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# Remarks on Local Solvability of Operators with Principal Symbol $\xi_1^2+\cdots+\xi_{n-1}^2+x_n^2\xi_n^2$

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## 1. Definitions and main results

Many authors have studied local solvability in the spaces of distributions and ultradistributions. In the framework of distributions Hörmander [6] gave a necessary condition of local solvability, that is, for a differential operator $P$ he proved that the transposed operator ${}^tP$ of $P$ satisfies some estimates if $P$ is locally solvable. Conversely, Treves [15] and Yoshikawa [19] proved that the same type of estimates implies that $P$ is locally solvable. In the frameworks of ultradistributions and hyperfunctions the corresponding treatment is possible (see [4], [1], [3] and [16]).

In this article we shall study local solvability of pseudodifferential operators with principal symbol $\xi_1^2+\cdots+\xi_{n-1}^2+x_n^2\xi_n^2$ in the spaces of distributions and ultradistributions. In [5] Funakoshi proved that these operators are locally solvable in the space of hyperfunctions ( see, also, [16]). Our purpose is to illustrate, with these examples, how to study local solvability in the spaces of distributions and ultradistributions. For the details we refer to [17].

Let us first define Gevrey classes and symbol classes. Let $K$ be a regular compact set in $\mathbb{R}^n$, and let $\kappa>1$ and $h>0$. We denote by $\mathscr{E}^{\{\kappa\},h}(K)$ the space of all $f\in C^\infty(K)$ satisfying, with some $C>0$,

$$|D^\alpha f(x)|\le Ch^{|\alpha|}|\alpha|!^\kappa \quad \text{for any } x\in K \text{ and } \alpha\in(\mathbb{Z}_+)^n, \tag{1.1}$$

where $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, $D=i^{-1}\partial=i^{-1}(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$ and $|\alpha|=\sum_{j=1}^n\alpha_j$ for $\alpha=(\alpha_1,\cdots,\alpha_n)\in(\mathbb{Z}_+)^n$. We also denote by $\mathscr{D}_K^{\{\kappa\},h}$ the space of all $f\in C^\infty(\mathbb{R}^n)$ with supp $f\subset K$ satisfying (1.1). $\mathscr{E}^{\{\kappa\},h}(K)$ and $\mathscr{D}_K^{\{\kappa\},h}$

are Banach spaces under the norm defined by

$$|f;\mathscr{E}^{\{\kappa\},h}(K)| := \sup_{x\in K,\alpha\in(\mathbb{Z}_+)^n} |D^\alpha f(x)|/(h^{|\alpha|}|\alpha|!^\kappa).$$

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We introduce the following locally convex spaces ( Gevrey classes):

$$\mathscr{E}^{(\kappa)}(\Omega) := \varprojlim_{K\Subset\Omega} \mathscr{E}^{(\kappa)}(K), \qquad \mathscr{E}^{(\kappa)}(K) := \varprojlim_{h\to 0} \mathscr{E}^{\{\kappa\},h}(K),$$
$$\mathscr{E}^{\{\kappa\}}(\Omega) := \varprojlim_{K\Subset\Omega} \mathscr{E}^{\{\kappa\}}(K), \qquad \mathscr{E}^{\{\kappa\}}(K) := \varinjlim_{h\to\infty} \mathscr{E}^{\{\kappa\},h}(K),$$
$$\mathscr{D}^{(\kappa)}(\Omega) := \varinjlim_{K\Subset\Omega} \mathscr{D}_K^{(\kappa)}, \qquad \mathscr{D}_K^{(\kappa)} := \varprojlim_{h\to 0} \mathscr{D}_K^{\{\kappa\},h},$$
$$\mathscr{D}^{\{\kappa\}}(\Omega) := \varinjlim_{K\Subset\Omega} \mathscr{D}_K^{\{\kappa\}}, \qquad \mathscr{D}_K^{\{\kappa\}} := \varinjlim_{h\to\infty} \mathscr{D}_K^{\{\kappa\},h},$$

