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A fresh glimpse into the Stokes geometry of the Berk-Nevins-Roberts equation through a singular coordinate transformation*

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§0. Introduction

In the discussion with Professor A. Shudo, the following question occurred to us:

What if we kill a part of the symmetry of the Stokes geometry for the Berk-Nevins-Roberts operator P_{BNR} given by

$$\eta^{-3}\frac{\partial^3}{\partial x^3}+3\eta^{-1}\frac{\partial}{\partial x}+2ix, \tag{0.1}$$

by considering a singular coordinate transformation

$$z=x^2\ ? \tag{0.2}$$

Here and in what follows, η denotes a large parameter.

As is now well-known ([BNR],[AKKSST]), the Stokes geometry for P_{BNR} is as follows when $\arg\eta=0$:

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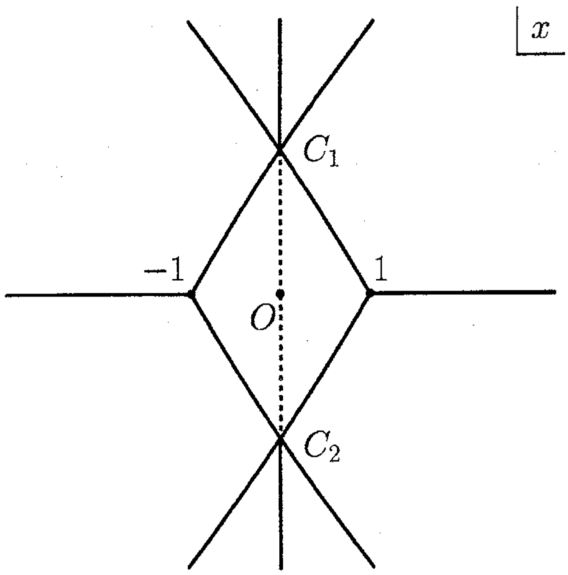


Fig. 0.1

Here $x = +1$ and -1 are simple turning points of the operator P_{BNR}, $x = 0$ is its (unique) virtual turning point, and the dotted line indicates that the portion C_1C_2 of the Stokes curve is inert in the sense that no Stokes phenomena are observed in any solutions of the equation $P_{\mathrm{BNR}}\psi = 0$. By the transformation (0.2), we find

$$\frac{1}{2x}P_{\mathrm{BNR}} = 4\eta^{-3}z\frac{\partial^3}{\partial z^3} + 6\eta^{-3}\frac{\partial^2}{\partial z^2} + 3\eta^{-1}\frac{\partial}{\partial z} + i. \tag{0.3}$$

In the sequel we let $P_{\mathrm{BNR}'}$ denote the operator that appears in the right-hand side of (0.3). With the help of a computer, one can readily find the following Stokes geometry of the equation $P_{\mathrm{BNR}'}\psi = 0$ with $\arg\eta = 0$:

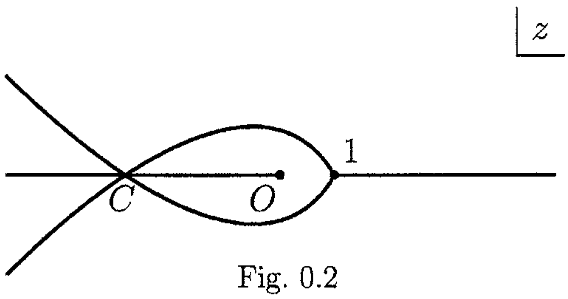


Fig. 0.2

Geometrically speaking, $z = 1$ is, under the transformation (0.2), the image of simple turning points $x = \pm 1$, and $z = C$ is the image of $x = C_1$ and C_2. The Stokes curve (half line) starting at the point $z = 0$ originates from the vanishing factor $z\cdot$ in front of $\partial^3/\partial z^3$; the precise definition of this Stokes curve will be given in our forthcoming paper [KKT] with the help of a decomposition theorem to be announced in Section 3 below. The results in Section 2 will also convince the reader of the assertion that the point $z = 0$ plays a role of a turning point of the operator $P_{\mathrm{BNR}'}$. In parenthesis, we note that there is no virtual turning point of the operator $P_{\mathrm{BNR}'}$; this fact can be readily seen by explicitly solving the Hamilton-Jacobi equation with the Hamiltonian determined by $P_{\mathrm{BNR}'}$, i.e.,

$$4z\zeta^3 + 3\zeta\eta^2 + i\eta^3 \tag{0.4}$$

with the initial condition

$$(z(0), y(0); \zeta(0), \eta(0)) = (1, y_0; -\frac{i}{2}, 1), \tag{0.5}$$

where y_0 is an arbitrary complex number.

