

# Tropical Affine Crystals

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## 1 Introduction

The notion of geometric crystals and unipotent crystals has been introduced by Berenstein and Kazhdan([1]) for reductive algebraic groups and it is extended to Kac-Moody setting in [11]. It seems to be a geometric lifting of the Kashiwara's crystal base theory. They are related to each other by "tropicalization/ultra-discretization" procedures.

Theory of perfect crystals([6],[7]) plays an important role in studying vertex type solvable lattice models and certain limit (denoted by  $B_\infty$ ) has been treated in [5].

In the mean while, Schubert varieties/cells associated with Kac-Moody groups have a canonical geometric/unipotent crystal structures([11]). Indeed, the geometric crystal and the crystal  $B_\infty$  are related by tropicalization/ultra-discretization procedures.

In this article, we review [11] and in the last section, we see some relation between geometric crystal on an affine Schubert cell and a crystal  $B_\infty$ .

## 2 Kac-Moody groups and Ind-varieties

In this section, we review Kac-Moody groups following [9],[10],[12].

### 2.1 Kac-Moody algebras and Kac-Moody groups

Fix a symmetrizable generalized Cartan matrix  $A = (a_{ij})_{i,j \in I}$ , where  $I$  be a finite index set. Let  $(\mathfrak{t}, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I})$  be the associated root data, where  $\mathfrak{t}$  be the vector space over  $\mathbb{C}$  with dimension  $|I| + \text{corank}(A)$ , and  $\{\alpha_i\}_{i \in I} \subset \mathfrak{t}^*$  and  $\{h_i\}_{i \in I} \subset \mathfrak{t}$  are linearly independent indexed sets satisfying  $\alpha_j(h_i) = a_{ij}$ .

The Kac-Moody Lie algebra  $\mathfrak{g} = \mathfrak{g}(A)$  associated with  $A$  is the Lie algebra over  $\mathbb{C}$  generated by  $\mathfrak{t}$ , the Chevalley generators  $e_i$  and  $f_i$  ( $i \in I$ ) with the usual defining relations ([9],[10]). There is the root space decomposition  $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$ . Denote the set of roots by  $\Delta := \{\alpha \in \mathfrak{t}^* | \alpha \neq 0, \mathfrak{g}_\alpha \neq (0)\}$ . Set  $Q = \sum_i \mathbb{Z}\alpha_i$ ,  $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$  and  $\Delta_+ := \Delta \cap Q_+$ . An element of  $\Delta_+$  is called a positive root.

Define simple reflections  $s_i \in \text{Aut}(\mathfrak{t})$  ( $i \in I$ ) by  $s_i(h) := h - \alpha_i(h)h_i$ , which generate the Weyl group  $W$ . We also define the action of  $W$  on  $\mathfrak{t}^*$  by  $s_i(\lambda) := \lambda - \alpha(h_i)\alpha_i$ . Set  $\Delta^{\text{re}} := \{w(\alpha_i) | w \in W, i \in I\}$ , whose element is called a real root.

Let  $\mathfrak{g}'$  be the derived Lie algebra of  $\mathfrak{g}$  and  $G$  be the Kac-Moody group associated with  $\mathfrak{g}'$  ([10]). Let  $U_\alpha := \exp \mathfrak{g}_\alpha$  ( $\alpha \in \Delta^{\text{re}}$ ) be an one-parameter subgroup of  $G$ . The group  $G$  is generated by  $U_\alpha$  ( $\alpha \in \Delta^{\text{re}}$ ). Let  $U^\pm$  be the subgroups generated by  $U_{\pm\alpha}$  ( $\alpha \in \Delta_+^{\text{re}} = \Delta^{\text{re}} \cap Q_+$ ), i.e.,  $U^\pm := \langle U_{\pm\alpha} | \alpha \in \Delta_+^{\text{re}} \rangle$ .

For any  $i \in I$ , there exists a unique homomorphism;  $\phi_i : SL_2(\mathbb{C}) \rightarrow G$  such that

$$\phi_i \left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp t e_i, \quad \phi_i \left( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp t f_i \quad (t \in \mathbb{C}).$$

Set  $x_i(t) := \exp t e_i$ ,  $y_i(t) := \exp t f_i$ ,  $T_i := \phi_i(\{\text{diag}(t, t^{-1}) | t \in \mathbb{C}\})$  and  $N_i := N_{G_i}(T_i)$ . Let  $T$  (resp.  $N$ ) be the subgroup of  $G$  generated by  $T_i$  (resp.  $N_i$ ), which is called a *maximal torus* in  $G$  and  $B^\pm = U^\pm T$  be the Borel subgroup of  $G$ . We have the isomorphism  $\phi : W \xrightarrow{\sim} N/T$  defined by  $\phi(s_i) = N_i T/T$ . An element  $\bar{s}_i := x_i(-1)y_i(1)x_i(-1)$  is in  $N_G(T)$ , which is a representative of  $s_i \in W = N_G(T)/T$ . Define  $R(w)$  for  $w \in W$  by

$$R(w) := \{(i_1, i_2, \dots, i_l) \in I^l | w = s_{i_1} s_{i_2} \cdots s_{i_l}\},$$

where  $l$  is the length of  $w$ . We associate to each  $w \in W$  its standard representative  $\bar{w} \in N_G(T)$  by  $\bar{w} = \bar{s}_{i_1} \bar{s}_{i_2} \cdots \bar{s}_{i_l}$ , for any  $(i_1, i_2, \dots, i_l) \in R(w)$ .

## 2.2 Ind-variety and Ind-group

Let us recall the notion of ind-varieties and ind-groups. (see [8]).

