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Local estimates and Maximum Principle for fully nonlinear equations in unbounded domains

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open, connected and possibly *unbounded* subset of $\mathbb{R}^N$, and let $u(x)$ be a bounded from above and upper semicontinuous function on the closure of Ω, in symbols

$$\sup_{\overline{\Omega}} u < +\infty\,, \qquad u \in USC(\overline{\Omega})\,,$$

satisfying in the viscosity sense a second order fully nonlinear differential inequality of the form

$$F(x,u,Du,D^2u) \geq 0 \qquad \text{in } \ \Omega\,. \tag{1}$$

In the recent paper [5], we gave an answer to the following question:
when the Maximum Principle - **MP** *in short - holds for inequality* (1), *that is what assumptions on the domain* Ω *and/or on the operator* F *can ensure the validity of the implication*

$$u \leq 0 \quad on \ \partial\Omega \quad \Longrightarrow \quad u \leq 0 \quad in \ \ \Omega\,?$$

When looking at previous results about **MP** for unbounded domains, one can distinguish basically two kinds of results

♣ general comparison principles, which include **MP** as a special case, between viscosity subsolutions and supersolutions of fully nonlinear equations. Within this approach, the operator $F(x,u,p,X) : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ is assumed to satisfy, besides the degenerate ellipticity inequality, some structural growth conditions and the *strict monotonicity* with respect to the u variable. On the other hand, no assumptions on the domain Ω are required, and even the case $\Omega = \mathbb{R}^N$ is allowed.

♣ for *strong solutions* of *linear* uniformly elliptic second order differential inequalities with bounded coefficients, that is for functions u satisfying

$$\begin{cases} \mathrm{tr}(A(x)\,D^2u) + \underline{b}(x)\cdot Du + c(x)\,u \geq 0 \quad \text{a.e. in } \Omega\,, \\ u \in W^{2,N}_{\mathrm{loc}}(\Omega)\,, \quad \sup_\Omega u < +\infty\,, \end{cases}$$

MP has been obtained as a consequence of the (improved) Alexandrov-Bakelman-Pucci (**ABP** in short) estimate. In this case, a *large monotonicity* in the zero order term is allowed, namely the requirement $c(x) \leq 0$ holds, but some geometric restrictions on the domain Ω are assumed.

For the former approach, we refer to the results obtained by R. Jensen, P.L. Lions & P.E. Souganidis [9] and by H. Ishii [8], and included in the celebrated "User's guide" of M. Crandall, H. Ishii & P.L. Lions [6]. In the latter case, we refer to the results of H. Berestycki, L. Nirenberg & S.R.S. Varadhan [1] and of X. Cabré [2], as well as to the further extensions by V. Cafagna & A. Vitolo [3] and by A. Vitolo [12].

Let us observe that, in general, **MP** does *not* hold for even linear uniformly elliptic inequality not strictly monotone with respect to the u variable. As a simple example, $u(x) = 1 - 1/|x|^{N-2}$, with $N \geq 3$, is a bounded subharmonic (actually, harmonic) function in the exterior domain $\Omega = \mathbb{R}^N \setminus \overline{B}_1(0)$ and constantly equals zero on the boundary, while being strictly positive inside Ω. Thus, widely speaking, some extra assumptions are needed in order to obtain **MP** .

In this notes, after recalling the method pursued for linear operators, we present the results obtained in [5], which extend it to viscosity solutions of fully nonlinear inequalities.

2 ABP estimate in the linear case.

Let u be a bounded from above strong solution of the following linear differential inequality

$$\begin{cases} \mathrm{tr}(A(x)\,D^2u) + \underline{b}(x)\cdot Du + c(x)\,u \geq f(x) \quad \text{a.e. in } \Omega\,, \\ u \in W^{2,N}_{\mathrm{loc}}(\Omega)\,, \quad \sup_\Omega u < +\infty\,, \end{cases}$$

with *bounded* coefficients satisfying

$$\lambda\, I_N \leq A(x) \leq \Lambda\, I_N\,, \quad c(x) \leq 0 \qquad \text{for a.e. } x \in \Omega\,,$$

and source term such that

$$f \in L^N(\Omega)\,.$$

The **ABP** estimate assumes different forms according to the boundedness properties of the domain Ω.

