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Maillet Type Theorem for Singular First Order Nonlinear Partial Differential Equations of Totally Characteristic Type

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1 Introduction.

Let $(t,x) \in \mathbf{C}_t^d \times \mathbf{C}_x^n$, $\mathbf{N} = \{0,1,2,\ldots\}$ and $\mathbf{N}_+ = \{1,2,\ldots\}$. We consider the following first order nonlinear partial differential equation:

$$(1.1) \qquad \begin{cases} f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x)) = 0, \\ u(0,x) \equiv 0, \end{cases}$$

where $\partial_t u = (\partial_{t_1} u, \ldots, \partial_{t_d} u)$, $\partial_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$.

In this paper, we always assume the following assumptions:

(H1) $f(t,x,u,\tau,\xi)$ $(\tau = (\tau_j) \in \mathbf{C}^d,\ \xi = (\xi_k) \in \mathbf{C}^n)$ is holomorphic in a neighbourhood of the origin, and is an entire function in τ variables for any fixed t, x, u and ξ.

(H2) **(Singular Equation)** The holomorphic function $f(t,x,u,\tau,\xi)$ satisfies

$$(1.2) \qquad f(0,x,0,\tau,0) \equiv 0$$

for $x \in \mathbf{C}^n$ near the origin and $\tau \in \mathbf{C}^d$.

(H3) **(Existence of Formal Solution)** The equation (1.1) has a formal solution of the form

$$(1.3) \qquad u(t,x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|\alpha|\geq 2, |\beta|\geq 0} u_{\alpha\beta} t^\alpha x^\beta \in \mathbf{C}[[t,x]],$$

where $|\alpha|$ and $|\beta|$ denote the sum of multi-indices $\alpha \in \mathbf{N}^d$ and $\beta \in \mathbf{N}^n$, respectively. Moreover we assume that all $\varphi_j(x)$ are holomorphic in a neighborhood of the origin.

We would like to consider the equation (1.1) under the condition which is called *Totally characteristic type*, that is, we assume

(H4) (Totally Characteristic Type) The equation (1.1) is totally characteristic type, which is defined by

$$\left\{\begin{array}{l} f_{\xi_k}(0,x,0,\{\varphi_j(x)\},0) \not\equiv 0 \\ f_{\xi_k}(0,0,0,\{\varphi_j(0)\},0) = 0 \end{array}\right. \quad \text{for } k=1,\dots,n. \tag{1.4}$$

Remark 1.1 By the above assumptions, $\{\varphi_j(x)\}$ satisfy the following system of equations:

$$\begin{aligned} &\frac{\partial}{\partial t_i} f(t,x,u(t,x),\{\partial_{t_j}u(t,x)\},\{\partial_{x_k}u(t,x)\})\Big|_{t=0} \\ &= \frac{\partial f}{\partial t_i}(0,x,0,\{\varphi_j(x)\},0) + \frac{\partial f}{\partial u}(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x) \\ &+ \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0,x,0,\{\varphi_j(x)\},0)\frac{\partial \varphi_i}{\partial x_k}(x) = 0, \end{aligned} \tag{1.5}$$

with $f_{\xi_k}(0,0,0,\{\varphi_j(0)\},0) = 0$ for $k = 1,2,\dots,d$. The formal solution of this system is not convergent in general, but we have a sufficient condition for the formal solution of system (1.5) to be convergent, which is found in [S2, Section 6]. By this reason we assumed the convergence of $\varphi_j(x)$ in (H3).

Now we put $\mathbf{a}(x) = (0,x,0,\{\varphi_j(x)\},0)$ for simplicity, and define

$$a_{ij}(x) := \frac{\partial^2 f}{\partial t_i \partial \tau_j}(\mathbf{a}(x)) + \frac{\partial^2 f}{\partial u \partial \tau_j}(\mathbf{a}(x))\varphi_i(x) + \sum_{k=1}^{n} \frac{\partial^2 f}{\partial \tau_j \partial \xi_k}(\mathbf{a}(x))\frac{\partial \varphi_i}{\partial x_k}(x), \tag{1.6}$$

for $i,j = 1,2,\dots,d$. Moreover we define

$$b_k(x) := \frac{\partial f}{\partial \xi_k}(\mathbf{a}(x)), \quad \text{for } k = 1,2,\dots,n. \tag{1.7}$$

We remark that the functions $a_{ij}(x)$ and $b_k(x)$ are determined as holomorphic functions in a neighborhood of the origin and $b_k(x) \not\equiv 0$, $b_k(0) = 0$ by the assumptions.