**Definition 2.1.** Let  $k$  be an algebraically closed field.

- (i) A set  $X$  is an *ind-variety* over  $k$  if there exists a filtration  $X_0 \subset X_1 \subset X_2 \subset \cdots$  such that
  - (a)  $\bigcup_{n \geq 0} X_n = X$ .
  - (b) Each  $X_n$  is a finite-dimensional variety over  $k$  such that the inclusion  $X_n \hookrightarrow X_{n+1}$  is a closed embedding.
- (ii) A *Zariski topology* on an ind-variety  $X$  is defined as follows; a set  $U \subset X$  is open if and only if  $U \cap X_n$  is open in  $X_n$  for any  $n \geq 0$ .
- (iii) Let  $X$  and  $Y$  be two ind-varieties with filtrations  $\{X_n\}$  and  $\{Y_n\}$  respectively. A map  $f : X \rightarrow Y$  is a *morphism* if for any  $n \geq 0$ , there exists  $m$  such that  $f(X_n) \subset Y_m$  and  $f|_{X_n} : X_n \rightarrow Y_m$  is a morphism. A morphism  $f : X \rightarrow Y$  is said to be an *isomorphism* if  $f$  is bijective and  $f^{-1} : Y \rightarrow X$  is also a morphism.
- (iv) Let  $X$  and  $Y$  be two ind-varieties. A *rational morphism*  $f : X \rightarrow Y$  is an equivalence class of morphisms  $f_U : U \rightarrow Y$  where  $U$  is an open dense subset of  $X$ , and two morphisms  $f_U : U \rightarrow Y$  and  $f_V : V \rightarrow Y$  are equivalent if they coincide on  $U \cap V$ .

**Definition 2.2.** An ind-variety  $H$  is called an *ind (algebraic)-group* if the underlying set  $H$  is a group and the maps

$$\begin{array}{ccc} H \times H & \longrightarrow & H \\ (x, y) & \longmapsto & xy \end{array} \quad \begin{array}{ccc} H & \longrightarrow & H \\ x & \longmapsto & x^{-1} \end{array}$$

are morphisms of ind-varieties.

We have the following facts:

- (i) A finite dimensional variety over  $k$  holds canonically an ind-variety structure.
- (ii) If  $X$  and  $Y$  are ind-varieties, then  $X \times Y$  is canonically an ind-variety by taking the filtration

$$(X \times Y)_n := X_n \times Y_n.$$

- (iii) Let  $G$  be a Kac-Moody group and  $U^\pm, B^\pm$  be its subgroups as above. Then  $G$  is an ind-group and  $U^\pm, B^\pm$  are its closed ind-subgroups.
- (iv) The multiplication maps

$$\begin{array}{ccc} T \times U & \longrightarrow & B \\ (t, u) & \mapsto & tu \end{array} \quad \begin{array}{ccc} U^- \times T & \longrightarrow & B^- \\ (v, t) & \mapsto & vt \end{array}$$

are isomorphisms of ind-varieties.

### 3 Geometric Crystals and Unipotent Crystals

In this section, we define geometric crystals and unipotent crystals associated with Kac-Moody groups, which is just a generalization of [1] to a Kac-Moody setting.

#### 3.1 Geometric Crystals

Let  $(a_{ij})_{i,j \in I}$  be a symmetrizable generalized Cartan matrix and  $G$  be the associated Kac-Moody group with the maximal torus  $T$ . An element in  $\text{Hom}(T, \mathbb{C}^\times)$  (resp.  $\text{Hom}(\mathbb{C}^\times, T)$ ) is called a *character* (resp. *co-character*) of  $T$ . We define a *simple co-root*  $\alpha_i^\vee \in \text{Hom}(\mathbb{C}^\times, T)$  ( $i \in I$ ) by  $\alpha_i^\vee(t) := T_i$ . We have a pairing  $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ .

**Definition 3.1.** (i) Let  $X$  be an ind-variety over  $\mathbb{C}$ ,  $\gamma : X \rightarrow T$  be a rational morphism and a family of rational  $\mathbb{C}$ -actions  $e_i : \mathbb{C}^\times \times X \rightarrow X$  ( $i \in I$ );

$$\begin{array}{ccc} e_i : \mathbb{C}^\times \times X & \longrightarrow & X \\ (c, x) & \mapsto & e_i^c(x). \end{array}$$

The triplet  $\chi = (X, \gamma, \{e_i\}_{i \in I})$  is a *geometric pre-crystal* if it satisfies  $\{1\} \times X \subset \text{dom}(e_i)$ ,  $e^1(x) = x$  and

$$\gamma(e_i^c(x)) = \alpha_i^\vee(c)\gamma(x). \quad (3.1)$$

(ii) Let  $(X, \gamma_X, \{e_i^X\}_{i \in I})$  and  $(Y, \gamma_Y, \{e_i^Y\}_{i \in I})$  be geometric pre-crystals. A rational morphism  $f : X \rightarrow Y$  is a *morphism of geometric pre-crystals* if  $f$  satisfies that

$$f \circ e_i^X = e_i^Y \circ f, \quad \gamma_X = \gamma_Y \circ f.$$

In particular, if a morphism  $f$  is a birational isomorphism of ind-varieties, it is called an *isomorphism of geometric pre-crystals*.

Let  $\chi = (X, \gamma, \{e_i\}_{i \in I})$  be a geometric pre-crystal. For a word  $\mathbf{i} = (i_1, i_2, \dots, i_l) \in R(w)$  ( $w \in W$ ), set  $\alpha^{(l)} := \alpha_{i_l}$ ,  $\alpha^{(l-1)} := s_{i_l}(\alpha_{i_{l-1}})$ ,  $\dots$ ,  $\alpha^{(1)} := s_{i_l} \dots s_{i_2}(\alpha_{i_1})$ . Now for a word  $\mathbf{i} = (i_1, i_2, \dots, i_l) \in R(w)$  we define a rational morphism  $e_{\mathbf{i}} : T \times X \rightarrow X$  by

$$(t, x) \mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \dots e_{i_l}^{\alpha^{(l)}(t)}(x).$$

**Definition 3.2.** (i) A geometric pre-crystal  $\chi$  is called a *geometric crystal* if for any  $w \in W$ , and any  $\mathbf{i}, \mathbf{i}' \in R(w)$  we have

$$e_{\mathbf{i}} = e_{\mathbf{i}'}. \quad (3.2)$$

- (ii) Let  $(X, \gamma_X, \{e_i^X\}_{i \in I})$  and  $(Y, \gamma_Y, \{e_i^Y\}_{i \in I})$  be geometric crystals. A rational morphism  $f : X \rightarrow Y$  is called a *morphism* (resp. an *isomorphism*) of *geometric crystals* if it is a morphism (resp. an isomorphism) of geometric pre-crystals.

The following lemma is a direct result from [1][Lemma 2.1] and the fact that the Weyl group of any Kac-Moody Lie algebra is a Coxeter group [3][Proposition 3.13].