where $A \Subset B$ means that the closure $\overline{A}$ of $A$ is compact and included in the interior $\mathring{B}$ of $B$. We denote by $\mathscr{D}^{*\prime}(\Omega)$ and $\mathscr{E}^{*\prime}(\Omega)$ the strong dual spaces of $\mathscr{D}^*(\Omega)$ and $\mathscr{E}^*(\Omega)$, respectively, where $*$ denotes $(\kappa)$ or $\{\kappa\}$. Elements of these spaces are called ultradistributions ( see, *e.g.*, [11]). We also write $\mathscr{E}^*, \cdots$, instead of $\mathscr{E}^*(\mathbb{R}^n)$, $\cdots$. Let us define symbol classes $S_*^{m;\delta}$, where $m, \delta \in \mathbb{R}$. We say that a symbol $p(x,\xi)$ belongs to $S_{(\kappa)}^{m;\delta}$ ( resp. $S_{\{\kappa\}}^{m;\delta}$) if $p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and for any $A$ there is $C \equiv C_A > 0$ ( resp. there are $A > 0$ and $C > 0$) such that

$$\text{(1.2)} \qquad |p_{(\beta)}^{(\alpha)}(x,\xi)| \le CA^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa \langle\xi\rangle^{m-|\alpha|}e^{\delta\langle\xi\rangle}$$

for any $x,\xi \in \mathbb{R}^n$ and $\alpha,\beta \in (\mathbb{Z}_+)^n$, where $p_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta p(x,\xi)$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. We define

$$S_{(\kappa)}^{0;\infty} := \bigcup_{\delta>0} S_{(\kappa)}^{0;\delta}, \qquad S_{\{\kappa\}}^{+} := \bigcap_{\delta>0} S_{\{\kappa\}}^{0;\delta}.$$

We also use the usual symbol classes $S_{\rho,\delta}^m$, where $0 \le \rho,\delta \le 1$ and $m \in \mathbb{R}$. We say that $p(x,\xi) \in S_{\rho,\delta}^m$ if $p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and there are positive constants $C_{\alpha,\beta}$ ( $\alpha,\beta \in (\mathbb{Z}_+)^n$) such that

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \quad \text{for any } x,\xi \in \mathbb{R}^n \text{ and } \alpha,\beta \in (\mathbb{Z}_+)^n.$$

Next we shall define the Fourier transformation and pseudodifferential operators in the space of ultradistributions. Let $\kappa > 1$ and $\varepsilon \in \mathbb{R}$, and define

$$\widehat{\mathscr{S}_{\kappa,\varepsilon}} := \{v(\xi) \in C^\infty(\mathbb{R}^n);\ \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi) \in \mathscr{S}\},$$

where $\mathscr{S}$ denotes the Schwartz space. We introduce the topology in $\widehat{\mathscr{S}_{\kappa,\varepsilon}}$ so that the mapping $\widehat{\mathscr{S}_{\kappa,\varepsilon}} \ni v(\xi) \mapsto \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi) \in \mathscr{S}$ is a homeomorphism. Since $\mathscr{D}\ (= C_0^\infty(\mathbb{R}^n))$ is dense in $\widehat{\mathscr{S}_{\kappa,\varepsilon}}$, the dual space $\widehat{\mathscr{S}'_{\kappa,\varepsilon}}$ of $\widehat{\mathscr{S}_{\kappa,\varepsilon}}$ is identified with $\{\exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi) \in \mathscr{D}';\ v \in \mathscr{S}'\}$. Let $\varepsilon \geq 0$, and define