Remark 1.2 If $b_k(x) \equiv 0$ for all $k = 1,2,\dots,n$, the equation (1.1) is called *the Fuchsian type.* In this case, the convergence or divergence criterion is obtained by Gérard-Tahara in [GT] or Miyake-Shirai in the forthcoming paper [MS2].

We put $v(t,x) = u(t,x) - \sum_{j=1}^{d} \varphi_j(x) t_j = O(|x|^K), (K \geq 2)$. Then $v(t,x)$ satisfies the following equation:

$$(1.8)\quad \begin{cases} \left(\displaystyle\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} + \sum_{k=1}^{n} b_k(x) \partial_{x_k} + \frac{\partial f}{\partial u}(\mathbf{a}(x)) \right) v(t,x) \\ \qquad\qquad = \displaystyle\sum_{|\alpha|=K} d_\alpha(x) t^\alpha + f_{K+1}(t, x, v(t,x), \partial_t v(t,x), \partial_x v(t,x)), \\ v(t,x) = O(|t|^K), \end{cases}$$

where $d_\alpha(x)$ are holomorphic in a neighborhood of the origin, and the function $f_{K+1}(t,x,v,\tau,\xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$(1.9)\qquad f_{K+1}(t,x,v,\tau,\xi) = \sum_{V(\alpha,p,q,r) \geq K+1} f_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r,$$

where $p \in \mathbf{N}$, $q \in \mathbf{N}^d$ and $r \in \mathbf{N}^n$ and

$$(1.10)\qquad V(\alpha,p,q,r) = |\alpha| + Kp + (K-1)|q| + K|r|.$$

The problem in this paper is to obtain the Gevrey order of formal solution of (1.1) or (1.8) in the case where all the eigenvalues of the Jacobi matrix of $\{b_k(x)\}$ at $x = 0$ are equal to zero. For this problem, Chen-Luo-Tahara [CLT] obtained the answer in the case where $(t,x) \in \mathbf{C}_t \times \mathbf{C}_x$, and they proved that their Gevrey order is the best constant in general. The purpose of this paper is to give a generalization of Chen-Luo-Tahara's result to the case of several (t,x) variables.

We put the Jordan canonical form of the Jacobi matrix of $\{b_1(x), \ldots, b_n(x)\}$ at $x = 0$ by

$$(1.11)\qquad \left.\frac{\partial(b_1, \ldots, b_n)}{\partial(x_1, \ldots, x_n)}\right|_{x=0} \sim \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_I \end{pmatrix}$$

where

$$N_j = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

denotes the nilpotent matrix block of size k_j.

The following theorem is the main theorem in this paper.

Theorem 1.1 *Suppose* (H1), (H2), (H3) *and* (H4). *If the following condition*

$$(1.12) \quad \left|\sum_{j=1}^{d}\lambda_j\alpha_j+\frac{\partial f}{\partial u}(\mathbf{a}(0))\right|\geq C|\alpha|, \quad (\textit{Non-resonance Poincaré condition})$$

holds by some positive constant $C>0$ *for all* $|\alpha|\geq K$, *then the formal solution* (1.3) *of the equation* (1.1) *belongs to the formal Gevrey class of order at most* 2σ *by*

$$(1.13) \qquad \sigma=\begin{cases}\max\{k_1,k_2,\ldots,k_I\} & (\textit{if } I<n)\\ \dfrac{p}{2(p-1)} & (\textit{if } I=n),\end{cases}$$

where $p=\min_{k=1,2,\ldots,n}\{m_k\geq 2\ ;\ b_k(x)=O(|x|^{m_k})\}$. *Namely, the power series*

$$\sum_{|\alpha|\geq 1,|\beta|\geq 0}\frac{u_{\alpha\beta}}{(|\alpha|+|\beta|)!^{2\sigma-1}}t^\alpha x^\beta$$

converges in a neighborhood of the origin.

2 Refinement of Theorem 1.1

After a linear change of independent variables which reduces the matrices $(a_{ij}(0))$ and $\frac{\partial(b_1,\ldots,b_n)}{\partial(x_1,\ldots,x_n)}(0)$ to the Jordan canonical forms, we can obtain more precise estimates of the Gevrey order in each variable. In order to state the result, we prepare some notation and definitions.