**Lemma 3.3.** *The relations (3.2) are equivalent to the following relations:*

$$\begin{aligned} e_i^{c_1} e_j^{c_2} &= e_j^{c_2} e_i^{c_1} && \text{if } \langle \alpha_i^\vee, \alpha_j \rangle = 0, \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} && \text{if } \langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_j^\vee, \alpha_i \rangle = -1, \\ e_i^{c_1} e_j^{c_1^2 c_2} e_i^{c_1 c_2} e_j^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^2 c_2} e_i^{c_1} && \text{if } \langle \alpha_i^\vee, \alpha_j \rangle = -2, \langle \alpha_j^\vee, \alpha_i \rangle = -1, \\ e_i^{c_1} e_j^{c_1^2 c_2} e_i^{c_1^3 c_2} e_j^{c_1 c_2} e_j^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1^2 c_2} e_i^{c_1} && \text{if } \langle \alpha_i^\vee, \alpha_j \rangle = -3, \langle \alpha_j^\vee, \alpha_i \rangle = -1, \end{aligned}$$

*Remark.* If  $\langle \alpha_i^\vee, \alpha_j \rangle \langle \alpha_j^\vee, \alpha_i \rangle \geq 4$ , there is no relation between  $e_i$  and  $e_j$ .

## 3.2 Unipotent Crystals

In the sequel, we denote the unipotent subgroup  $U^+$  by  $U$ . We define unipotent crystals (see [1],[11]) associated to Kac-Moody groups.

**Definition 3.4.** Let  $X$  be an ind-variety over  $\mathbb{C}$  and  $\alpha : U \times X \rightarrow X$  be a rational  $U$ -action such that  $\alpha$  is defined on  $\{e\} \times X$ . Then, the pair  $\mathbf{X} = (X, \alpha)$  is called a  *$U$ -variety*. For  $U$ -varieties  $\mathbf{X} = (X, \alpha_X)$  and  $\mathbf{Y} = (Y, \alpha_Y)$ , a rational morphism  $f : X \rightarrow Y$  is called a  *$U$ -morphism* if it commutes with the action of  $U$ .

Now, we define the  $U$ -variety structure on  $B^- = U^-T$ . As in Sect.2,  $B^-$  is an ind-subgroup of  $G$  and hence an ind-variety over  $\mathbb{C}$ . The multiplication map in  $G$  induces the open embedding;  $B^- \times U \hookrightarrow G$ , which is a birational isomorphism. Let us denote the inverse birational isomorphism by  $g$ ;

$$g : G \longrightarrow B^- \times U.$$

Then we define the rational morphisms  $\pi^- : G \rightarrow B^-$  and  $\pi : G \rightarrow U$  by  $\pi^- := \text{proj}_{B^-} \circ g$  and  $\pi := \text{proj}_U \circ g$ . Now we define the rational  $U$ -action  $\alpha_{B^-}$  on  $B^-$  by

$$\alpha_{B^-} := \pi^- \circ m : U \times B^- \longrightarrow B^-,$$

where  $m$  is the multiplication map in  $G$ . Then we obtain  $U$ -variety  $\mathbf{B}^- = (B^-, \alpha_{B^-})$ .

**Definition 3.5.** (i) Let  $\mathbf{X} = (X, \alpha)$  be a  $U$ -variety and  $f : X \rightarrow \mathbf{B}^-$  be a  $U$ -morphism. The pair  $(\mathbf{X}, f)$  is called a *unipotent  $G$ -crystal* or, for short, *unipotent crystal*.

- (ii) Let  $(\mathbf{X}, f_X)$  and  $(\mathbf{Y}, f_Y)$  be unipotent crystals. A  $U$ -morphism  $g : X \rightarrow Y$  is called a *morphism of unipotent crystals* if  $f_X = f_Y \circ g$ . In particular, if  $g$  is a birational isomorphism of ind-varieties, it is called an *isomorphism of unipotent crystals*.

We define a product of unipotent crystals following [1]. For unipotent crystals  $(\mathbf{X}, f_X)$ ,  $(\mathbf{Y}, f_Y)$ , define a morphism  $\alpha_{X \times Y} : U \times X \times Y \rightarrow X \times Y$  by

$$\alpha_{X \times Y}(u, x, y) := (\alpha_X(u, x), \alpha_Y(\pi(u \cdot f_X(x)), y)). \quad (3.3)$$

If there is no confusion, we use abbreviated notation  $u(x, y)$  for  $\alpha_{X \times Y}(u, x, y)$ .

**Theorem 3.6** ([1]). (i) *The morphism  $\alpha_{X \times Y}$  defined above is a rational  $U$ -morphism on  $X \times Y$ .*

(ii) *Let  $\mathbf{m} : B^- \times B^- \rightarrow B^-$  be a multiplication morphism and  $f = f_{X \times Y} : X \times Y \rightarrow B^-$  be the rational morphism defined by*

$$f_{X \times Y} := \mathbf{m} \circ (f_X \times f_Y).$$

*Then  $f_{X \times Y}$  is a  $U$ -morphism and  $(\mathbf{X} \times \mathbf{Y}, f_{X \times Y})$  is a unipotent crystal, which we call a product of unipotent crystals  $(\mathbf{X}, f_X)$  and  $(\mathbf{Y}, f_Y)$ .*

(iii) *Product of unipotent crystals is associative.*

### 3.3 From unipotent crystals to geometric crystals

For  $i \in I$ , set  $U_i^\pm := U^\pm \cap \bar{s}_i U^\mp \bar{s}_i^{-1}$  and  $U_\pm^i := U^\pm \cap \bar{s}_i U^\pm \bar{s}_i^{-1}$ . Indeed,  $U_i^\pm = U_{\pm\alpha_i}$ . Set

$$Y_{\pm\alpha_i} := \langle x_{\pm\alpha_i}(t) U_\alpha x_{\pm\alpha_i}(-t) \mid t \in \mathbb{C}, \alpha \in \Delta_\pm^{\text{re}} \setminus \{\pm\alpha_i\} \rangle.$$