2.1 ABP for bounded domains.

In the standard case of a bounded domain, the **ABP** estimate states that (see e. g. [7])

$$\sup_\Omega u \leq \limsup_{x\to\partial\Omega} u + C\,\mathrm{diam}(\Omega)\,\|f^-\|_{L^N(\Omega)}\,,$$

where f^- is the negative part of the function f and $C > 0$ is a constant depending on N, on the ellipticity constants λ and Λ, and on the product $\mathrm{diam}(\Omega)\,\|\underline{b}\|_{L^\infty(\Omega)}$.

2.2 ABP for domains having finite measure.

In this case, by assuming further that $f \in L^\infty(\Omega)$, H. Berestiycki, L. Nirenberg & S.R.S. Varadhan [1] proved that

$$\sup_\Omega u \leq \limsup_{x\to\partial\Omega} u + C\,\mathrm{meas}(\Omega)^{\frac{2}{N}}\,\|f^-\|_{L^\infty(\Omega)}\,,$$

with $C>0$ depending on N, λ, Λ, and on the product $\mathrm{meas}(\Omega)^{\frac{1}{N}}\,\|\underline{b}\|_{L^\infty(\Omega)}$.

2.3 ABP for certain unbounded domains.

The general case of an unbounded domain has been considered by X. Cabré [2], under the following geometric condition that will be referred to as condition **(G)** :
there exist constants $\sigma,\tau \in (0,1)$ and $R(\Omega)>0$ such that, for all $y\in\Omega$, there is a ball B_{R_y}, containing y and having radius $R_y \leq R(\Omega)$, which satisfies

$$\mathrm{meas}\left(B_{R_y}\setminus\Omega_{y,\tau}\right) \geq \sigma\,\mathrm{meas}\left(B_{R_y}\right)\,,$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega\cap B_{R_y/\tau}$ containing y.
Roughly speaking, the requirement $R_y \leq R(\Omega)$ for all $y\in\Omega$ imposes in a measure theoretic sense that there is "enough boundary" uniformly near to every point of Ω. The positive constant $R(\Omega)$ plays the role of the diameter for un unbounded domain. Examples of domains satisfying condition **(G)** include all the domain having finite measure, in which case we have $R(\Omega) = (2\,\mathrm{meas}(\Omega)/\mathrm{meas}(B_1))^{1/N}$, and all the cylinders, for which $R(\Omega)$ equals the diameter of their bounded projections.
If Ω satisfies **(G)** , the improved **ABP** estimate obtained in [2] states that

$$\sup_\Omega u \leq \limsup_{x\to\partial\Omega} u + C\,R(\Omega)\,\|f^-\|_{L^N(\Omega)}\,,$$

with $C>0$ depending on N, λ, Λ, and on the product $R(\Omega)\,\|\underline{b}\|_{L^\infty(\Omega)}$.

3 ABP and MP in the fully nonlinear case.

Let u be a bounded from above viscosity solution of the following fully nonlinear differential inequality

$$\begin{cases} F(x,u,Du,D^2u) \geq f(x) & \text{in } \Omega\,, \\ u\in USC(\overline{\Omega})\,, \quad \sup_{\overline{\Omega}} u < +\infty\,. \end{cases} \tag{P}$$

Here we assume that $f\in C(\Omega)\cap L^\infty(\Omega)$. Furthermore, the continuous real valued function $F:\Omega\times\mathbb{R}\times\mathbb{R}^N\times S^N \to \mathbb{R}$ (with S^N being the set of $N\times N$ real symmetric matrices) is assumed to satisfy, besides the degenerate ellipticity inequality

$$F(x,t,p,X) \geq F(x,t,p,Y) \tag{$\mathbf{F_1}$}$$

for all $x\in\Omega$, $t\in\mathbb{R}$, $p\in\mathbb{R}^N$ and $X,\ Y\in S^N$ with $X-Y\geq O$, the following bound from above

$$F(x,t,p,X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X) + b(x)\,|p| \tag{$\mathbf{F_2}$}$$