Definition 2.1 (s-Borel transformation) Let $\mathbf{s}=(\mathbf{s}',\widetilde{\mathbf{s}})\in(\mathbf{R}_{\geq 1})^d\times(\mathbf{R}_{\geq 1})^n$ where $\mathbf{R}_{\geq 1}=\{x\in\mathbf{R}\ ;\ x\geq 1\}$. The s-Borel transformation $\mathcal{B}^{\mathbf{s}}_{t,x}(f)(t,x)$ of $f(t,x)=\sum_{|\alpha|+|\beta|\geq 0}f_{\alpha\beta}t^\alpha x^\beta$ is defined by

$$(2.1) \qquad \mathcal{B}^{\mathbf{s}}_{t,x}(f)(t,x)=\sum_{|\alpha|+|\beta|\geq 0}f_{\alpha\beta}\frac{|\alpha|!|\beta|!}{(\mathbf{s}'\cdot\alpha)!(\widetilde{\mathbf{s}}\cdot\beta)!}t^\alpha x^\beta,$$

where $(\mathbf{s}'\cdot\alpha)!$ and $(\widetilde{\mathbf{s}}\cdot\beta)!$ denote the Gamma functions $\Gamma(\mathbf{s}'\cdot\alpha+1)$ and $\Gamma(\widetilde{\mathbf{s}}\cdot\beta+1)$, respectively.

Definition 2.2 (Gevrey class $\mathcal{G}^{\mathbf{s}}_{t,x}$) We say that $f(t,x)=\sum_{\alpha\in\mathbf{N}^d,\beta\in\mathbf{N}^n}f_{\alpha\beta}t^\alpha x^\beta\in\mathcal{G}^{\mathbf{s}}_{t,x}$, if the s-Borel transformation $\mathcal{B}^{\mathbf{s}}_{t,x}(f)(t,x)$ converges in a neighborhood of the origin, and s is called the Gevrey order.

Remark 2.1 (i) If two Gevrey orders $\mathbf{s} = \{s_j\}$ and $\bar{\mathbf{s}} = \{\bar{s_j}\}$ satisfy $s_j \leq \bar{s_j}$ for all $j = 1, 2, \ldots, d+n$, then $\mathcal{G}^{\mathbf{s}}_{t,x} \subset \mathcal{G}^{\bar{\mathbf{s}}}_{t,x}$.

(ii) If $\mathbf{s} = (s, s, \ldots, s) \in (\mathbf{R}_{\geq 1})^n$, then $f(t,x) \in \mathcal{G}^{\mathbf{s}}_{t,x}$ if and only if

$$\sum \frac{f_{\alpha\beta}}{(|\alpha| + |\beta|)!^{s-1}} t^\alpha x^\beta$$

converges in a neighborhood of the origin.

(iii) For a formal power series $u(t,x) \in \mathbf{C}[[t,x]]$, if $\mathcal{B}^{\mathbf{s}}_{t,x}(u)(t,x) \in \mathcal{G}^{\widehat{\mathbf{s}}}_{t,x}$, then we have $u(t,x) \in \mathcal{G}^{\mathbf{s}+\widehat{\mathbf{s}}-\mathbf{1}_{d+n}}_{t,x}$ with $\mathbf{1}_{d+n} = (1, 1, \ldots, 1) \in \mathbf{N}^{d+n}$.

Let us give a refined form of Theorem 1.1. By a linear change of independent variables which brings the matrices $(a_{ij}(0))$ and $\frac{\partial(b_1,\ldots,b_n)}{\partial(x_1,\ldots,x_n)}|_{x=0}$ to the Jordan canonical forms, the equation (1.8) is reduced to the following form:

$$(2.2)\quad (\Lambda + \Delta + N)u = \sum_{i,j=1}^{d} \alpha_{ij}(x) t_i \partial_{t_j} u + \sum_{j=1}^{I} \sum_{k=1}^{k_j} \beta_{jk}(x) \partial_{x_{j,k}} u + \eta(x) u + \sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, u, \partial_t u, \partial_x u)$$

where $u = u(t,x)$, $(t,x) = (t, \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I) \in \mathbf{C}^d \times \mathbf{C}^{k_1} \times \mathbf{C}^{k_2} \times \cdots \times \mathbf{C}^{k_I}$, $\mathbf{x}^j = \{x_{j,k}\}_{k=1,2,\ldots,k_j} \in \mathbf{C}^{k_j}$, and Λ, Δ and N denote the following linear operators:

$$(2.3)\quad \Lambda = \sum_{j=1}^{d} \lambda_j t_j \partial_{t_j} + c(0), \qquad \Delta = \sum_{j=1}^{d-1} \delta_j t_{j+1} \partial_{t_j},$$

$$(2.4)\quad N = \sum_{j=1}^{I} \sum_{k=1}^{k_j - 1} \delta x_{j,k+1} \partial_{x_{j,k}}.$$