$$\mathscr{S}_{\kappa,\varepsilon} := \mathscr{F}^{-1}[\widehat{\mathscr{S}_{\kappa,\varepsilon}}]\,(= \mathscr{F}[\widehat{\mathscr{S}_{\kappa,\varepsilon}}] = \{u \in \mathscr{S};\ \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]\hat{u}(\xi) \in \mathscr{S}\}),$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transformation and the inverse Fourier transformation on $\mathscr{S}$ ( or $\mathscr{S}'$), respectively, and $\hat{u}(\xi) \equiv \mathscr{F}[u](\xi) := \int e^{-ix\cdot\xi}u(x) \times dx$ for $u \in \mathscr{S}$. We introduce the topology in $\mathscr{S}_{\kappa,\varepsilon}$ so that $\mathscr{F}$: $\widehat{\mathscr{S}_{\kappa,\varepsilon}} \to \mathscr{S}_{\kappa,\varepsilon}$ is a homeomorphism. Denote by $\mathscr{S}'_{\kappa,\varepsilon}$ the dual space of $\mathscr{S}_{\kappa,\varepsilon}$. Then we can define the transposed operators ${}^t\mathscr{F}$ and ${}^t\mathscr{F}^{-1}$ of $\mathscr{F}$ and $\mathscr{F}^{-1}$ which map $\mathscr{S}'_{\kappa,\varepsilon}$ and $\widehat{\mathscr{S}'_{\kappa,\varepsilon}}$ onto $\widehat{\mathscr{S}'_{\kappa,\varepsilon}}$ and $\mathscr{S}'_{\kappa,\varepsilon}$, respectively. Since $\widehat{\mathscr{S}_{\kappa,-\varepsilon}} \subset \widehat{\mathscr{S}'_{\kappa,\varepsilon}}$ ( $\subset \mathscr{D}'$), we can define $\mathscr{S}_{\kappa,-\varepsilon} := {}^t\mathscr{F}^{-1}[\widehat{\mathscr{S}_{\kappa,-\varepsilon}}]$, and introduce the topology so that ${}^t\mathscr{F}^{-1} : \widehat{\mathscr{S}_{\kappa,-\varepsilon}} \to \mathscr{S}_{\kappa,-\varepsilon}$ is a homeomorphism. $\mathscr{S}'_{\kappa,-\varepsilon}$ denotes the dual space of $\mathscr{S}_{\kappa,-\varepsilon}$. Then we have $\mathscr{S}'_{\kappa,-\varepsilon} = \mathscr{F}[\widehat{\mathscr{S}'_{\kappa,-\varepsilon}}]$. From the definitions it follows that (i) $\widehat{\mathscr{S}'_{\kappa,-\varepsilon}} \subset \mathscr{S}' \subset \widehat{\mathscr{S}'_{\kappa,\varepsilon}}$ and $\mathscr{S}'_{\kappa,-\varepsilon} \subset \mathscr{S}' \subset \mathscr{S}'_{\kappa,\varepsilon}$ for $\varepsilon \geq 0$, (ii) $\mathscr{F} = {}^t\mathscr{F}$ on $\mathscr{S}'$, (iii) $\mathscr{D}^{(\kappa)}$ is a dense subspace of $\mathscr{S}_{\kappa,\varepsilon}$, (iv) $\mathscr{D}^{\{\kappa\}} \subset \mathscr{S}_{\kappa,+} := \bigcup_{\varepsilon>0}\mathscr{S}_{\kappa,\varepsilon}$ and $\mathscr{F}_{\{\kappa\}} := \bigcap_{\varepsilon>0}\mathscr{S}'_{\kappa,\varepsilon} \subset \mathscr{D}^{\{\kappa\}\prime}$, (v) $\mathscr{E}^{(\kappa)\prime} \subset \mathscr{E}_{\kappa,\infty} := \bigcup_{\varepsilon>0}\mathscr{S}_{\kappa,-\varepsilon}$ and $\mathscr{E}^{\{\kappa\}\prime} \subset \mathscr{E}_{\kappa,0} := \bigcap_{\varepsilon>0}\mathscr{S}_{\kappa,-\varepsilon}$, and (vi) $\mathscr{D}^{(\kappa)} \subset \mathscr{S}_{\kappa,\varepsilon} \subset \mathscr{S}_{\kappa,\varepsilon'} \subset \mathscr{S}'_{\kappa,-\varepsilon''} \subset \mathscr{F}_{(\kappa)} := \bigcup_{\varepsilon>0}\mathscr{S}'_{\kappa,\varepsilon} \subset \mathscr{D}^{(\kappa)\prime}$ if $\varepsilon \geq \varepsilon' \geq \varepsilon''$ ( see, *e.g.*, [10]). So we write ${}^t\mathscr{F}$ as $\mathscr{F}$. Let $p(\xi,y,\eta)$ be a symbol satisfying

$$|\partial_\xi^\alpha D_y^\beta \partial_\eta^\gamma p(\xi,y,\eta)| \leq C_{\alpha,\gamma}A^{|\beta|}|\beta|!^\kappa \exp[\delta_1\langle\xi\rangle^{1/\kappa} + \delta_2\langle\eta\rangle^{1/\kappa}]$$

