gamma^P_{t,T}} \sup_{b \in \mathcal{B}_{t,T}} J(t, x; \alpha[b], b)$$
$$V^{SP}(t,x) \equiv \inf_{\alpha \in \Gamma^{SP}_{t,T}} \sup_{b \in \mathcal{B}_{t,T}} J(t, x; \alpha[b], b).$$

Note that under (A.1)–(A.3), V^P and V^{SP} are bounded and Lipschitz continuos on $[0, T] \times \mathbb{R}^N$

We first give the form of the generator for progressive case. The proof is implicitly done in [5].

Proposition 2.5. *For $\varphi \in C^1((0,T) \times \mathbb{R}^N)$,*

$$\frac{1}{\delta}(F^P_{t,t+\delta}\varphi(t+\delta, \cdot)(x) - \varphi(t,x)) \to \frac{\partial \varphi}{\partial t}(t,x) + \underline{H}(t, x, \nabla\varphi(t,x)), \ \delta \to 0+ \tag{2.6}$$

uniformly on each compact set in $(0,T) \times \mathbb{R}^N$.

In the next result, we shall prove the infinitesimal generator associated to strictly progressive strategy corresponds to upper Hamiltonian $\overline{H}(t, x, p)$.

Proposition 2.6. *For $\varphi \in C^1((0,T) \times \mathbb{R}^N)$,*

$$\frac{1}{\delta}(F^{SP}_{t,t+\delta}\varphi(t+\delta, \cdot)(x) - \varphi(t,x)) \to \frac{\partial \varphi}{\partial t}(t,x) + \overline{H}(t, x, \nabla\varphi(t,x)), \ \delta \to 0+ \tag{2.7}$$

uniformly on each compact set in $(0,T) \times \mathbb{R}^N$.

3 Dynamic programming principle and Isaacs equations

In the present section, we shall study relationships between Dynamic Programming Principle (DPP) and its infinitesimal generators. More precisely, if DPP holds and the infinitesimal generator is identified, the inf-sup type game value is a viscosity solution of the corresponding Isaacs equation. Furthermore, if the Hamiltonian of the Isaacs equation satisfies certain structural conditions, the value is characterized as the unique viscosity solution.

In progressive and strictly progressive case, we can show DPPs hold. By combining results in Section 2, we can prove that value defined by progressive (*resp.* strictly progressive) strategy is characterized as the unique viscosity solution for lower Isaacs equation (*resp.* upper Isaacs equation).

Firstly, we give a general result on relationship between DPP and the Isaacs equation. For given class of strategies $\Gamma_{t,s} \subset \Gamma^0_{t,s}$, $0 \le t < s \le T$, we define inf-sup type game value $V(t,x)$ as (2.1). For given class $\Gamma_{t,s}$, DPP is described as follows: For $0 \le t < t+\delta \le T$, $x \in \mathbb{R}^N$,

$$V(t,x) = \inf_{\alpha \in \Gamma_{t,t+\delta}} \sup_{b \in \mathcal{B}_{t,t+\delta}} [\int_t^{t+\delta} l(s, x(s), \alpha[b](s), b(s))ds + V(t+\delta, x(t+\delta))].$$

This is reformulated in terms of (2.2):

$$F_{t,t+\delta}V(t+\delta,\cdot)(x) = V(t,x) \tag{3.1}$$

Proposition 3.1. *Suppose that* (3.1) *holds and the infinitesimal generator is identified, i.e., there exists* $H(t,x,p)$ *such that for* $\varphi \in C^1((0,T)\times\mathbb{R}^N)$,

$$\frac{1}{\delta}(F_{t,t+\delta}\varphi(t+\delta,\cdot)(x) - \varphi(t,x)) \to \frac{\partial \varphi}{\partial t}(t,x) + H(t,x,\nabla\varphi(t,x)),\ \delta \to 0+. \tag{3.2}$$

Then, $V(t,x)$ *is a viscosity solution of the equation:*

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + H(t,x,\nabla V(t,x)) = 0, \ (t,x) \in (0,T)\times\mathbb{R}^N, \\ V(T,x) = \Phi(x), \ x \in \mathbb{R}^N. \end{cases} \tag{3.3}$$

We shall next apply Proposition 3.1 to progressive and strictly progressive cases. In Section 2, we showed that infinitesimal generators for progressive strategies and strictly progressive strategies have explicit forms. To associate the values defined by these classes with the corresponding Isaacs equation, we need to prove DPP.