The coefficients $\alpha_{ij}(x)$, $\beta_{jk}(x)$, $\gamma_k(x)$, $\eta(x)$ and $\zeta_\alpha(x)$ are holomorphic in a neighborhood of the origin satisfying

$$(2.5)\quad \alpha_{ij}(x),\ \eta(x) = O(|x|), \qquad \beta_{jk}(x) = O(|x|^2).$$

and $g_{K+1}(t, x, u, \tau, \xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$g_{K+1}(t, x, u, \tau, \xi) = \sum_{V(\alpha,p,q,r) \geq K+1} g_{\alpha pqr}(x) t^\alpha u^p \tau^q \xi^r.$$

Remark 2.2 By the linear change of variables as above, we have $\delta_j = 0$ or 1 and $\delta = 1$, which appeared in the operators Δ and N. However, we may assume that the constants δ_j and δ are as small as we want. Indeed, we introduce new independent variables $\widehat{t}_j = \varepsilon^j t_j$ and $\widehat{x}_{j,k} = \varepsilon^{k_1+\cdots+k_{j-1}+k} x_{j,k}$. Then δ_j and δ are changed by $\varepsilon\delta_j$ and $\varepsilon\delta$, respectively. Therefore, by choosing $\varepsilon > 0$ small enough, we may assume that δ_j and δ are arbitrary small.

In order to state more precise result for Theorem 1.1, we put the valuation of $\beta_{jk}(y, z)$ by

$$\beta_{jk}(x) = O(|x|^{\ell_{jk}}), \quad (\ell_{jk} \geq 2).$$

For a vector $\mathbf{p} = (p_1, \ldots, p_h)$ and a constant a, the notaion $\mathbf{p}(a)$ is defined by

$$\mathbf{p}(a) = (p_1 + a, p_2 + a, \ldots, p_h + a).$$

Theorem 1.1 is obtained immediately from the following:

Theorem 2.1 *Let $K_0 = \max\{k_1, \ldots, k_I\}$. The equation (2.2) has a unique formal solution which belongs to the Gevrey class $\mathcal{G}^{\mathbf{s}}_{t,x}$ of order at most* $\mathbf{s}$ *by*

$$\mathbf{s} = (\mathbf{1}_d(\sigma_1), \mathbf{s}^1(\sigma_2), \ldots, \mathbf{s}^I(\sigma_2)), \tag{2.6}$$

where $\mathbf{1}_d = (1, 1, \ldots, 1) \in \mathbf{N}^d$ and

$$\widehat{\mathbf{s}} = (\mathbf{1}_d, \mathbf{s}^1, \ldots, \mathbf{s}^I), \qquad \mathbf{s}^j = (1, 2, \ldots, k_j), \tag{2.7}$$

$$\sigma_1 = \max_{\alpha,p,q,r} \left\{ \frac{\sigma_2 + K_0 - 1}{V(\alpha, p, q, r) - K} \ ; \ g_{\alpha pqr}(x) \not\equiv 0 \right\}, \tag{2.8}$$

$$\sigma_2 = \max_{\substack{j=1,2,\ldots,I \\ k=1,\ldots,k_j}} \left\{ \frac{k}{\ell_{jk} - 1} \right\}. \tag{2.9}$$

Proof of Theorem 1.1. If $K_0 = 1$, then we have $\sigma_2 \leq 1/(p-1)$ and $\sigma_1 \leq 1/(p-1)$ where p is the constant which appeared in Theorem 1.1. In the case $K_0 \geq 2$, we have $\sigma_2 \leq K_0$ and $\sigma_1 \leq 2K_0 - 1$. Therefore, the maximal component $||\mathbf{s}||$ of $\mathbf{s}$ which appeared in (2.6) is estimated by

$$||\mathbf{s}|| \leq \begin{cases} 2K_0 & (\text{if } K_0 \geq 2) \\ \dfrac{p}{p-1} & (\text{if } K_0 = 1) \end{cases} = 2\sigma.$$

Thus, Theorem 1.1 is proved. ∎

3 Sketch of the Proof of Theorem 2.1.

In this section, we shall prove Theorem 2.1 by assuming lemmas which are tools to prove Theorem 2.1.

3.1 Construction of Majorant equations.

First we give the following lemma:

Lemma 3.1 *Let $P = \Lambda + \Delta + N$. Then the following propositions hold:*

(i) $P : \mathbf{C}[t]_L[x]_M \longrightarrow \mathbf{C}[t]_L[x]_M$ *is invertible for all* $L \geq K$ *and* $M \geq 0$.