We have the unique decomposition;

$$U^- = U_i^- \cdot Y_{\pm\alpha_i} = U_{-\alpha_i} \cdot U_-^i.$$

By using this decomposition, we get the canonical projection  $\xi_i : U^- \rightarrow U_{-\alpha_i}$ . Now, we define the function on  $U^-$  by

$$\chi_i := y_i^{-1} \circ \xi_i : U^- \longrightarrow U_{-\alpha_i} \xrightarrow{\sim} \mathbb{C},$$

and extend this to the function on  $B^-$  by  $\chi_i(u \cdot t) := \chi_i(u)$  for  $u \in U^-$  and  $t \in T$ . For a unipotent  $G$ -crystal  $(\mathbf{X}, \mathbf{f}_X)$ , we define a function  $\varphi_i := \varphi_i^X : X \rightarrow \mathbb{C}$  by

$$\varphi_i := \chi_i \circ \mathbf{f}_X,$$

and a rational morphism  $\gamma_X : X \rightarrow T$  by

$$\gamma_X := \text{proj}_T \circ \mathbf{f}_X : X \rightarrow B^- \rightarrow T, \quad (3.4)$$

where  $\text{proj}_T$  is the canonical projection. Suppose that the function  $\varphi_i$  is not identically zero on  $X$ . We define a morphism  $e_i : \mathbb{C}^\times \times X \rightarrow X$  by

$$e_i^c(x) := x_i \left( \frac{c-1}{\varphi_i(x)} \right) (x). \quad (3.5)$$

**Theorem 3.7** ([1]). *For a unipotent  $G$ -crystal  $(\mathbf{X}, \mathbf{f}_X)$ , suppose that the function  $\varphi_i$  is not identically zero for any  $i \in I$ . Then the rational morphisms  $\gamma_X : X \rightarrow T$  and  $e_i : \mathbb{C}^\times \times X \rightarrow X$  as above define a geometric  $G$ -crystal  $(X, \gamma_X, \{e_i\}_{i \in I})$ , which is called the induced geometric  $G$ -crystals by unipotent  $G$ -crystal  $(\mathbf{X}, \mathbf{f}_X)$ .*

Note that in [1], the cases  $\varphi_i \equiv 0$  for some  $i \in I$  are treated by considering Levi subgroups of  $G$ . But here we do not treat such things.

The following product structure on geometric crystals are most important results in the sense of comparison with the tensor product theorem in Kashiwara's crystal theory.

**Proposition 3.8.** For unipotent  $G$ -crystals  $(\mathbf{X}, f_X)$  and  $(\mathbf{Y}, f_Y)$ , set the product  $(\mathbf{Z}, f_Z) := (\mathbf{X}, f_X) \times (\mathbf{Y}, f_Y)$ , where  $Z = X \times Y$ . Let  $(Z, \gamma_Z, \{e_i\})$  be the induced geometric  $G$ -crystal from  $(\mathbf{Z}, f_Z)$ . Then we obtain;

(i)  $\gamma_Z = \mathbf{m} \circ (\gamma_X \times \gamma_Y)$ .

(ii) For each  $i \in I$ ,  $(x, y) \in Z$ ,

$$\varphi_i^Z(x, y) = \varphi_i^X(x) + \frac{\varphi_i^Y(y)}{\alpha_i(\gamma_X(x))}. \quad (3.6)$$

(iii) For any  $i \in I$ , the action  $e_i : \mathbb{C}^\times \times Z \rightarrow Z$  is given by:  $e_i^c(x, y) = (e_i^{c_1}(x), e_i^{c_2}(y))$ , where

$$c_1 = \frac{c\alpha_i(\gamma_X(x))\varphi_i^X(x) + \varphi_i^Y(y)}{\alpha_i(\gamma_X(x))\varphi_i^X(x) + \varphi_i^Y(y)}, \quad c_2 = \frac{\alpha_i(\gamma_X(x))\varphi_i^X(x) + \varphi_i^Y(y)}{\alpha_i(\gamma_X(x))\varphi_i^X(x) + c^{-1}\varphi_i^Y(y)} \quad (3.7)$$

Here note that  $c_1 c_2 = c$ . The formula  $c_1$  and  $c_2$  in [1] seem to be different from ours.

## 4 Crystal structure on Schubert varieties

### 4.1 Highest weight modules and Schubert varieties

As in Sect.2, let  $G$  be a Kac-Moody group,  $B^\pm = U^\pm T$  (resp.  $U^\pm$ ) be the Borel (resp. unipotent) subgroups in  $G$  and  $W$  be the associated Weyl group. Here, we have the following Bruhat decomposition and Birkhoff decomposition;

**Proposition 4.1** ([8],[10],[12]). We have

$$G = \bigcup_{w \in W} B^+ \bar{w} B^+ = \bigcup_{w \in W} U^+ \bar{w} B^+ \quad (\text{Bruhat decomposition}), \quad (4.1)$$

$$G = \bigcup_{w \in W} B^- \bar{w} B^+ = \bigcup_{w \in W} U^- \bar{w} B^+ \quad (\text{Birkhoff decomposition}). \quad (4.2)$$

Let  $J \subset I$  be a subset of the index set  $I$  and  $W_J := \langle s_i | i \in J \rangle$  be the subgroup of  $W$  associated with  $J$ . Set  $P_J := B^+ W_J B^+$  and call it a (standard) parabolic subgroup of  $G$  associated with  $J \subset I$ . We denote the set of the minimal coset representatives of  $W/W_J$  in  $W$  by  $W^J$ . There exist the following parabolic Bruhat/Birkhoff decompositions:

**Proposition 4.2** ([8],[10],[12]). Let  $J$  be a subset of  $I$  and,  $W_J$  and  $W^J$  be as above. Then we have

$$G = \bigcup_{w^* \in W^J} U^+ \bar{w}^* P_J, \quad G = \bigcup_{w^* \in W^J} U^- \bar{w}^* P_J.$$

### 4.2 Unipotent crystal structure on Schubert variety

For  $\Lambda \in P_+$  ( $P_+$  is the set of dominant integral weight), let us denote an integral highest weight simple module with the highest weight  $\Lambda$  by  $L(\Lambda)$  ([3]) and its projective space by



**Lemma 4.6** ([12, 2.2]). *For any  $w \in W^{J_\Lambda}$  ( $\Lambda \in P_+$ ), there exists an isomorphism of ind (algebraic)-varieties*

$$\begin{array}{ccc} \delta : U_w & \xrightarrow{\sim} & X(\Lambda)_w \\ u & \mapsto & u \cdot v_\Lambda \end{array}$$

Define an isomorphism of ind (algebraic)-varieties

$$\begin{array}{ccc} \zeta : X(\Lambda)_w & \xrightarrow{\sim} & U_w \bar{w} \\ v & \mapsto & \zeta(v) := \delta^{-1}(v) \bar{w}, \end{array}$$

where  $w \in W^{J_\Lambda}$  and  $\Lambda \in P_+$ . Since  $X(\Lambda)_w$  is  $U$ -orbit of  $\rho(\bar{w} \cdot P_{J_\Lambda}/P_{J_\Lambda})$ ,  $U$  acts rationally on  $X(\Lambda)_w$ . We denote the action of  $x \in U$  on  $v \in X(\Lambda)_w$  by  $x(v)$ .