for any $(\xi,y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha,\beta,\gamma \in (\mathbb{Z}_+)^n$, where $A > 0$, $\delta_1, \delta_2 \in \mathbb{R}$ and the positive constants $C_{\alpha,\gamma}$ are independent of $\beta$. Throughout this paper we denote by $C_{a,b,\cdots}$ and $C_{a,b,\cdots}(A,B,\cdots)$ constants depending on $a$, $b$, $\cdots$ and $a$, $b$, $\cdots$, $A$, $B$, $\cdots$, respectively. Define

$$p(D_x,y,D_y)u(x) := (2\pi)^{-n}\mathscr{F}_\xi^{-1}\Big[\int e^{-iy\cdot\xi}\Big(\int e^{iy\cdot\eta}p(\xi,y,\eta)\hat{u}(\eta)\,d\eta\Big)\,dy\Big](x)$$

for $u \in \mathscr{S}_{\kappa,\infty} := \bigcap_{\varepsilon>0}\mathscr{S}_{\kappa,\varepsilon}$.

**Proposition 1.1 (Proposition 2.3 of [10]).** *$p(D_x,y,D_y)$ maps continuously $\mathscr{S}_{\kappa,\varepsilon_2}$ to $\mathscr{S}_{\kappa,\varepsilon_1}$ and $\mathscr{S}'_{\kappa,-\varepsilon_2}$ to $\mathscr{S}'_{\kappa,-\varepsilon_1}$ if $\delta_2 - \kappa(nA)^{-1/\kappa} < \varepsilon_2$, $\varepsilon_1 \leq \varepsilon_2 - \delta_1 - \delta_2$ and $\varepsilon_1 < \kappa(nA)^{-1/\kappa} - \delta_1$.*

Let $p(x,\xi) \in S_{(\kappa)}^{0;\infty}$. From Proposition 1.1 we can define $p(x,D)$ and ${}^tp(x,D)$ by

$$p(x,D) = p(D_x,y,D_y), \qquad {}^tp(x,D) = q(D_x,y,D_y), \tag{1.3}$$

where $p(\xi,y,\eta)=p(y,\eta)$ and $q(\xi,y,\eta)=p(y,-\xi)$. It follows from Proposition 1.1 that $p(x,D)$ and ${}^tp(x,D)$ map continuously $\mathscr{S}_{\kappa,\varepsilon}$ to $\mathscr{S}_{\kappa,\varepsilon-\delta}$ and $\mathscr{S}'_{\kappa,\varepsilon}$ to $\mathscr{S}'_{\kappa,\varepsilon+\delta}$ for any $\varepsilon\in\mathbb{R}$ if $p(x,\xi)\in S^{0;\delta}_{(\kappa)}$, and that $p(x,D)$ and ${}^tp(x,D)$ map $\mathscr{S}_{\kappa,\infty}$ to $\mathscr{S}_{\kappa,\infty}$ and $\mathscr{F}_{(\kappa)}$ to $\mathscr{F}_{(\kappa)}$. Let $p(x,\xi)\in S^{+}_{\{\kappa\}}$. Similarly, we can define $p(x,D)$ and ${}^tp(x,D)$ by (1.3), which map $\mathscr{S}_{\kappa,+}$ to $\mathscr{S}_{\kappa,+}$, $\mathscr{E}_{\kappa,0}$ to $\mathscr{E}_{\kappa,0}$ and $\mathscr{F}_{\{\kappa\}}$ to $\mathscr{F}_{\{\kappa\}}$. In order to state our main results we give definitions of local solvability adopted here.