(ii) *For* $\widehat{\mathbf{s}} = (\mathbf{1}_d, \mathbf{s}^1, \dots, \mathbf{s}^I) = (\mathbf{1}_d, \widetilde{\mathbf{s}}) \in (\mathbf{R}_{\geq 1})^d \times (\mathbf{R}_{\geq 1})^{k_1+\cdots+k_I}$, *if a majorant relation* $\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(u)(t,x) \ll W_{LM}T^L X^M$ ($T = |t|$, $X = |x|$) *does hold, then there exists a positive constants* $C_0 > 0$ *independent of* L *and* M *such that*

$$(3.1) \qquad \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(P^{-1}u)(t,x) \ll \frac{C_0}{L} W_{LM}T^L X^M = C_0(TD_T)^{-1} W_{LM} T^L X^M.$$

We put $Pv(t,x) = U(t,x)$ as a new unknown function. Then the equation (2.5) is reduced to the following:

$$(3.2)\ U(t,x) = \sum_{i,j=1}^{d} \alpha_{ij}(x) t_i \partial_{t_j} P^{-1}U + \sum_{j=1}^{I}\sum_{k=1}^{k_j} \beta_{jk}(x)\partial_{x_{j,k}} P^{-1}U + \eta(x)P^{-1}U + \sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U).$$

We apply the $\widehat{\mathbf{s}}$-Borel transformation for (3.2), where $\widehat{\mathbf{s}}$ is the same vector which appeared in Lemma 3.1 (ii).

$$\begin{aligned}\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(U)(t,x) &= \sum_{i,j=1}^{d} \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\alpha_{ij}(x) t_i \partial_{t_j} P^{-1}U\} + \sum_{j=1}^{I}\sum_{k=1}^{k_j} \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\beta_{jk}(x)\partial_{x_{j,k}} P^{-1}U\} \\ &\quad + \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\eta(x)P^{-1}U\} + \sum_{|\alpha|=K} \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\zeta_\alpha(x)t^\alpha\} \\ &\quad + \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_y P^{-1}U)\}.\end{aligned}$$

In order to construct a majorant equation of (3.3), we prepare a lemma.

Lemma 3.2 *Let $u(t,x)$, $v(t,x)$ be formal power series, and the multi-index $\widehat{\mathbf{s}} = (\mathbf{1}_d, \mathbf{s}^1, \ldots, \mathbf{s}^I)$ be the same constants as (2.7). Then the following majorant relations hold:*

(i) There exists a positive constant $C_1 > 0$ independent of u and v, such that

$$\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(uv)(t,x) \ll C_1 \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(|u|)(t,x) \times \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(|v|)(t,x). \tag{3.3}$$

(ii) Let $T = |t|$ and $X = |x|$. If a majorant relation $\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(u)(t,x) \ll W(T,X)$ $= \sum_{L \geq K, M \geq 0} W_{LM} T^L X^M$ does hold, then there exists a positive constant $C_2 > 0$ which depends only on $\widehat{\mathbf{s}}$, such that

$$\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(\partial_{t_j} P^{-1} u)(t,x) \ll C_2 \partial_T (T\partial_T)^{-1} W(T,X), \tag{3.4}$$

$$\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(\partial_{x_{jk}} P^{-1} u)(t,x) \ll C_2 (T\partial_T)^{-1} \partial_X (X\partial_X)^{k-1} W(T,X), \tag{3.5}$$

By Lemmas 3.1 and 3.2, if $\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}(u)(t,x) \ll W(T,X)$, then we have the following majorant relations by the positive constant $C_3 = C_0 C_1 C_2$:

- $\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\alpha_{ij}(x) t_i \partial_{t_j} P^{-1} U\} \ll C_3 |\alpha_{ij}|(\mathbf{X}) W(T,X)$,
- $$\begin{aligned}\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\beta_{jk}(x) \partial_{x_{j,k}} P^{-1} U\} &\ll C_3 |\beta_{jk}|(\mathbf{X}) (T\partial_T)^{-1} \partial_X (X\partial_X)^{k-1} W(T,X) \\ &\ll C_3 |\beta_{jk}|(\mathbf{X}) \partial_X (X\partial_X)^{k-1} W(T,X),\end{aligned}$$
- $\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\eta(x) P^{-1} U\} \ll C_3 |\eta|(\mathbf{X}) W(T,X)$,
- $\displaystyle\sum_{|\alpha|=K} \mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{\zeta_\alpha(x) t^\alpha\} \ll \left(\sum_{|\alpha|=K} |\zeta_\alpha|(\mathbf{X})\right) T^K$,
- $\mathcal{B}^{\widehat{\mathbf{s}}}_{t,x}\{g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U)\}$