Now, in view of the geometrical correspondence between Fig. 0.1 and Fig. 0.2, we were tempted to believe that the segment CO of the Stokes curve emanating from $z = 0$ should be inert. This belief is validated in two ways; in Section 1 we confirm this fact numerically (i.e., with the help of a computer) by applying the steepest descent method to an integral that represents a solution of the equation $P_{\mathrm{BNR}'}\psi = 0$, and in Section 2 we confirm this fact analytically (i.e., without the help of a computer) by reducing the problem to the connection problem for a second order operator with simple poles in the coefficients that was analyzed in [K1] and [K2]. In either case, the reason for our success is very subtle, or rather miraculous; in the steepest descent method approach two integrals cancel out, and in the approach of reducing the problem to that of a second order operator, some parameter determined by $P_{\mathrm{BNR}'}$ kills the relevant Stokes multiplier that is given by [K1] and [K2].

To show the full scope of applicability of the method employed in Section 2, we present in Section 3 a general decomposition theorem for a class of operators that includes $P_{\mathrm{BNR}'}$. The details of the results in Section 3 shall be given elsewhere.

§1. Steepest descent method approach

We begin our discussion by noting that the equation $P_{\mathrm{BNR}'}\psi = 0$ admits an integral representation of solutions; that is,

$$\psi = \int_{\gamma} \zeta^{-\frac{3}{2}} \exp(\eta(z\zeta - (\frac{3}{4\zeta} + \frac{i}{8\zeta^2})))d\zeta \tag{1.1}$$

is a solution of the equation

$$(4\eta^{-3} z \frac{\partial^3}{\partial z^3} + 6\eta^{-3}\frac{\partial^2}{\partial z^2} + 3\eta^{-1}\frac{\partial}{\partial z} + i)\psi = 0 \tag{1.2}$$

for a properly chosen contour γ. The saddle point $\zeta_j(z)$ ($j = 1, 2, 3$) is determined by the equation

$$z + \frac{3}{4\zeta^2} + \frac{i}{4\zeta^3} = 0, \tag{1.3}$$

which coincides with the characteristic equation of (1.2). Hence a WKB solution of $P_{\mathrm{BNR}'}\psi = 0$ is obtained if we choose γ to be the steepest descent path γ_j that passes through the saddle point $\zeta_j(z)$. Then, as is well known ([U]), some topological change of the configuration of steepest descent paths passing through saddle points is a necessary condition for the occurrence of Stokes phenomena of WKB solutions. Hence we trace with the help of a computer the topological change of configurations when z moves around the point C in Fig. 0.2. We choose 20 points, $\rho_1, \rho_2, \cdots, \rho_{20}, \rho_{21} = \rho_1$, near $z = C$ as is designated just by a number j in Fig. 1.1 below.

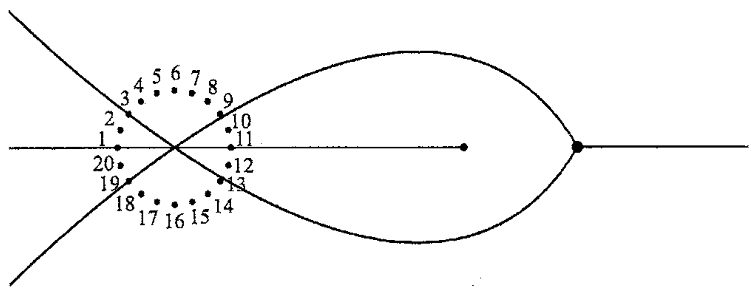


Fig. 1.1

The configuration of steepest descent paths at $z = \rho_j$ with $\arg \eta = 0$ is given in Fig. 1.1.j below; the tiny dot indicates $\zeta = 0$ and other dots correspond to saddle points $\zeta_l(\rho_j)$ $(l = 1, 2, 3)$.