**Lemma 4.7.** *The isomorphism  $\zeta : X(\Lambda)_w \rightarrow U_w \bar{w}$  is a  $U$ -morphism.*

Define a rational morphism  $f_w : X(\Lambda)_w \rightarrow B^-$  by  $f_w = \pi^- \circ \zeta$ . The following is one of the main results of this article.

**Theorem 4.8.** *For  $\Lambda \in P_+$  and  $w \in W^{J_\Lambda}$ , let  $X(\Lambda)_w$  be a finite Schubert cell and  $f_w : X(\Lambda)_w \rightarrow B^-$  be as defined above. Then the pair  $(X(\Lambda)_w, f_w)$  is a unipotent  $G$ -crystal.*

In the sense of Definition 3.5(ii),  $\zeta$  is an isomorphism of unipotent crystals on  $X(\Lambda)_w$  and  $U_w \bar{w}$ .

Since  $X(\Lambda)_w \hookrightarrow \bar{X}(\Lambda)_w$  is an open embedding, they are birationally equivalent. Let  $\omega : \bar{X}(\Lambda)_w \rightarrow X(\Lambda)_w$  be the inverse birational isomorphism. Thus,  $\bar{f}_w := f_w \circ \omega : \bar{X}(\Lambda)_w \rightarrow B^-$  is a  $U$ -morphism. Then we have

**Corollary 4.9.** *Let  $\bar{X}(\Lambda)_w$  be a finite Schubert variety and  $\bar{f}_w$  be defined as above. Then the pair  $(\bar{X}(\Lambda)_w, \bar{f}_w)$  is a unipotent  $G$ -crystal.*

*Remark.* Note that for all  $w \leq w'$ , we have the closed embedding  $\bar{X}(\Lambda)_w \hookrightarrow \bar{X}(\Lambda)_{w'}$  ([12]), and the isomorphism

$$X(\Lambda) \xrightarrow{\sim} \varinjlim_{w \in W^{J_\Lambda}} \bar{X}(\Lambda)_w.$$

Nevertheless, in general, we do not obtain a unipotent crystal structure on  $X(\Lambda)$  by using this direct limit since for  $y < w$ , the rational morphism  $\bar{f}_w : \bar{X}(\Lambda)_w \rightarrow B^-$  is not defined on  $\bar{X}(\Lambda)_y$ .

### 4.3 Geometric Crystal structure on $X(\Lambda)_w$

As we have seen in 3.3, we can associate geometric crystal structure with the finite Schubert cell (resp. variety)  $X(\Lambda)_w$  (resp.  $\bar{X}(\Lambda)_w$ ) since we have seen that they are unipotent  $G$ -crystals.

By Theorem 3.7, we have

**Theorem 4.10.** *For  $w \in W$ , suppose that  $I = I(w)$ . We can associate the geometric  $G$ -crystal structure with the finite Schubert cell  $X(\Lambda)_w$  (resp. variety  $\bar{X}(\Lambda)_w$ ) by setting (see (3.4) and (3.5))*

$$\gamma_w := \text{proj}_T \circ f_w \quad (\text{resp. } \bar{\gamma}_w := \text{proj}_T \circ \bar{f}_w), \quad e_i^c(x) = x_i \left( \frac{c-1}{\varphi_i(x)} \right) (x),$$

where  $\text{proj}_T : B^- = U^- T \rightarrow T$ .



We denote this induced geometric crystal by  $(X(\Lambda)_w, \gamma_w, \{e_i\}_{i \in I})$  (resp.  $(\overline{X}(\Lambda)_w, \overline{\gamma}_w, \{e_i\}_{i \in I})$ ). This geometric/unipotent crystal  $(X(\Lambda)_w, \gamma_w, \{e_i\}_{i \in I})$  is realized in  $B^-$  in the following sense.

**Proposition 4.11.** *For  $w = s_{i_1} \cdots s_{i_k}$ , define*

$$B_w^- := \{Y_w(c_1, \dots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_k}(c_k) \in B^- \mid c_i \in \mathbb{C}^\times\}.$$

where  $Y_i(c) = y_i(\frac{1}{c})\alpha_i^\vee(c)$  and  $U$ -actions on  $B_w^-$  by

$$u(Y_w(c_1, \dots, c_k)) := \pi^-(u \cdot Y_w(c_1, \dots, c_k)) \quad (u \in U).$$

Then  $X(\Lambda)_w$  and  $B_w^-$  are birationally equivalent via  $f_w$  and isomorphic as unipotent crystals. Moreover, they are isomorphic as induced geometric crystals.

## 5 Tropicalization of Crystals and Schubert Varieties

### 5.1 Positive structure and Ultra-discretizations/Tropicalizations

Let us recall the notion of “positive structure”  $([1],[11])$ .

The setting below is simpler than the ones in  $([1],[11])$ , since it is sufficient for our purpose.

Let  $T = (\mathbb{C}^\times)^l$  be an algebraic torus over  $\mathbb{C}$  and  $X^*(T) \cong \mathbb{Z}^l$  (resp.  $X_*(T) \cong \mathbb{Z}^l$ ) be the lattice of characters (resp. co-characters) of  $T$ . Set  $R := \mathbb{C}(c)$  and define

$$\begin{aligned} v : R \setminus \{0\} &\longrightarrow \mathbb{Z} \\ f(c) &\longmapsto \deg(f(c)). \end{aligned}$$

Here note that for  $f_1, f_2 \in R \setminus \{0\}$ , we have

$$v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2) \quad (5.1)$$

Let  $f = (f_1, \dots, f_n) : T \rightarrow T'$  be a rational morphism between two algebraic tori  $T = (\mathbb{C}^\times)^m$  and  $T' = (\mathbb{C}^\times)^n$ . We define a map  $\widehat{f} : X_*(T) \rightarrow X_*(T')$  by

$$(\widehat{f}(\xi))(c) := (c^{v(f_1(\xi(c)))}, \dots, c^{v(f_n(\xi(c)))}),$$

where  $\xi \in X_*(T)$ . Since  $v$  satisfies (5.1), the map  $\widehat{f}$  is an additive group homomorphism. Identifying  $X_*(T)$  (resp.  $X_*(T')$ ) with  $\mathbb{Z}^m$  (resp.  $\mathbb{Z}^n$ ) by  $\xi(c) = (c^{l_1}, \dots, c^{l_m}) \leftrightarrow (l_1, \dots, l_m) \in \mathbb{Z}^m$ , we write

$$\widehat{f}(l_1, \dots, l_m) := (v(f_1(\xi(c))), \dots, v(f_n(\xi(c)))).$$

A rational function  $f(c) \in \mathbb{C}(c)$  ( $f \neq 0$ ) is *positive* if  $f$  can be expressed as a ratio of polynomials with positive coefficients.