$$\ll |g_{K+1}|\Big(\mathbf{T}, \mathbf{X}, C_3 W, \Big\{C_3 \partial_T (T\partial_T)^{-1}(W)\Big\}, \Big\{C_3 (T\partial_T)^{-1} \partial_X (X\partial_X)^{k-1} W\Big\}\Big),$$

where $\mathbf{T} = (T, T, \ldots, T) \in \mathbf{C}^d$ and $\mathbf{X} = (X, X, \ldots, X) \in \mathbf{C}^n$.

Here we consider the following equation:

$$
\begin{aligned}
(3.6)\quad W(T,X) &= \left(\sum_{|\alpha|=K} |\zeta_\alpha|(\mathbf{X})\right) T^K + F(X)W(T,X) \\
&+\sum_{j=1}^{I}\sum_{k=1}^{k_j} C_3|\beta_{jk}|(\mathbf{X})\partial_X (X\partial_X)^{k-1}W(T,X) \\
&+|g_{K+1}|\Big(\mathbf{T},\mathbf{X},C_3W,\Big\{C_3\partial_T(T\partial_T)^{-1}W\Big\},\Big\{C_3(T\partial_T)^{-1}\partial_X(X\partial_X)^{k-1}W\Big\}\Big),
\end{aligned}
$$

where $F(X)$ is a holomorphic function given by

$$
F(X)=\sum_{i,j=1}^{d} C_3|\alpha_{ij}|(\mathbf{X}) + C_3|\eta|(\mathbf{X}) = O(X).
$$

By the construction of the equation (3.6), the following majorant relation is clearly holds:

$$
(3.7)\qquad \mathcal{B}^{\hat{\mathbf{s}}}_{t,x}(U)(t,x) \ll W(T,X).
$$

By multiplying $(1-F(X))^{-1}\in \mathbf{C}\{X\}$ on the both hands side of (3.6), the equation (3.6) is rewritten by

$$
\begin{aligned}
(3.8)\quad W &= Z(X)T^K + \sum_{j=1}^{I}\sum_{k=1}^{k_j} B_{jk}(X)(X\partial_X)^k W \\
&+G_{K+1}\Big(T,X,C_3W,\Big\{C_3\partial_T(T\partial_T)^{-1}W\Big\},\Big\{C_3(T\partial_T)^{-1}\partial_X(X\partial_X)^{k-1}W\Big\}\Big),
\end{aligned}
$$

where $W=W(T,X)$ and

$$
Z(X)=\frac{1}{1-F(X)}\sum_{|\alpha|=K}|\zeta_\alpha|(\mathbf{X}),\quad B_{jk}(X)=\frac{C_3}{1-F(X)}\frac{|\beta_{jk}|(\mathbf{X})}{X}=O(X^{\ell_{jk}-1}),
$$

$$
G_{K+1}(T,X,u,\tau,\xi)=\frac{1}{1-F(X)}|g_{K+1}|(\mathbf{T},\mathbf{X},u,\tau,\xi).
$$

Remark 3.1 From now on, we shall prove that the formal solution $W(T,X)$ of (3.8) belongs to the formal Gevrey class $\mathcal{G}^{\sigma_1+1,\sigma_2+1}_{T,X}$, where σ_1 and σ_2 are the constants which appeared in (2.8) and (2.9). If we can prove this fact, then we obtain the consequence of Theorem 2.1. Indeed, by Remark 2.1, Lemma 3.1 (ii) and (3.7), we have $v(t,x)\in\mathcal{G}^{\mathbf{s}}_{t,x}$ where $\mathbf{s}$ is the vector introduced by (2.6).

3.2 The estimate of Gevrey order in X.

By substituting $W(T,X) = \sum_{L\geq K} W_L(X)T^L$ into (3.8), we have the following recursion formulas:

$$(3.9)\qquad W_L(X) = \sum_{j=1}^{I}\sum_{k=1}^{k_j} B_{jk}(X)(X\partial_X)^k W_L(X) \\ +H_L\left(X, \{W_j(X)\}_{j=K,\dots,L-1}, \{\partial_X(X\partial_X)^k W_j(X)\}_{\substack{k=1,2,\dots,K_0\\ j=K,\dots,L-1}}\right),$$

where $H_L(X,\{\xi_j\},\{\eta_{jk}\})$ denotes a polynomial in $\{\xi_j\}$ and $\{\eta_{jk}\}$ variables in the case $L>K$ or $H_K = Z(X)$, and K_0 is a positive integer defined by

$$K_0 = \max\{k_1,\dots,k_I\}.$$

In order to construct majorant recursion formulas of (3.9), we prepare a lemmas.