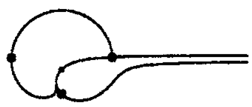


Fig. 1.1.1

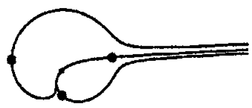


Fig. 1.1.2

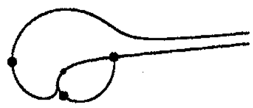


Fig. 1.1.3

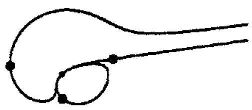


Fig. 1.1.4

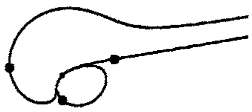


Fig. 1.1.5

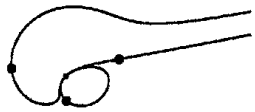


Fig. 1.1.6

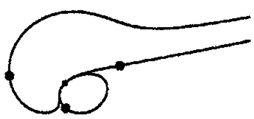


Fig. 1.1.7

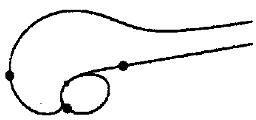


Fig. 1.1.8

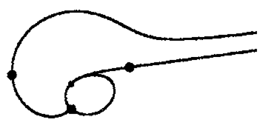


Fig. 1.1.9

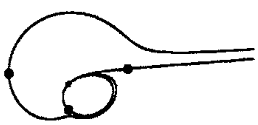


Fig. 1.1.10

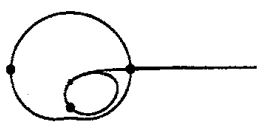


Fig. 1.1.11

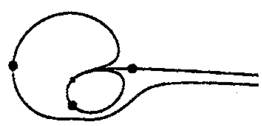
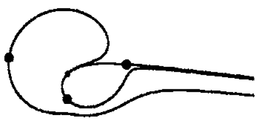


Fig. 1.1.12



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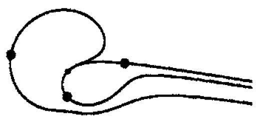

Fig. 1.1.14

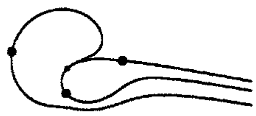

Fig. 1.1.15

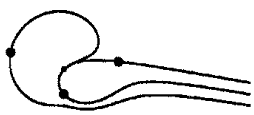

Fig. 1.1.16

Fig. 1.1.17

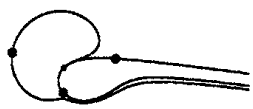

Fig. 1.1.18

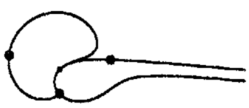

Fig. 1.1.19

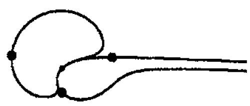

Fig. 1.1.20

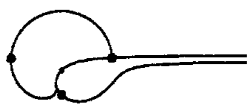

Fig. 1.1.21

The reader will notice some topological changes at $z = \rho_1 \, (= \rho_{21}), \rho_3, \rho_9, \rho_{13}$ and ρ_{19}, as is expected. And, one notices a topological change also at $z = \rho_{11}$! Thus one might think that there should occur some Stokes phenomena across the segment CO in Fig. 0.2. But, if one carefully compare the configurations of steepest descent paths in Fig. 1.1.10 and those in Fig. 1.1.12, one finds that in the course of analytic continuation from $z = \rho_{10}$ to $z = \rho_{12}$ the integral I_1 along the steepest descent path passing through the saddle point $\zeta_1(z)$ acquires the integral I_2 along the steepest descent path γ_2 passing through the saddle point $\zeta_2(z)$, and at the same time loses another integral $\widetilde{I}_2$ along the same path γ_2 with the opposite orientation. One might then conclude the net contribution to I_1 is $I_2 + I_2$; the conclusion is erroneous, because the branch of the integrand of I_2 and that of $\widetilde{I}_2$ are different due to the factor $\zeta^{-3/2}$ in the integrand of (1.1). Then the net contribution to I_1 is $I_2 - I_2 = 0$!

This implies that the segment CO is inert, validating our belief.