**Definition 1.2.** Let $x^0\in\mathbb{R}^n$. (i) For $p(x,\xi)\in S^{0;\infty}_{(\kappa)}$ ( resp. $S^{+}_{\{\kappa\}}$) we say that $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}^{*\prime}$ if there is an open neighborhood $U$ of $x^0$ such that for any $f\in\mathscr{D}^{*\prime}$ there is $u\in\mathscr{F}_*$ satisfying $p(x,D)u=f$ in $U$ ( in $\mathscr{D}^{*\prime}(U)$), where $*=(\kappa)$ ( resp. $*=\{\kappa\}$). Moreover, we say that $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}^{*\prime}$ in a germ sense if for any $f\in\mathscr{D}^{*\prime}$ there are an open neighborhood $U$ of $x^0$ and $u\in\mathscr{F}_*$ satisfying $p(x,D)u=f$ in $U$ ( in $\mathscr{D}^{*\prime}(U)$). (ii) For $p(x,\xi)\in S^m_{1,0}$ we say that $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}'$ if there is an open neighborhood $U$ of $x^0$ such that for any $f\in\mathscr{D}'$ there is $u\in\mathscr{S}'$ satisfying $p(x,D)u=f$ in $U$ ( in $\mathscr{D}'(U)$). Similarly, we define local solvability at $x^0$ in $\mathscr{D}'$ in a germ sense.

*Remark.* (i) We remark that the above definitions of local solvability are slightly different from usual ones. (ii) In $\mathscr{D}^{\{\kappa\}\prime}$ "local solvability in a germ sense" implies "local solvability" for properly supported pseudodifferential operators ( see [17]).

Let $\kappa>1$. We denote $(\kappa)$ or $\{\kappa\}$ by $*$. Let $\alpha(x,\xi)\in S^{1;0}_*$, and let

$$L(x,\xi)=|\xi'|^2+x_n^2\xi_n^2+\alpha(x,\xi),$$

where $\xi'=(\xi_1,\cdots,\xi_{n-1})$ for $\xi=(\xi_1,\cdots,\xi_n)\in\mathbb{R}^n$. Then we have the following

**Theorem 1.3.** (i) *If $\kappa\le 2$ when $*=(\kappa)$, and if $\kappa<2$ when $*=\{\kappa\}$, then $L(x,D)$ is locally solvable at the origin in $\mathscr{D}^{*\prime}$.* (ii) *Assume that $\alpha(x,\xi)$ can be written as*

$$\alpha(x,\xi)=\sum_{k=1}^{n-1}\alpha_k(x,\xi)\xi_k+x_n\alpha_n(x,\xi)+\alpha_0(x,\xi),$$

*where $\alpha_j(x,\xi)\in S^{0;0}_*$ ( $0\le j\le n-1$) and $\alpha_n(x,\xi)\in S^{1;0}_*$. Then $L(x,D)$ is locally solvable at the origin in $\mathscr{D}^{*\prime}$.*

*Remark.* It was shown that $L(x,D)$ is locally solvable at the origin in the space of hyperfunctions if $\alpha(x,\xi)$ is an analytic symbol ( see, *e.g.*, Chapter V of [16]).

Let $P(x,D)$ be a differential operator of the form

$$P(x,D) = D_1^2 + \cdots + D_{n-1}^2 + x_n^2 D_n^2 - x_n a(x) D_n \\ - (1 + 2ix_1 + x_1^2 b(x)) D_n - \sum_{k=1}^{n-1} c_k(x) D_k + d(x).$$

Then we have the following theorem which gives necessary conditions of local solvability.

**Theorem 1.4.** (i) *Assume that $a(x)$, $b(x)$, the $c_k(x)$ and $d(x)$ are analytic near the origin. Then $P(x,D)$ is not locally solvable at the origin in $\mathscr{D}^{*\prime}$ if $\kappa > 2$.* (ii) *Assume that $a(x), b(x), c_k(x), d(x) \in C^\infty(\mathbb{R}^n)$. Then $P(x,D)$ is not locally solvable at the origin in $\mathscr{D}'$.*

*Remark.* From Hörmander [7] and Olejnik and Radkevic [12] it follows that the operator

$$D_1^2 + \cdots + D_{n-1}^2 + x_n^2 D_n^2 + (i\alpha(x) + x_n a(x)) D_n + \sum_{k=1}^{n-1} b_k(x) D_k + c(x)$$

is ( hypoelliptic and ) locally solvable at the origin in $\mathscr{D}'$ if $\alpha(x), a(x), b_k(x), c(x) \in C^\infty(\mathbb{R}^n)$, $\alpha(x)$ is real-valued and there is $\gamma \in (\mathbb{Z}_+)^n$ such that $\gamma_n = 0$ and $(D^\gamma \alpha)(0) \neq 0$ ( see, also, [18]).