Lemma 3.3 *Let $\{a_j(x)\}$ ($x\in\mathbf{C}$) be holomorphic functions and the vanishing order of $a_j(x)$ be $m_j\in\mathbf{N}_+$ for $j=1,2,\dots,n$, and $f(x)\in\mathcal{G}_x^{\sigma+1}$ where $\sigma = \max_{j=1,\dots,n}\{j/m_j\}$. Then the formal solution $u(x)$ of linear ordinary differential equation*

$$(3.10)\qquad u(x) = \sum_{j=1}^{n} a_j(x)\left(x\frac{d}{dx}\right)^j u(x) + f(x)$$

belongs to the formal Gevrey class of order $\sigma+1$. Moreover, the formal solution $U(x)$ of linear functional equation

$$(3.11)\qquad U(x) = \sum_{j=1}^{n} |a_j|(x)U(x) + \mathcal{B}_x^{\sigma+1}(|f|)(x)$$

is a majorant function of $\mathcal{B}_x^{\sigma+1}(u)(x)$.

For $s\geq 0$, we define the formal s differential operator $(X\partial_X)^s$ by

$$(3.12)\qquad (X\partial_X)^s(X^M) := M^s X^M,$$

for all $M\in\mathbf{N}$.

By Lemma 3.3, the formal solutions $\{W_L(X)\}_{L\geq K}$ belong to the formal Gevrey class of order σ_2+1 where σ_2 is the same constant as (2.9). Namely, by this observation, we see that the Gevrey order of $W(T,X)$ in X variable is $\sigma_2 + 1$. Furthermore, by the second statemant of Lemma 3.3, the formal solutions $\{V_L(X)\}_{L\geq K}$ of recursion formulas

$$(3.13)\quad V_L(X) = \sum_{j=1}^{I}\sum_{k=1}^{k_j} B_{jk}(X)V_L(X) \\ +H_L\left(X, \{V_j(X)\}_{j=K,\dots,L-1}, \{\partial_X(X\partial_X)^{\sigma_2+k-1}V_j(X)\}_{\substack{k=1,2,\dots,K_0\\ j=K,\dots,L-1}}\right)$$

are the majorant functions of $\{\mathcal{B}_X^{\sigma_2+1}(W_L)(X)\}_{L\geq K}$, that is,

$$(3.14)\qquad \mathcal{B}_X^{\sigma_2+1}(W_L)(X) \ll V_L(X), \qquad \text{for all } L \geq K.$$

Because by Lemma 3.2, we can obtain the following majorant relation:
If $\mathcal{B}_X^{\sigma_2+1}(W_j)(X) \ll V_j(X)$ for $j \geq K$, then

$$\mathcal{B}_X^{\sigma_2+1}\left\{H_L\left(X, \{W_j(X)\}_{j=K,\dots,L-1}, \{\partial_X(X\partial_X)^k W_j(X)\}_{\substack{k=1,2,\dots,N\\ j=K,\dots,L-1}}\right)\right\} \\ \ll H_L\left(X, \{V_j(X)\}_{j=K,\dots,L-1}, \{\partial_X(X\partial_X)^{\sigma_2+k}V_j(X)\}_{\substack{k=1,2,\dots,K_0\\ j=K,\dots,L-1}}\right).$$

Now we consider the following equation:

$$(3.15)\, V(T,X) = Z(X)T^K + \sum_{j=1}^{I}\sum_{k=1}^{k_j} B_{jk}(X)(T\partial_T)^{-1}V(T,X) \\ +G_{K+1}\Big(T, X, C_3V, \Big\{C_3\partial_T(T\partial_T)^{-1}V\Big\}, \Big\{C_3(T\partial_T)^{-1}\partial_X(X\partial_X)^{\sigma_2+k-1}V\Big\}\Big).$$