In progressive case, DPP and characterization of $V^P(t,x)$ as viscosity solution is proved in [5] (See Theorems 3.1, 4.1, [5]). By using DPP in [5], we can apply Propositions 2.5 and 3.1 to obtain characterization result for V^P.

Proposition 3.2 (Theorem 3.1, [5]). $V^P(t,x)$ *satisfies DPP:*

$$\begin{aligned} V^P(t,x) &= F^P_{t,t+\delta}V^P(t+\delta,\cdot)(x) \\ &= \inf_{\alpha \in \Gamma^P_{t,t+\delta}} \sup_{b \in \mathcal{B}_{t,t+\delta}} [\int_t^{t+\delta} l(s,x(s),\alpha[b](s),b(s))ds + V^P(t+\delta, x(s))]. \end{aligned}$$

We have another proof for characterization of V^P.

Theorem 3.3 (cf. Theorem 4.1, [5]). *$V^P(t,x)$ is the unique viscosity solution in $BUC([0,T]\times \mathbb{R}^N)$ for the lower Isaacs equation:*

$$\begin{cases} \dfrac{\partial V^P}{\partial t}(t,x) + \underline{H}(t,x,\nabla V^P(t,x)) = 0, \ (t,x) \in (0,T)\times\mathbb{R}^N, \\ \qquad\qquad V^P(T,x) = \Phi(x), \ x \in \mathbb{R}^N, \end{cases} \tag{3.4}$$

where $BUC([0,T]\times\mathbb{R}^N)$ is the set of bounded, Lipschitz continuous functions on $[0,T]\times \mathbb{R}^N$.

We consider value for strictly progressive strategies $V^{SP}(t,x)$. In [8], relationship between V^{SP} and upper Isaacs equation is studied and V^{SP} is identified with upper game value which is a viscosity solution of upper Isaacs equation. On the other hand, we directly prove DPP and apply Propositions 2.6 and 3.1.

Proposition 3.4. *$V^{SP}(t,x)$ satisfies DPP: For $0 \le t < t+\delta \le T$, $x \in \mathbb{R}^N$,*

$$\begin{aligned} V^{SP}(t,x) &= F^{SP}_{t,t+\delta} V^{SP}(t+\delta,\cdot)(x) \\ &= \inf_{\alpha\in\Gamma^{SP}_{t,t+\delta}} \sup_{b\in\mathcal{B}_{t,t+\delta}} [\int_t^{t+\delta} l(s,x(s),\alpha[b](s),b(s))ds + V^{SP}(t+\delta, x(t+\delta))]. \end{aligned} \tag{3.5}$$

Then, we have characterization for V^{SP}.

Theorem 3.5. *$V^{SP}(t,x)$ is the unique viscosity solution in $BUC([0,T]\times\mathbb{R}^N)$ for the upper Isaacs equation:*

$$\begin{cases} \dfrac{\partial V^{SP}}{\partial t}(t,x) + \overline{H}(t,x,\nabla V^{SP}(t,x)) = 0, \ (t,x) \in (0,T)\times\mathbb{R}^N, \\ \qquad\qquad V^{SP}(T,x) = \Phi(x), \ x \in \mathbb{R}^N. \end{cases} \tag{3.6}$$

4 Remarks on unbounded control

In the above sections, we studied the case that the control spaces for both of players are compact. However, as we saw in Introduction, if we would like to treat differential games arising from small noise limit problem in risk-sensitive control or totally risk-averse limit in optimal investment problems, we need to consider the case that B is unbounded. See [3], [9] for small noise limit and [6], [7] for total risk averse limit. In this section, we shall provide an extension of the previous sections to (1.12) without details of argument and proof. We just point out that a main difference of the arguments could be some estimate of ϵ-optimal control of the maximizing player uniform on ϵ. This could be proved because of the linear growth condition on $F(t,x,a)$, $L(t,x,a)$ as assumed below.