Let $A$ be an operator defined by $Au(x) = (x_n D_n u(x) + D_n(x_n u(x)))/2$, *i.e.*, $A = x_n D_n - i/2$. Moreover, let $Q(x,D) = D_1^m + \sum_{|\alpha| \le m, \alpha_1 < m} a_\alpha D^{\alpha'} A^{\alpha_n}$, where $m \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, $\alpha' = (\alpha_1, \cdots, \alpha_{n-1})$ for $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{Z}_+)^n$ and $D^{\alpha'} = D_1^{\alpha_1} \cdots D_{n-1}^{\alpha_{n-1}}$.

**Theorem 1.5.** *$Q(x,D)$ is locally solvable at the origin in $\mathscr{D}'$.*

*Remark.* By the above theorem the operator

$$P \equiv D_1^2 + \cdots + D_{n-1}^2 + x_n^2 D_n^2 + \sum_{k=1}^{n-1} a_k D_k + a_n x_n D_n + b$$

is locally solvable at the origin in $\mathscr{D}'$, where $a_k, b \in \mathbb{C}$. (ii) In [13] and [14] Tahara studied more general operators and proved local solvability of those operators in $\mathscr{D}'$ in a germ sense. (iii) The argument used in the proof of Theorem 1.5 gives an alternative proof of local solvability of differential operators with constant coefficients.

In §2 we shall give criteria ( abstruct necessary conditions and sufficient conditions) for local solvability. Using these results one can prove Theorems 1.3 and 1.4. In §3 we shall prove Theorem 1.5.

## 2. Outline of the proofs of Theorems 1.3 and 1.4

We begin with well-known results on local solvability in $\mathscr{D}'$ ( see, *e.g.*, [15], [19] and [6]).

**Proposition 2.1.** *Let $x^0 \in \mathbb{R}^n$ and $p(x,\xi)$ be a symbol in $S^m_{1,0}$, where $m \in \mathbb{R}$.* (i) *If there is an open neighborhood $U$ of $x^0$ such that for any $s \geq 0$ there are $\ell \in \mathbb{R}$ and $C > 0$ satisfying*

$$\|\langle D\rangle^s u\| \leq C\{\|\langle D\rangle^{\ell}\, {}^t p(x,D)u\| + \|u\|\}$$

*for any $u \in C_0^\infty(U)$, then $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}'$. Here $\|f\|$ denotes the $L^2$-norm of $f$, i.e., $\|f\| = (\int |f(x)|^2\,dx)^{1/2}$ for $f \in L^2(\mathbb{R}^n)$.* (ii) *If $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}'$, then there is an open neighborhood $U$ of $x^0$ such that for any $s \geq 0$ there are $\ell \in \mathbb{R}$ and $C > 0$ satisfying*

$$\|\langle D\rangle^s u\| \leq C\|\langle D\rangle^{\ell}\, {}^t p(x,D)u\| \quad \textit{for any } u \in C_0^\infty(U).$$

Repeating the same argument as in the proof of Proposition 2.1 we shall prove Theorems 2.4 and 2.5 below which give criteria for local solvability in $\mathscr{D}^{*\prime}$. In doing so, we need the following

**Lemma 2.2 (Lemma 5.1.8 in [16]).** *Let $f(t)$ be a continuous functions on $[0,\infty)$ such that $f(t) \geq 0$ ( $t \in [0,\infty)$) and $\lim_{t\to\infty} f(t)/t = 0$. Then there is an analytic function $F(t)$ defined in $\mathbb{C}\setminus(-\infty,0]$ satisfying the following:* (i) $F(t) \geq \max_{0\leq s\leq t} f(s)$ *for $t \geq 0$.* (ii) $\lim_{t\to+\infty} F(t)/t = 0$. (iii) $\lim_{t\to+\infty} t/(F(t)(1+\log t)) = 0$. (iv) $0 < F'(t) \leq F(t)/t$ *for $t > 0$.* (v) *There is $C > 0$ such that $F(t)/t \leq CF'(t)$ for $t \geq C$.* (vi) $F''(t) < 0$ *for $t > 0$.* (vii) $\lim_{t\to+\infty} t^2F''(t)/F(t) = 0$. (viii) *There is $C > 0$ such that*

$$|(d/dt)^k F(t)| \leq C(2/t)^k k!F(t) \quad \textit{for } t > 0 \textit{ and } k \in \mathbb{Z}_+.$$