We put $V(T,X) = \sum_{L\geq K} V_L(X)T^L$. Then the coefficients $\{V_L(X)\}$ are determined as holomorphic functions by the same recursion formulas as (3.13). Therefore, the formal solution $V(T,X)$ is a majorant function of the power series $\sum_{L\geq K}\mathcal{B}_X^{\sigma_2+1}(W_L)(X)T^L$, that is,

$$(3.16)\qquad \sum_{L\geq K}\mathcal{B}_X^{\sigma_2+1}(W_L)(X)T^L \ll V(T,X).$$

3.3 The estimate of Gevrey order in T.

We give the Gevrey order in T variable. We take holomorphic majorant functions $Q(X)$ and $R_{K+1}(T,X,u,\tau,\xi,\eta)$ by

$$Q(X) = \frac{A}{(R-X)^K} \gg P(X)Z(X),$$

and

$$
\begin{aligned}
R_{K+1}(T,X,u,\tau,\xi,\eta) &= \sum_{V(\alpha,p,q,r)\geq K+1} \frac{A_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} T^{|\alpha|} u^p \tau^q \xi^r \\
&\gg P(X) G_{K+1}(T,X,u,\tau,\xi)
\end{aligned}
$$

by some positive constants R, A and $\{A_{\alpha pqr}\}$ where

$$
P(X) = \frac{1}{1 - \sum_{j=1}^{I} \sum_{k=1}^{k_j} B_{jk}(X)}.
$$

By the above majorant relations, we can easily see that the formal solution of the following equation satisfies $Y(T,X) \gg V(T,X)$:

$$
\begin{aligned}
(3.17)\, Y(T,X) &= Q(X)T^K \\
&+ R_{K+1}\Big(T, X, C_3 Y, \Big\{C_3 \partial_T (T\partial_T)^{-1} Y\Big\}, \Big\{C_3 (T\partial_T)^{-1} \partial_X (X\partial_X)^{\sigma_2+k-1} Y\Big\}\Big).
\end{aligned}
$$

Put $y(T,X) = (T\partial_T)^{-1} Y(T,X)$. Then $y(T,X)$ satisfies the following equation:

$$
(3.18) \begin{cases} T\partial_T y(T,X) = Q(X)T^K \\ \qquad + R_{K+1}\Big(T, X, C_3 T\partial_T y, \Big\{C_3 \partial_T y\Big\}, \Big\{C_3 \partial_X (X\partial_X)^{\sigma_2+k-1} y\Big\}\Big), \\ y(T,X) = O(T^K). \end{cases}
$$

We know that the formal solution $y(T,X)$ belongs to the Gevrey class $\mathcal{G}_T^{\sigma_1+1}$. Indeed, by drawing the Newton polygon for the nonlinear equation, we have

$$
\text{The Gevrey order in } T = 1 + \max_{\alpha,p,q,r} \left\{ \frac{\sigma_2 + K_0 - 1}{V(\alpha,p,q,r) - K} \right\} = 1 + \sigma_1.
$$

The definition of the Newton polygon for nonlinear equation is found in [S1]. The details are omitted here.

Therefore, we have

$$
V(T,X) \in \mathcal{G}_T^{\sigma_1+1} \Longrightarrow W(T,X) \in \mathcal{G}_{T,X}^{\sigma_1+1,\sigma_2+1} \Longrightarrow U(t,x) \in \mathcal{G}_{t,x}^{\boldsymbol{s}}.
$$

References

[CL] Chen H. and Luo Z., On the Holomorphic Solution of Non-linear Totally Characteristic Equations with Several Space Variables, *Preprint* 99/23 *November* 1999, *Institut für Mathematik, Universität Potsdam.*

[CLT] Chen H. and Luo Z. and Tahara H., Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity, *Ann. Inst. Fourier (Grenoble)*, **51** (2001), No.6, 1599—1620.

[CT] Chen H. and Tahara H., On Totally Characteristic Type Non-linear Partial Differential Equations in Complex Domain, *Publ. RIMS, Kyoto Univ.* **35** (1999), 621–636.

[GT] Gérard R. and Tahara H., Singular nonlinear partial differential equations, *Vieweg Verlag*, 1996.

[MS1] Miyake M. and Shirai A., Convergence of formal solutions of first order singular nonlinear partial differential equations in complex domain, *Annales Polonic Mathmatici*, **74** (2000), 215—228.

[MS2] Miyake M. and Shirai A., Structure of formal solutions of nonlinear first order singular partial differential equations in complex domain, *to appear in Funkcial. Ekvac.*

[S1] Shirai A., Maillet type theorem for nonlinear partial differential equations and Newton polygons, *J. Math. Soc. Japan.*, **53** (2001), 565—587.

[S2] Shirai A., Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type, *Funkcial. Ekvac.*, **45** (2002), 187—208.

[S3] Shirai A., Maillet type theorem for first order singular nonlinear partial differential equations, *Publ. RIMS. Kyoto Univ.*, **39** (2003), 275—296.