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Uniqueness in the Cauchy problem for systems with partial analytic coefficients

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The problem of the uniqueness in the Cauchy problem is a fundamental problem in a theory of partial differential equations. In this paper, we consider the uniqueness in the Cauchy problem for systems with partial analytic coefficients. In the case that the coefficients are analytic, by Holmgren's theorem the uniqueness hold for any non-characteristic initial hypersurface. On the other hand, It is well known that when the coefficients is merely C^{∞} function, the uniqueness is false for some non-characteristic initial hypersurface, for example Alihac and Baouendi [AB] showd that there exists some second order hyperbolic operator $P=\partial_{t}^{2}-A(t,x,D_{x})$, where A is an second elliptic operator with C^{∞} coefficients, and some time-like initial hypersurface for which the uniqueness result is false, where time-like means that if hypersurface S is locally defined by $S=\{x|\varphi(x)=0\}$, φ satisfies that

$$\varphi_{t}'(0)^{2}-A(0,0,\varphi_{x}'(0))<0.$$

In [T] Tataru showed that on the assumption of partial analyticity of the coefficients, uniqueness result holds for any non-characteristic initial hypersurface. But Tataru considered the uniqueness only for simple differential operators. The purpose of this paper is to show the result of uniqueness for differential systems with partial analytic coefficeints, which was obtained in [Tam]

We introduce some notation. Let n_{a}, n_{b} be non negative integers with $n=n_{a}+n_{b}\geq 1$.We set $\mathbb{R}^{n}=\mathbb{R}^{n_{a}}\times\mathbb{R}^{n_{b}}$and, for x or ξ in $\mathbb{R}^{n}$, $x=(x_{a},x_{b}),\xi=(\xi_{a},\xi_{b})$. Let $P(x,D_{x})=(p_{ij}(x,D_{x}))_{1\leq i,j\leq N}=\sum_{|\alpha|\leq m}A_{\alpha}(x)D_{x}^{\alpha}$ be a linear differential system with the principal part $P_{m}(x,\xi)=\sum_{|\alpha|=m}\xi^{\alpha}A_{\alpha}(x)$. Let S be a C^{2} hypersurface through 0 locally given by

$$S=\{x:\varphi(x)=0\},\varphi(0)=0,\varphi'(0)=(\varphi_{a}'(0),\varphi_{b}'(0))\neq 0.$$

Our result is as follows;

Theorem 0.1
Let $P(x,D_{x})$ be a differential systems of order m with C^{∞} coefficinents. We assume that all coefficients of P are analytic with respect to x_{a} in a neighborhood of 0, and that The principal symbol $P_{m}(x,\xi)$ satisfies the following coniditions.

1. *For any* $\xi_b \in \mathbb{R}^{n_b} \backslash \{0\}$

$$\det P_m(0, 0, \xi_b) \neq 0. \tag{1}$$

2. *For any* $\xi_b \in \mathbb{R}^{n_b}$

$$\det P_m(0, i\varphi_a'(0), i\varphi_b'(0) + \xi_b) \neq 0. \tag{2}$$

Let V *be a neighborhood of* 0 *and* $u = (u_1, u_2, \cdots, u_N) \in C^\infty(V)^N$ *be such that*

$$\begin{cases} P(x, D_x)u(x) = 0, \quad x \in V \\ \operatorname{supp} u = \cup_{k=1}^N \operatorname{supp} u_k \subset \{x \in V : \varphi(x) \leq 0\}. \end{cases}$$

Then there exists a neighborhood W *of 0 in which* $u \equiv 0$.

We make some remarks on this result. This theorem contains Holmgren's theorem for differential systems. In fact when we set $n_a = n, n_b = 0$, the condition of $P_m(x, \xi)$ in this theorem means that $P(x, D_x)$ is non-characteristic with respect to the initial hypersurface S. Moreover by this theorem we can show that uniqueness holds in the following differential systems.

Example 1
We set $(t, x) \in \mathbb{R}^2$. *Let* $a, b, c \in C^\infty(\mathbb{R}^2)$,*and* $A(x) \in C^\infty(\mathbb{R}, M_2(\mathbb{C}))$. *We assume that* a, b, c *satisfy the following conditions.*

1. $a(t, x), b(t, x), c(t, x)$ *is analytic with respect of* t *in some neighborhood of* 0
2. $a_0 b_0 \neq 0$, *where* $a_0 = a(0, 0), b_0 = b(0, 0), c_0 = c(0, 0)$.

Then the equation

$$\partial_t u = \begin{pmatrix} c\partial_x & a\partial_x \\ b\partial_x & 0 \end{pmatrix} u + A(x)u$$

has a unique continuation proparty with respect to the initial surface $S = \{x | \varphi(x) = 0\}$ *where* φ *satisfies*

1. $\varphi_t^2 + c_0\varphi_t\varphi_x + a_0 b_0 \varphi_x^2 \neq 0$
2. $c_0\varphi_t' + 2a_0 b_0 \varphi_x \neq 0$.

Our proof is based on Carleman method and FBI transformation theory, basically the same as the proof given by[RZ]. By the Sjöstrand's theory of FBI transformation we microlocalize the symbols of $P(x, D_x)$ with respect to x_a and by using semi-classical pseudo differential symbolic calculus and the Gårding's inequality, we construct Carleman estimate of $P(x, D_x)$.If you want to know the detail of our proof, refer [Tam].

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