Remark 1.1. The above reasoning tells us that, if we start with the following operator $\widetilde{P}$ instead of $P_{\mathrm{BNR}'}$,

$$\widetilde{P} = 4\eta^{-3} z \frac{\partial^3}{\partial z^3} + a\eta^{-3} \frac{\partial^2}{\partial z^2} + 3\eta^{-1} \frac{\partial}{\partial z} + i, \tag{1.4}$$

where a is a complex number, the inert character of the segment CO is observed only when

$$a \equiv 2 \mod 4. \tag{1.5}$$

We will encounter this condition again in the next section. (Cf. Remark 2.1.)

§2. Reduction to a second order operator

Let P denote the operator $\eta^3 (4z)^{-1} P_{\mathrm{BNR}'}$; that is,

$$P = \frac{\partial^3}{\partial z^3} + \eta^2 \frac{3}{4z} \frac{\partial}{\partial z} + \frac{i}{4z} \eta^3 + \frac{3}{2z} \frac{\partial^2}{\partial z^2}. \tag{2.1}$$

Then we can find pre-Borel summable series

$$q(z, \eta) = q_0(z) + \eta^{-1} q_1(z) + \cdots \tag{2.2}$$

and

$$a_j(z, \eta) = a_{j,0}(z) + \eta^{-1} a_{j,1}(z) + \cdots \tag{2.3}$$

with $j = 1, 2$ on a punctured disc $V \backslash \{0\}$ for some open neighborhood V of $z = 0$ so that the following conditions (2.4) $\sim$ (2.8) are satisfied:

$$P = (\frac{\partial}{\partial z} - \eta q(z, \eta))(\frac{\partial^2}{\partial z^2} + \eta a_1(z, \eta) \frac{\partial}{\partial z} + \eta^2 a_2(z, \eta)), \tag{2.4}$$

(2.5) $q_j(z)$ $(j \neq 1)$ is holomorphic on V,

(2.6) $q_1(z)$ has a simple pole at $z = 0$ with residue -1,

(2.7) $a_{1,k}(z)$ $(k \neq 1)$ is holomorphic on V,

(2.8) $z a_{1,1}$ and $z a_{2,k}$ $(k \geq 0)$ are holomorphic on V.

The proof of this decomposition result is a straightforward one, and we omit it here. Since only two saddle points $\zeta_1(z)$ and $\zeta_2(z)$ are relevant to the possible Stokes phenomena near $z = 0$, the decomposition (2.4) enables us to reduce the connection problem for the operator P to that of the second order equation

$$(\frac{\partial^2}{\partial z^2} + \eta a_1(z,\eta)\frac{\partial}{\partial z} + \eta^2 a_2(z,\eta))\psi = 0. \tag{2.9}$$

After the change of the unknown function ψ to

$$\varphi = \exp(\frac{1}{2}\eta \int^z a_1(z,\eta)dz)\psi, \tag{2.10}$$

we can use the theory of formal coordinate transformation (cf. [KT, §2.3]) to reduce the problem to the connection problem for a second order operator with simple poles in the sense of [K2]. Then the results in [K1] and [K2] assert that the Stokes multiplier along the segment CO in Fig. 0.2 is given by

$$2i\cos(\pi\sqrt{1+4\lambda}) \tag{2.11}$$

with

$$\lambda = \frac{c^2 - 2c}{4}, \tag{2.12}$$

where $c = \mathrm{Res}_{z=0}\, a_{1,1}$. (See [K1] and [K2] for the precise statement concerning the dominance relations of WKB solutions.)

Now, it immediately follows from (2.4) and (2.6) that

$$\operatorname*{Res}_{z=0} a_{1,1} = \frac{1}{2}. \tag{2.13}$$

Hence we see that the Stokes multiplier (2.11) vanishes. Thus we have again confirmed that the segment CO in Fig. 0.2 is inert!

Remark 2.1. If we start with the operator $\tilde{P}$ containing a parameter a, instead of $P_{\mathrm{BNR}'}$, as in Remark 1.1, we can easily check

$$\operatorname*{Res}_{z=0} a_{1,1} = \frac{a}{4} - 1. \tag{2.14}$$

Then the relevant Stokes multiplier (2.11) vanishes if

$$(\frac{a}{4} - 1) - 1 = \frac{l}{2}; \quad l\text{: odd}. \tag{2.15}$$

This is exactly the same as (1.5).