for all $x \in \Omega$, $p \in \mathbb{R}^N$, $X \in \mathcal{S}^N$ and $t \geq 0$. We assume that $b \in C(\Omega) \cap L^\infty(\Omega)$ is a nonnegative function and we denote by $\mathcal{P}^+_{\lambda,\Lambda}$ the Pucci maximal operator, defined as (see [4, 6])

$$\mathcal{P}^+_{\lambda,\Lambda}(X) := \sup_{A \in \mathcal{A}} \mathrm{Tr}\,(AX) = \Lambda\,\mathrm{Tr}(X^+) - \lambda\,\mathrm{Tr}\,(X^-)\,,$$

where $\mathcal{A} = \mathcal{A}(\lambda, \Lambda) = \left\{A \in \mathcal{S}^N \,:\, \lambda I \leq A \leq \Lambda I\right\}$, and X^+ and X^- are nonnegative definite matrices such that $X = X^+ - X^-$ and $X^+ X^- = O$.
Let us point out that assumptions $(\mathbf{F_1})$, $(\mathbf{F_2})$ are satisfied by any uniformly elliptic proper operator F having linear growth with respect to first order terms. Furthermore, if F satisfies $(\mathbf{F_1})$ and its principal part $F(x,0,0,X)$ is linear with respect to X, then condition $(\mathbf{F_2})$ implies the uniform ellipticity of $F(x,0,0,X)$. Indeed, by using $(\mathbf{F_2})$ with $X = \pm Q$ and $Q \geq O$, it follows that

$$F(x,0,0,Q) \leq \mathcal{P}^+_{\lambda,\Lambda}(Q) = \Lambda\,\mathrm{Tr}\,(Q)\,,\ F(x,0,0,-Q) \leq \mathcal{P}^+_{\lambda,\Lambda}(-Q) = -\lambda\,\mathrm{Tr}\,(Q)\,,$$

and then, by linearity,

$$\lambda\,\mathrm{Tr}\,(Q) \leq F(x,0,0,Q) \leq \Lambda\,\mathrm{Tr}\,(Q)\,, \qquad \forall\, Q \geq O\,.$$

On the other hand, assumptions $(\mathbf{F_1})$, $(\mathbf{F_2})$ include also nonlinear, possibly degenerate, elliptic operators, such as the following one

$$F(x,t,p,X) = \Lambda\left(\sum_{i=1}^N \varphi(\mu_i^+)\right) - \lambda\left(\sum_{i=1}^N \psi(\mu_i^-)\right) + H(x,t,p)\,,$$

where μ_i, $i = 1, \ldots N$, are the eigenvalues of the matrix $X \in \mathcal{S}^N$, φ, ψ : $[0,+\infty) \to [0,+\infty)$ are continuous and nondecreasing functions such that $\varphi(s) \leq s$ and $\psi(s) \geq s$ for all $s \geq 0$, and $H(x,t,p)$ is a continuous function such that $H(x,t,p) \leq b(x)\,|p|$ for all $x \in \Omega$, $t \geq 0$ and $p \in \mathbb{R}^N$.

3.1 ABP for bounded domains.

When the domain Ω is bounded, the **ABP** estimate has also in the fully nonlinear case the form

$$\sup_\Omega u \leq \sup_{x \in \partial\Omega} u^+ + C\,\mathrm{diam}(\Omega)\,\|f^-\|_{L^N(\Omega)}\,,$$

where $C > 0$ is a constant depending on N, on the ellipticity constants λ and Λ, and on the product $\mathrm{diam}(\Omega)\,\|b\|_{L^\infty(\Omega)}$. This has been proved by L. Caffarelli & X. Cabré [4] if the operator F does not depend on the gradient variable, and then extended by A. Persello [11] to complete fully nonlinear operators satisfying $(\mathbf{F_1})$–$(\mathbf{F_2})$, and by S. Koike & T. Takahashi [10] to operators having superlinear growth with respect to the gradient.

3.2 ABP and MP for certain unbounded domains

Here and henceforth we assume that the domain Ω satisfies the following condition, which will be referred to as **(wG)**
there exist constants $\sigma, \tau \in (0,1)$ such that, for all $y \in \Omega$, there is a ball B_{R_y} containing y which satisfies

$$\mathrm{meas}\left(B_{R_y} \setminus \Omega_{y,\tau}\right) \geq \sigma\,\mathrm{meas}\left(B_{R_y}\right)\,,$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y.