*Remark.* A rational function  $f(c) \in \mathbb{C}(c)$  is positive if and only if  $f(a) > 0$  for any  $a > 0$

If  $f_1, f_2 \in R$  are positive, then we have (5.1) and

$$v(f_1 + f_2) = \max(v(f_1), v(f_2)). \quad (5.2)$$

**Definition 5.1** ([1]). Let  $f = (f_1, \dots, f_n) : T \rightarrow T'$  between two algebraic tori  $T, T'$  be a rational morphism as above. It is called *positive*, if the following two conditions are satisfied:

- (i) For any co-character  $\xi : \mathbb{C}^\times \rightarrow T$ , the image of  $\xi$  is contained in  $\text{dom}(f)$ .
- (ii) For any co-character  $\xi : \mathbb{C}^\times \rightarrow T$ , any  $f_i(\xi(c))$  ( $i \in I$ ) is a positive rational function.

Denote by  $\text{Mor}^+(T, T')$  the set of positive rational morphisms from  $T$  to  $T'$ .

**Lemma 5.2** ([1]). *For any positive rational morphisms  $f \in \text{Mor}^+(T_1, T_2)$  and  $g \in \text{Mor}^+(T_2, T_3)$ , the composition  $g \circ f$  is in  $\text{Mor}^+(T_1, T_3)$ .*

By Lemma 5.2, we can define a category  $\mathcal{T}_+$  whose objects are algebraic tori over  $\mathbb{C}$  and arrows are positive rational morphisms.

**Lemma 5.3** ([1]). *For any algebraic tori  $T_1, T_2, T_3$ , and positive rational morphisms  $f \in \text{Mor}^+(T_1, T_2)$ ,  $g \in \text{Mor}^+(T_2, T_3)$ , we have  $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$ .*

By this lemma, we obtain a functor

$$\begin{array}{ccc} \mathcal{UD} : & \mathcal{T}_+ & \longrightarrow & \mathfrak{Set} \\ & T & \mapsto & X_*(T) \\ & (f : T \rightarrow T') & \mapsto & (\widehat{f} : X_*(T) \rightarrow X_*(T')) \end{array}$$

**Definition 5.4** ([1]). Let  $\chi = (X, \gamma, \{e_i\}_{i \in I})$  be a geometric crystal,  $T'$  be an algebraic torus and  $\theta : T' \rightarrow X$  be a birational isomorphism. The isomorphism  $\theta$  is called *positive structure* on  $\chi$  if it satisfies

- (i) the rational morphism  $\gamma \circ \theta : T' \rightarrow T$  is positive.
- (ii) For any  $i \in I$ , the rational morphism  $e_{i,\theta} : \mathbb{C}^\times \times T' \rightarrow T'$  defined by  $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$  is positive.

Let  $\theta : T \rightarrow X$  be a positive structure on a geometric crystal  $\chi = (X, \gamma, \{e_i\}_{i \in I})$ . Applying the functor  $\mathcal{UD}$  to positive rational morphisms  $e_{i,\theta} : \mathbb{C}^\times \times T' \rightarrow T'$  and  $\gamma \circ \theta : T' \rightarrow T$  (the notations are as above), we obtain

$$\begin{aligned} \tilde{e}_i &:= \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T) \rightarrow X_*(T) \\ \tilde{\gamma} &:= \mathcal{UD}(\gamma \circ \theta) : X_*(T') \rightarrow X_*(T). \end{aligned}$$

Now, for given positive structure  $\theta : T' \rightarrow X$  on a geometric pre-crystal  $\chi = (X, \gamma, \{e_i\}_{i \in I})$ , we associate the triplet  $(X_*(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i \in I})$  with a free pre-crystal structure (see [1, 2.2]) and denote it by  $\mathcal{UD}_{\theta, T'}(\chi)$ . By Lemma 3.3, we have the following theorem:

**Theorem 5.5.** *For any geometric crystal  $\chi = (X, \gamma, \{e_i\}_{i \in I})$  and positive structure  $\theta : T' \rightarrow X$ , the associated pre-crystal  $\mathcal{UD}_{\theta, T'}(\chi) = (X_*(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i \in I})$  is a free  $W$ -crystal (see [1, 2.2])*

We call the functor  $\mathcal{UD}$  “*ultra-discretization*” instead of “*tropicalization*” unlike in [1]. And for an object  $B$  in  $\mathfrak{Set}$ , if there exists a geometric crystal  $\chi$ , an algebraic torus  $T$  in  $\mathcal{T}_+$  and a positive structure  $\theta$  on  $\chi$  such that  $\mathcal{UD}_{\theta, T}(\chi) \cong B$  as crystals, we call  $\chi$  a *tropicalization* of  $B$ .

Now, we define certain positive structure on geometric crystal  $B_w^-$  ( $I = I(w)$ , and  $w \in W^{J_\Lambda}$ ) and see that it turns out to be a tropicalization of (Langlands dual of) some Kashiwara's crystal.