§3. A decomposition theorem for operators with simple poles in their coefficients

In this section we introduce a class (K) of operators with simple poles in their coefficients and then present a decomposition theorem for operators in class (K). Although we do not give the proof of the theorem in this report, we explain its background in a heuristic manner. As our discussion in this section is not of immediate relevance to P_{BNR}, but rather of a general character, we use the variable x, not z.

Definition 3.1. Let V be an open neighborhood of the origin of $\mathbb{C}_x$, and let $A_{j,k}(x)$ $(j = 1, 2, \cdots, m(\geq 2); k = 1, 2, \cdots)$ be a meromorphic function on V having a pole at $x = 0$. Assume further that

$$A_j(x,\eta) = \sum_{k\geq 0} A_{j,k}(x)\eta^{-k} \tag{3.1}$$

is pre-Borel summable on $V\backslash\{0\}$. Then the operator P given by

$$\frac{\partial^m}{\partial x^m} + \eta A_1(x,\eta)\frac{\partial^{m-1}}{\partial x^{m-1}} + \cdots + \eta^m A_m(x,\eta) \tag{3.2}$$

is in class (K) if the following conditions (3.3), (3.4) and (3.5) are satisfied:

(3.3) $A_{1,k}$ $(k \neq 1)$ is holomorphic on V,

(3.4) $xA_{1,1}$ and $xA_{j,k}$ $(2 \leq j \leq m, k \geq 0)$ are holomorphic on V,

(3.5) Letting α_j $(2 \leq j \leq m)$ denote $\operatorname*{Res}_{x=0} A_{j,0}$, we find

(3.5.i) $\alpha_2 \neq 0$,

(3.5.ii) $\alpha_m \neq 0$,

(3.5.iii) The equation $\sum_{j=2}^{m} \alpha_j \xi^{m-j} = 0$ has mutually distinct $(m-2)$ solutions.

An important property of the class (K) is that the class is stable under the decomposition of the form (2.4); to be more precise we have the following

Theorem 3.1. *Let*

$$P = \frac{\partial^m}{\partial x^m} + \eta A_1(x,\eta)\frac{\partial^{m-1}}{\partial x^{m-1}} + \cdots + \eta^m A_m(x,\eta) \tag{3.6}$$

be an operator in class (K) *on a neighborhood* V *of the origin of* $\mathbb{C}_x$. *Assume that* $m \geq 3$. *Then we can find an open neighborhood* W *of the origin, a pre-Borel summable series*

$$q(x,\eta) = q_0(x) + \eta^{-1} q_1(x) + \cdots \tag{3.7}$$

on $V \backslash \{0\}$, *and an operator* R *of order* $(m-1)$ *in class* (K) *on* W *so that they satisfy the following:*

$$P = (\frac{\partial}{\partial x} - \eta q(x,\eta))R, \tag{3.8}$$

(3.9) q_j $(j \neq 1)$ *is holomorphic on* W,

(3.10) xq_1 *is holomorphic on* W *and the residue of* q_1 *at* $x = 0$ *is* -1.

Once this theorem is obtained, we can repeatedly use it to find the following decomposition:

$$P = (\frac{\partial}{\partial x} - \eta q^{(1)})(\frac{\partial}{\partial x} - \eta q^{(2)}) \cdots (\frac{\partial}{\partial x} - \eta q^{(m-2)})R, \tag{3.11}$$

where $q^{(j)}$ $(j = 1, \ldots, m-2)$ is a pre-Borel summable series satisfying (3.9) and (3.10) and R is a second order operator in class (K); thus we can reduce the connection problem for the operator P to the connection problem for the second order operator R, which was essentially discussed in [K1] and [K2].

In order to explain the intuitive meaning of Theorem 3.1, we prepare the following

Remark 3.1. If we introduce another class $(\widetilde{K})$ of operators by replacing (3.3) with

$$A_{1,0} \text{ and } xA_{1,k} \ (k \geq 1) \text{ are holomorphic on } V, \tag{3.12}$$