Note that condition (**wG**) is exactly the same as (**G**) but not requiring any uniform bound on the radii R_y's. Typical examples of domains satisfying (**wG**) (and failing (**G**)) are *cones*, for which $R_y = O(|y|)$ as $|y| \to \infty$.

Under assumption (**wG**) we have the following localized version of **ABP** estimate.

Theorem 1 (see [5]) *Let u be a solution of* (**P**), *with F and Ω satisfying respectively assumptions* ($\mathbf{F_1}$)–($\mathbf{F_2}$) *and* (**wG**) . *Then, for every $y \in \Omega$, there exists a costant $\theta_y \in (0,1)$, depending on N, λ, Λ, σ, τ and on y through the quantity $R_y\,\|b\|_{L^\infty(\Omega_{y,\tau})}$, such that*

$$w^+(y) \leq (1-\theta_y)\,\sup_{\Omega} w^+ + \theta_y\,\sup_{\partial\Omega} w^+ + R_y\,\|f^-\|_{L^N(\Omega_{y,\tau})}\,. \tag{2}$$

If either the operator F does *not* depend on the gradient or the domain Ω satisfies condition (**G**) , then the costant θ_y appearing in (2) is independent of y. In this case, from (2) with $f \equiv 0$ we immediately obtain the following

Corollary 2 *Assume that F satisfies* ($\mathbf{F_1}$)–($\mathbf{F_2}$) *and that* (**wG**) *holds true for Ω. If either* ($\mathbf{F_2}$) *is satisfied with $b \equiv 0$ or Ω satisfies* (**G**) , *then* **MP** *holds for the operator F in the domain Ω.*

In order to obtain a global **ABP** estimate for fully nonlinear inequalities in unbounded domains we need to assume, besides condition (**wG**) on Ω and assumptions ($\mathbf{F_1}$)–($\mathbf{F_2}$) on F, a further requirement coupling the geometry of the domain with the growth of the first order coefficients. Precisely, we have the following result.

Theorem 3 (see [5]) *Let Ω, F and u be as in Theorem 1. If further*

$$\sup_{y\in\Omega} R_y\,\|b\|_{L^\infty(\Omega_{y,\tau})} < \infty\,, \tag{$*$}$$

where R_y and $\Omega_{y,\tau}$ are as in (**wG**) *and b is as in* ($\mathbf{F_2}$), *then*

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w^+ + C\,\sup_{y\in\Omega} R_y\,\|f^-\|_{L^N(\Omega_{y,\tau})}$$

for some positive constant C depending on N, λ, Λ, σ, τ and $\sup_{y\in\Omega} R_y\,\|b\|_{L^\infty(\Omega_{y,\tau})}$.

For $f \geq 0$, Theorem 3 immediately yields the following

Corollary 4 *Under the same assumptions of Theorem 3, the Maximum Principle holds for the operator F in the domain Ω.*

4 Examples and further extensions.

4.1 On the necessity of condition ($*$) for MP .

For a complete second order operator condition (**wG**) alone is in general not enough for **MP** to hold. A counterexample (see [12]) is given by the function

$$u(x) = u(x_1,x_2) = \left(1-\mathrm{e}^{1-x_1^\alpha}\right)\,\left(1-\mathrm{e}^{1-x_2^\alpha}\right)\,,$$

with $0 < \alpha < 1$. Indeed, u is bounded and strictly positive in the plane cone

$$\Omega = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 > 1, \ x_2 > 1 \right\},$$

and it satisfies

$$u \equiv 0 \quad \text{on } \partial\Omega, \quad \Delta u + \underline{b}(x) \cdot Du = 0 \quad \text{in } \Omega,$$

where the vector–field $\underline{b}$ is given by

$$\underline{b}(x) = \underline{b}(x_1, x_2) = \left(\frac{\alpha}{x_1^{1-\alpha}} + \frac{1-\alpha}{x_1} \, , \, \frac{\alpha}{x_2^{1-\alpha}} + \frac{1-\alpha}{x_2} \right).$$

Notice that Ω satisfies **(wG)** with $R_y = O(|y|)$ as $|y| \to \infty$ and, on the other hand, condition $(\mathbf{F_2})$ holds with $b(x) = |\underline{b}(x)|$. Since for every $y \in \Omega$ and for any choice of B_{R_y} we have $\|b\|_{L^\infty(\Omega_{y,\tau})} \geq 1$ and $\sup_{y\in\Omega} R_y = +\infty$, condition $(*)$ clearly fails in this example.