**Definition 2.3.** (i) We say that a symbol $\omega(\xi) \in C^\infty(\mathbb{R}^n)$ belongs to $\mathscr{W}_{(\kappa)}$ if there is $\varepsilon \geq 1$ such that $\omega(\xi) = \varepsilon\langle\xi\rangle^{1/\kappa}$. (ii) We say that a symbol $\omega(\xi) \in C^\infty(\mathbb{R}^n)$ belongs to $\mathscr{W}_{\{\kappa\}}$ if there is a real analytic function $F(t)$ defined near $[1,\infty)$ satisfying the following conditions: (0) $\omega(\xi) = F(\langle\xi\rangle^{1/\kappa})$. (i) $F(t) \geq t/(1+\log t)$ for $t \geq 1$. (ii) $\lim_{t\to+\infty} F(t)/t = 0$. (iii) $0 < F'(t) \leq F(t)/t$ for $t \geq 1$. (iv) There is $C > 1$ such that $F(t)/t \leq CF'(t)$ for $t \geq C$. (v) $F''(t) < 0$ for $t \geq 1$. (vi) $\lim_{t\to+\infty} t^2F''(t)/F(t) = 0$. (vii) There is $C > 0$ such that $|(d/dt)^k F(t)| \leq C(2/t)^k k!F(t)$ for $t \geq 1$ and $k \in \mathbb{Z}_+$.

Using the Hahn-Banach theorem and Poincaré's inequality we can prove the following

**Theorem 2.4.** *Let $x^0 \in \mathbb{R}^n$, and let $\Omega$ be an open neighborhood of $x^0$. Assume that for any $\omega(\xi) \in \mathscr{W}_*$ there are $\mu(\xi) \in \mathscr{W}_*$ and $C > 0$ such that*

$$\|e^{\omega(D)}v\| \le C\{\|e^{\mu(D)}\,{}^t p(x,D)v\| + \|v\|\}$$

*for any $v \in \mathscr{D}^{(\kappa)}(\Omega)$. Then $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}^{*\prime}$.*

**Theorem 2.5.** *Let $x^0 \in \mathbb{R}^n$.* (i) *Let $* = (\kappa)$, and assume that $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}^{(\kappa)\prime}$. Then there is an open neighborhood $U$ of $x^0$ such that for any $\varepsilon > 0$ there are $\delta > 0$ and $C > 0$ satisfying*

$$\|e^{\varepsilon\langle D\rangle^{1/\kappa}}v\| \le C\|e^{\delta\langle D\rangle^{1/\kappa}}\,{}^t p(x,D)v\| \quad \text{for any } v \in \mathscr{D}^{(\kappa)}(U). \tag{2.1}$$

(ii) *Let $* = \{\kappa\}$, and assume that $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}^{\{\kappa\}\prime}$. Then there is an open neighborhood $U$ of $x^0$ such that for any $\delta > 0$ with $\delta < \varepsilon_0$ there are $\varepsilon > 0$ and $C > 0$ satisfying*

$$\|e^{\varepsilon\langle D\rangle^{1/\kappa}}v\| \le C\{\|e^{\delta\langle D\rangle^{1/\kappa}}\,{}^t p(x,D)v\| + \|v\|\} \quad \text{for any } v \in \mathscr{D}^{(\kappa)}(U), \tag{2.2}$$

*where $\varepsilon_0$ is a positive constant determined by $p(x,\xi)$. If $p(x,D)$ is properly supported, then one can drop the term $\|v\|$ on the right-hand side of* (2.2).