Theorem 3.1 remains to hold for the class $(\widetilde{K})$ instead of (K). Let us further introduce another class $(\widetilde{\widetilde{K}})$ of operators as follows: An operator $\widetilde{P}$ is in class $(\widetilde{\widetilde{K}})$ if it has the form

$$x\frac{\partial^m}{\partial x^m} + \eta \widetilde{A}_1(x,\eta)\frac{\partial^{m-1}}{\partial x^{m-1}} + \cdots + \eta^m \widetilde{A}_m(x,\eta), \tag{3.13}$$

where $\widetilde{A}_j(x,\eta) = \sum_{k\geq 0} \widetilde{A}_{j,k}(x)\eta^{-k}$ is a pre-Borel summable series on V that satisfies the following conditions:

$$\widetilde{A}_{j,k} \ (1 \leq j \leq m, k \geq 0) \text{ is holomorphic on } V, \tag{3.14}$$

$$\widetilde{A}_{1,0}(0) = 0. \tag{3.15}$$

In what follows, with some abuse of languages, we say that the series $\widetilde{A}_j(x,\eta)$ is holomorphic on V when the condition (3.14) is satisfied.

With these definitions, $(\widetilde{K})$ and $(\widetilde{\widetilde{K}})$ are isomorphic by the correspondence $\widetilde{P} = xP$. It is also clear that the class $(\widetilde{\widetilde{K}})$ is closed under the operation of considering the adjoint operator: For an operator $\widetilde{P}$ in $(\widetilde{\widetilde{K}})$ we can find an operator Q in $(\widetilde{\widetilde{K}})$ such that $(-1)^m\widetilde{P} = Q^*$, the adjoint operator of Q. It is then clear from the above mentioned correspondence between operators in $(\widetilde{K})$ and those in $(\widetilde{\widetilde{K}})$ that $(-1)^m x^{-1}Q^*$ is in $(\widetilde{K})$. Then the modified version of Theorem 3.1 that adopts $(\widetilde{K})$ instead of (K) guarantees that

$$x^{-1}Q^* = (\frac{\partial}{\partial x} - \eta q)R \tag{3.16}$$

holds for some R in $\widetilde{K}$ with $q(x,\eta)$ satisfying (3.9) and (3.10). Hence we find

$$\begin{aligned} Q &= R^*(x\frac{\partial}{\partial x} - \eta xq)^* \\ &= R^*(-\frac{\partial}{\partial x}x \cdot -\eta xq) \\ &= R^*(-x\frac{\partial}{\partial x} - 1 - \eta xq). \end{aligned} \tag{3.17}$$

Letting $\tilde{q}$ denote $q + \eta^{-1}x^{-1}$, we see from (3.10) that $\tilde{q}(x,\eta)$ is holomorphic on W. Otherwise stated, the equation $Q\psi = 0$ admits a WKB solution ψ of the form

$$\exp(-\eta \int^x \tilde{q}(x,\eta)dx) \tag{3.18}$$

with $\tilde{q}$ holomorphic on W. Hence the somewhat clumsy condition (3.10) corresponds to the existence of a holomorphic WKB solution to the equation $Q\psi = 0$. But, the existence of a holomorphic WKB solution to the equation that has the form (3.18) is formally obvious because q_0 is holomorphic. Thus (3.17), and hence (3.16) also, may be understood as a consequence of existence of holomorphic WKB solutions.

Reversing the reasoning in the above Remark 3.1, we can give an intuitive and WKB-theoretic interpretation of the seemingly curious condition (3.10):

For an operator P in $(\widetilde{K})$, we consider an operator Q in $(\widetilde{\widetilde{K}})$ defined by $(xP)^*$. Then this operator is divisible from the right by the factor $(\partial/\partial x + \eta\tilde{q})$ for holomorphic $\tilde{q}$, reflecting the fact that the equation $Q\psi = 0$ admits a WKB solution of the form $\exp(-\eta \int^x \tilde{q}\, dx)$. Although some technical care is needed to justify the relation $(R^* x)^* = xR$ for an operator R in $(\widetilde{K})$, we can "prove without computation" the modified form of Theorem 3.1 by choosing $q = \tilde{q} - \eta^{-1}x^{-1}$. Further we note that the factor $(\partial/\partial x - \eta q)$ with $\operatorname{Res} q_1 = -1$ is a counterpart of the existence of a holomorphic WKB solution to the adjoint equation.

Remark 3.2. Theorem 3.1 applies to the equation (1.2) in [AKT], which plays a central role in [AKT]; this means that the reasoning near the point b_1 ([AKT, p.636 and p.637]) may be replaced by the reasoning similar to that given in Section 2. Actually the reader will notice the resemblance of the discussion in [AKT, p.636 and p.637] with that given in Section 1 of this report.

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