4.2 An application.

Let us look at some special non trivial cases in which condition $(*)$ is fulfilled.

(a) Consider the half cylinder $\Omega = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \ : \ |x'| < 1 \, , \ x_N > 0 \right\}$. Since Ω satisfies condition **(G)** , then $(*)$ is satisfied if b in assumption $(\mathbf{F_2})$ is any nonnegative bounded and continuous function.

(b) Ω is a convex set with "parabolic" boundary, i.e.

$$\Omega = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \ : \ x_N > |x'|^q \right\}$$

with $q > 1$. Then, Ω satisfies assumption **(wG)** with radii $R_y = O\left(|y|^{1/q}\right)$ as $|y| \to \infty$. In this case, requirement $(*)$ imposes to the function b a rate of decay $b(y) = O\left(1/|y|^{1/q}\right)$ as $|y| \to \infty$. If so, the balls B_{R_y} in **(wG)** can be chosen in such a way that $\|b\|_{L^\infty(\Omega_{y,\tau})} = O\left(1/|y|^{1/q}\right)$ as $|y| \to \infty$ and $(*)$ is fulfilled.

(c) Ω is the strictly convex cone $\left\{ x \in \mathbb{R}^N \setminus \{0\} \ : \ x/|x| \in \Gamma \right\}$ where Γ is a proper subset of the unit half-sphere $S_+^{N-1} = \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \ : \ |x| = 1 \, , \ x_N > 0 \right\}$. In this case, condition **(wG)** is satisfied with $R_y = O(|y|)$ for $|y| \to \infty$ and condition $(*)$ requires on the coefficient b the rate of decay $b(y) = O\left(1/|y|\right)$ as $|y| \to \infty$.

Note that cases (a) and (c) can be seen as limiting cases of situation (b) when, respectively, $q \to +\infty$ and $q = 1$.

4.3 MP for domains not satisfying (wG) .

The validity of **MP** can be extended to even more general domains, not satisfying **(wG)** , by repeatedly applying the argument of Corollary 4.
More precisely, let F be a second order operator satisfying $(\mathbf{F_1})$–$(\mathbf{F_2})$ and assume that there exists a closed set $H \subset \Omega$ with the following properties

(i) **MP** holds for F in each connected component of $\Omega \setminus H$;

(ii) (**wG**) holds for all points of H, i.e. there exist constants $\sigma, \tau \in (0,1)$ such that for all $y \in H$ there is a ball B_{R_y} of radius R_y containing y such that

$$|B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|,$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y;

(iii)

$$\sup_{y \in H} R_y \, \|b\|_{L^\infty(\Omega_{y,\tau})} < \infty,$$

where R_y and $\Omega_{y,\tau}$ are as in (ii) and b is as in ($\mathbf{F_2}$).

In this situation we have the following

Theorem 5 *(see [3, 5, 12]) Assume that F satisfies conditions ($\mathbf{F_1}$)–($\mathbf{F_2}$) and that assumptions (i), (ii) and (iii) above hold for Ω. Then,* **MP** *holds for operator F in Ω.*

As a consequence of the above result, **MP** can be obtained in non-convex, perhaps degenerate cones. For instance, if F satisfies ($\mathbf{F_2}$) with a coefficient $b(x)$ such that $b(x) = O(1/|x|)$ as $|x| \to \infty$, then **MP** holds for F in the cat plane $\Omega = \mathbb{R}^2 \setminus \{(x_1, 0) \in \mathbb{R}^2 \ : \ x_1 \leq 0\}$, as it follows from Theorem 5 with e.g. $H = \{(x_1, -x_1) \in \mathbb{R}^2 \ : \ x_1 < 0\}$.

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