For  $i \in I$ , let  $B_i$  be the crystal defined by (see e.g. [4])

$$\begin{aligned} B_i &:= \{(x)_i | x \in \mathbb{Z}\}, \\ \tilde{e}_i(x)_i &= (x+1)_i, \quad \tilde{f}_i(x)_i = (x-1)_i, \quad \tilde{e}_j(x)_i = \tilde{f}_j(x)_i = 0 \quad (i \neq j) \\ wt(x)_i &= x\alpha_i, \quad \varepsilon_i(x)_i = -x, \quad \varphi_i(x)_i = x, \quad \varepsilon_j(x)_i = \varphi_j(x)_i = -\infty \quad (i \neq j). \end{aligned}$$

For  $w = s_{i_1} s_{i_2} \cdots s_{i_k} \in W$  and  $\mathbf{i} = (i_1, i_2, \dots, i_k) \in R(w)$ , we define the morphism  $\theta_{\mathbf{i}} : (\mathbb{C}^\times)^k \rightarrow B_w^-$  by

$$\theta_{\mathbf{i}}(c_1, c_2, \dots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_k}(c_k) = y_{i_1} \left( \frac{1}{c_1} \right) \alpha_{i_1}^\vee(c_1) \cdots y_{i_k} \left( \frac{1}{c_k} \right) \alpha_{i_k}^\vee(c_k) \quad (5.3)$$

**Proposition 5.6.** (i) For any  $\mathbf{i} \in R(w)$  ( $w \in W$ ,  $I(w) = I$ ), the morphism  $\theta_{\mathbf{i}}$  defined in (5.3) is a positive structure on the geometric crystal  $B_w^-$ .

(ii) Geometric crystal  $B_w^-$  is a tropicalization of the Langlands dual of the crystal  $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$  with respect to the positive structure  $\theta_{\mathbf{i}}(c_1, c_2, \dots, c_k)$ , or equivalently  $\mathcal{UD}(B_w^-) \cong \text{Langlands dual}(B_{i_1} \otimes \cdots \otimes B_{i_k})$  as crystals.

Indeed, we have

$$\gamma \left( y_{i_1} \left( \frac{1}{c_1} \right) \alpha_{i_1}^\vee(c_1) \cdots y_{i_k} \left( \frac{1}{c_k} \right) \alpha_{i_k}^\vee(c_k) \right) = \alpha_{i_1}^\vee(c_1) \cdots \alpha_{i_k}^\vee(c_k),$$

and the explicit action of  $e_i^c$  on  $Y_w(c_1, \dots, c_k)$ :

$$e_i^c(Y_w(c_1, \dots, c_k)) = x_i \left( \frac{c-1}{\varphi_i(Y_w(c_1, \dots, c_k))} \right) (Y_w(c_1, \dots, c_k)) =: Y_w(\mathcal{C}_1, \dots, \mathcal{C}_k),$$

where

$$\mathcal{C}_j := c_j \cdot \frac{\sum_{1 \leq m \leq j, i_m = i} \frac{c}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m} + \sum_{j < m \leq k, i_m = i} \frac{1}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m}}{\sum_{1 \leq m < j, i_m = i} \frac{c}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m} + \sum_{j \leq m \leq k, i_m = i} \frac{1}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m}}. \quad (5.4)$$

Furthermore, we describe the action of  $\tilde{e}_i^c$  on  $B_{i_1} \otimes \cdots \otimes B_{i_k}$ . Take  $b_i = (b_1)_{i_1} \otimes \cdots \otimes (b_k)_{i_k}$  ( $\mathbf{i} = (i_1, \dots, i_k)$ ,  $b_j \in \mathbb{Z}$ ). Since the action of  $\tilde{e}_i$  on tensor products is described explicitly in [4], we obtain

$$\tilde{e}_i^c(b_i) = (\beta_1)_{i_1} \otimes \cdots \otimes (\beta_k)_{i_k},$$

where

$$\begin{aligned} \beta_j &= b_j + \max \left( \max_{\substack{1 \leq m \leq j, \\ i_m = i}} (c - b_m - \sum_{l < m} b_l a_{i, i_l}), \max_{\substack{j < m \leq k, \\ i_m = i}} (-b_m - \sum_{l < m} b_l a_{i, i_l}) \right) \\ &- \max \left( \max_{\substack{1 \leq m < j, \\ i_m = i}} (c - b_m - \sum_{l < m} b_l a_{i, i_l}), \max_{\substack{j \leq m \leq k, \\ i_m = i}} (-b_m - \sum_{l < m} b_l a_{i, i_l}) \right) \end{aligned} \quad (5.5)$$

Now, we know that (5.4) and (5.5) are related to each other by the tropicalization/ultra-discretization operations:

$$\begin{array}{ccc}
 C_j & \begin{array}{c} \xrightarrow{\text{ultra-discretization}} \\ \xleftarrow{\text{tropicalization}} \end{array} & \beta_j \\
 c_j & \longleftrightarrow & b_j \\
 x \cdot y & \longleftrightarrow & x + y \\
 \frac{x}{y} & \longleftrightarrow & x - y \\
 x + y & \longleftrightarrow & \max(x, y) \\
 a_{i,j} & \begin{array}{c} \xrightarrow{\text{Langlands dual}} \\ \xleftarrow{\text{Langlands dual}} \end{array} & a_{j,i}
 \end{array}$$

## 6 Affine perfect crystal $\text{Aff}(B_\infty)$ for $\widehat{\mathfrak{sl}}_{n+1}$

In this subsection, we see an application of ultra-discretization of geometric crystal on Schubert cells/varieties defined for  $\widehat{SL}_{n+1}$ .

### 6.1 Perfect crystals and their limit

Perfect crystals are defined for quantum affine algebras and they play an important role in studying solvable lattice models([6],[7]). In [5], certain limit of perfect crystals are introduced, which is denoted  $B_\infty$ .

Let  $\mathfrak{g}$  be an affine Lie algebra and  $P_{cl}$  be a classical weight lattice and set  $(P_{cl})_l^+ := \{\lambda \in P_{cl} | \langle c, \lambda \rangle = l, \langle h_i, \lambda \rangle \geq 0\}$  ( $l \in \mathbb{Z}_{>0}$ ).

**Definition 6.1.** A crystal  $B$  is a perfect of level  $l$  if

- (i)  $B \otimes B$  is connected.
- (ii) There exists  $\lambda_0 \in P_{cl}$  such that

$$wt(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i, \quad \#B_{\lambda_0} = 1$$

- (iii) There exists a finite dimensional  $U_q(\mathfrak{g})$ -module  $V$  with a crystal pseudo-base  $B_{ps}$  such that  $B \cong B_{ps}/\pm 1$
- (iv) We have  $\varepsilon, \varphi : B^{min} := \{b \in B | \langle c, \varepsilon(b) \rangle = l\} \xrightarrow{\sim} (P_{cl}^+)_l$  (bijective).

Now let us define the limit of perfect crystals. Let  $\{B_l\}_{l \geq 1}$  be a family of perfect crystals of level  $l$  and set  $J := \{(l, b) | l > 0, b \in B_l^{min}\}$ .