We shall consider a special case of system dynamics (1.1) with $f(x,a,b) = F(t,x,a) + \sigma(t,x,a)b$, *i.e.*,

$$\begin{cases} \dfrac{dx}{ds}(s) = F(t,x(s),a(s)) + \sigma(t,x(s),a(s))b(s) \ t \le s \le T, \\ x(t) = x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $F : [0,T] \times \mathbb{R}^N \times A \to \mathbb{R}^N$, $\sigma(t,x,a)$ is a mapping of $[0,T] \times \mathbb{R}^N \times A$ into set of $N \times m$-matrices. In the similar way to the previous sections, we denote by $\mathcal{A}_{t,s}$ the set of A-valued measurable functions on $[t,s]$. But we suppose that control space of maximizing player $B = \mathbb{R}^n$ and $\mathcal{B}_{t,s} = L^2(t,s;\mathbb{R}^m)$ which is set of square integrable functions from $[t,s]$ into $\mathbb{R}^m$. Game-payoff function (1.2) with $l(t,x,a) = L(t,x,a) - |b|^2/2$ is considered:

$$J(t,x;a,b) = \int_t^T (L(s,x(s),a(s)) - \frac{1}{2}|b(s)|^2)ds + \Phi(x(T)), \tag{4.2}$$

where $L : [0,T] \times \mathbb{R}^N \times A \to \mathbb{R}$, $\Phi : \mathbb{R}^N \to \mathbb{R}$.

We assume the following conditions:

$$A \text{ is compact in } \mathbb{R}^n, \tag{B.1}$$

$$F(t,x,a),\ \sigma(t,x,a),\ L(t,x,a),\ \Phi(x) \text{ are continuous,} \tag{B.2}$$

$$\begin{gathered} \text{There exists } C > 0 \text{ such that for each } s \in [0,T],\ x,y \in \mathbb{R}^N,\ a \in A \\ |F(s,x,a) - F(s,y,a)| \le C|x-y|, \\ |\sigma(s,x,a) - \sigma(s,y,a)| \le C|x-y|, \\ |L(s,x,a) - L(s,y,a)| \le C|x-y|, \\ |\Phi(x) - \Phi(y)| \le C|x-y|, \end{gathered} \tag{B.3}$$

$$\sigma(t,x,a) \text{ is bounded on } [0,T] \times \mathbb{R}^N \times A. \tag{B.4}$$

Under the conditions (B.1) – (B.4), by using the estimate for ϵ-optimal control of maximizing player, all the results in previous sections can be obtained except for uniqueness of viscosity solutions in (3.4) and (3.6). It is not obvious that inf-sup values $V^P(t,x)$ and $V^{SP}(t,x)$ are in $BUC([0,T] \times \mathbb{R}^N)$ and that Hamiltonians $\underline{H}(t,x,p)$ and $\overline{H}(t,x,p)$ have nice regularity to apply uniqueness results used in Theorems 3.3, 3.5 (cf. Theorem 9.1, Chapter II, [10], Exercise 3.9, Chapter II, [1]). However, we can prove that $V^P(t,x)$, $V^{SP}(t,x)$ are locally Lipschitz continuous on x uniformly on $t \in [t,T]$ at least and we have uniqueness results in this class including the case for $\underline{H}(t,x,p)$ and $\overline{H}(t,x,p)$ (cf. [12]). Thus, $V^P(t,x)$ and $V^{SP}(t,x)$ can be characterized as the unique viscosity solution of (3.4) and (3.6) respectively.

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