**Definition 6.2.** A crystal  $B_\infty$  with an element  $b_\infty$  is called a limit of  $\{B_l\}_{l \geq 1}$  if

- (i)  $wt(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$ .
- (ii) For any  $(l, b) \in J$ , there exists an embedding of crystals:

$$\begin{array}{ccc}
 f_{(l,b)} : & T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} & \hookrightarrow B_\infty \\
 & t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} & \mapsto b_\infty
 \end{array}$$

$$(iii) B_\infty = \bigcup_{(l,b) \in J} \text{Im} f_{(l,b)}.$$

Let  $B(\infty)$  be the crystal of the subalgebra  $U_q^-(\mathfrak{g})$ . Then we have the isomorphism of crystals:

$$B(\infty) \xrightarrow{\sim} B(\infty) \otimes B_\infty.$$

In the case  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ ,  $B_\infty$  and its affinization  $\text{Aff}(B_\infty)$  are given as follows ([5]):

$$\begin{aligned} B_\infty &:= \{b = (l_1, l_2, \dots, l_n) \mid l_i \in \mathbb{Z}\} (\cong \mathbb{Z}^n) \\ \text{Aff}(B_\infty) &:= \{b = (k, l_1, l_2, \dots, l_n) \mid k, l_i \in \mathbb{Z}\} (\cong \mathbb{Z}^{n+1}) \end{aligned}$$

$$\begin{cases} \tilde{e}_0(b) = (k+1, l_1-1, \dots), \\ \tilde{e}_i(b) = (\dots, l_i+1, l_{i+1}-1, \dots) \quad (i = 1, \dots, n), \\ \tilde{f}_i = \tilde{e}_i^{-1}, \\ \text{wt}(b) = k\delta + (-2l_1 - l_2 - \dots - l_n)\Lambda_0 + \sum_{i=1}^n (l_i - l_{i+1})\Lambda_i, \end{cases}$$

where  $\Lambda_i$  is a fundamental weight and  $\delta$  is a basis of null roots.

## 6.2 Alternative positive structure on certain affine Schubert cell

Now, for  $G = \widehat{SL}_{n+1}$ , let  $w^* = s_0 s_1 \dots s_{n-1} s_n$  and set

$$B_{w^*}^- = \{Y(c_0, c_1, \dots, c_n) := Y_0(c_0)Y_1(c_1) \dots Y_n(c_n) \mid c_0, c_1, \dots, c_n \in \mathbb{C}^\times\},$$

as in Sect.5, which is isomorphic to the Schubert cell  $X(\Lambda)_{w^*}$  as a geometric crystal. We define the following positive structure on  $B_{w^*}^-$ :

$$\begin{aligned} \Theta : \quad (\mathbb{C}^\times)^{n+1} &\longrightarrow B_{w^*}^- \\ (k, l_1, l_2, \dots, l_n) &\mapsto Y_0(k)Y_1(kl_1) \dots Y_n(kl_1 \dots l_n). \end{aligned}$$

Through this  $\Theta$ , on  $(\mathbb{C}^\times)^{n+1}$  we obtain

$$\begin{cases} e_{0,\Theta}^c(k, l_1, \dots) = (ck, c^{-1}l_1, \dots) \\ e_{i,\Theta}^c(\dots, l_i, l_{i+1}, \dots) = (\dots, cl_i, c^{-1}l_{i+1}, \dots) \quad (i = 1, \dots, n). \end{cases}$$

On the other hand, on  $\text{Aff}(B_\infty)$  we have

$$\begin{cases} \tilde{e}_0^c(k, l_1, \dots) = (k+c, l_1-c, \dots) \\ \tilde{e}_i^c(\dots, l_i, l_{i+1}, \dots) = (\dots, l_i+c, l_{i+1}-c, \dots) \quad (i = 1, \dots, n), \end{cases}$$

which imply

$$\mathcal{UD}_\Theta(B_{w^*}^-) \cong \text{Aff}(B_\infty). \quad (6.1)$$

Note that ultra-discretization of the function  $\varphi_i$  on L.H.S. of (6.1) does not coincide with  $\varepsilon_i$  on R.H.S. of (6.1), which will be resolved in a forthcoming paper.

## References

- [1] Berenstein A. and Kazhdan D., Geometric crystals and Unipotent crystals, GAFA 2000 (Tel Aviv,1999), *Geom. Funct. Anal.* 2000, Special Volume, Part I, 188–236.
- [2] Berenstein A. and Zelevinsky A., Tensor product multiplicities, Canonical bases and Totally positive varieties, *Invent.Math.*, **143**, 77–128 (2001).
- [3] Kac V.G., Infinite dimensional Lie algebras 3rd ed., Cambridge University Press.
- [4] Kashiwara M., Crystal base and Littelmann’s refined Demazure character formula. *Duke Math. J.* **71** (3), 839–858 (1993).
- [5] Kang S-J., Kashiwara M. and Misra K.C., Crystal bases of Verma modules for quantum affine Lie algebras, *Compositio Mathematica* **92** (1994), 299–345.
- [6] Kang S-J., Kashiwara M., Misra K.C., Miwa T., Nakashima T. and Nakayashiki A., Affine crystals and vertex models, *Int.J.Mod.Phys.*,A7 Suppl.1A (1992), 449–484.
- [7] Kang S-J., Kashiwara M., Misra K.C., Miwa T., Nakashima T. and Nakayashiki A., Perfect crystals of quantum affine Lie algebras, *Duke Math. J.*, **68**(3), (1992), 499-607.
- [8] Kumar S., Kac-Moody groups, their Flag varieties and Representation Theory, *Progress in Mathematics* 204, Birkhauser Boston, 2002.
- [9] Kac V.G. and Peterson D.H., Defining relations of certain infinite-dimensional groups; in “Arithmetic and Geometry”(Artin M.,Tate J.,eds), 141–166, Birkhäuser, Boston-Basel-Stuttgart, (1983).
- [10] Peterson D.H., and Kac V.G., Infinite flag varieties and conjugacy theorems, *Proc.Nat.Acad.Sci.USA*, **80**, 1778–1782, (1983).
- [11] Nakashima T., Geometric crystals on Schubert varieties, (to appear in *J.Gem. and Phys.*).
- [12] Slodowy P., On the geometry of Schubert varieties attached to Kac-Moody Lie algebras, *Can.Math.Soc.Conf.Proc. on ‘Algebraic geometry’ (Vancouver)* **6**, 405–442, (1986).