In the rest of this section we assume that $p(x,\xi) \in S_*^{m;0}$, where $m \in \mathbb{R}$. Let $\omega(\xi) \in \mathscr{W}_*$, and put

$$p_\omega(x,D) := e^{-\omega(D)}p(x,D)e^{\omega(D)}.$$

Then we have

$$p_\omega(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}e^{\omega(\xi)}(\partial^\alpha e^{-\omega(\xi)})p_{(\alpha)}(x,\xi).$$

Let $\rho > 0$, and let $p_\omega^\rho(x,\xi)$ be a symbol in $S_{1,0}^m$ satisfying

$$\tilde{p}_\omega(x,\xi) \equiv p_\omega^\rho(x,\xi) \qquad (\mathrm{mod}\ S_{1,0}^{m-\rho}).$$

Theorem 2.4 gives the following

**Theorem 2.6.** *Let $x^0 \in \mathbb{R}^n$, and let $\Omega$ be an open neighborhood of $x^0$. Assume that for any $\omega(\xi) \in \mathscr{W}_*$ and $a > 0$ there is $C > 0$ such that*

$$\|{}^t p_\omega^\rho(x,D)u\| \ge a\|\langle D\rangle^{m-\rho}u\| - C\|\langle D\rangle^{m-\rho-1}u\| \tag{2.3}$$

*for $u \in C_0^\infty(\Omega)$. Then $p(x,D)$ is locally solvable at $x^0$ in $\mathscr{D}^{*\prime}$.*

If one can obtain the estimates of type (2.3), one can prove Theorem 1.3, applying Theorem 2.6. For the detail we refer to [17]. Repeating the arguments in Cardoso-Treves [2], Ivrii-Petkov [9] and Ivrii [8] and constructing asymptotic solutions we can prove Theorem 1.4 ( see [17]).

## 3. Proof of Theorem 1.5

Let $X = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. So $X = L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $X$ is a Hilbert space with norm $\|(f,g)\|_X$ defined by $\|(f,g)\|_X = (\|f\|^2 + \|g\|^2)^{1/2}$. Let $\mathcal{T} : L^2(\mathbb{R}^n) \to X$ be a linear operator defined by

$$(\mathcal{T}u)(y) = (e^{y_n/2}u(y', e^{y_n}), e^{y_n/2}u(y', -e^{y_n})).$$

Then $\mathcal{T}$ is a unitary operator. We note that $\mathcal{T}(C_0^\infty(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$, since $|y_n|^\ell e^{y_n/2} \le C_{\ell,k}$ if $\ell, k \in \mathbb{Z}_+$ and $y_n \le k$. For $(f,g) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ we define

$$a(y)D^\alpha(f,g) = (a(y)D^\alpha f(y), a(y)D^\alpha g(y)).$$

Recall that $A = x_n D_n - i/2$ and $Q(x,D) = D_1^m + \sum_{|\alpha| \le m,\, \alpha_1 < m} a_\alpha D^{\alpha'} A^{\alpha_n}$. Since ${}^tA = -A$, we have ${}^tQ(x,D) = (-1)^m D_1^m + \sum_{|\alpha| \le m,\, \alpha_1 < m} (-1)^{|\alpha|} a_\alpha D^{\alpha'} A^{\alpha_n}$. Moreover, we have

$$\begin{aligned} &(\mathcal{T}(D_k u))(y) = D_k(\mathcal{T}u)(y) \quad (1 \le k \le n-1), \\ &(\mathcal{T}(Au))(y) = D_n(\mathcal{T}u)(y), \\ &(\mathcal{T}({}^tQ(x,D)u))(y) = (-1)^m \widetilde{Q}(D)(\mathcal{T}u)(y) \end{aligned} \tag{3.1}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, where $\widetilde{Q}(\eta) = \eta_1^m + \sum_{|\alpha| \le m,\, \alpha_1 < m} (-1)^{m-|\alpha|} a_\alpha \eta^\alpha$. Write

$$\widetilde{Q}(\eta) = \prod_{j=1}^m (\eta_1 - \lambda_j(\eta'')), \tag{3.2}$$

where $\{\lambda_j(\eta'')\}$ is enumerated as

$$\begin{aligned} &\mathrm{Re}\ \lambda_1(\eta'') \le \mathrm{Re}\ \lambda_2(\eta'') \le \cdots \le \mathrm{Re}\ \lambda_m(\eta''), \\ &\mathrm{Im}\ \lambda_j(\eta'') \le \mathrm{Im}\ \lambda_k(\eta'') \quad \text{if } \mathrm{Re}\ \lambda_j(\eta'') = \mathrm{Re}\ \lambda_k(\eta'') \text{ and } j < k. \end{aligned}$$

It is obvious that $\mathrm{Re}\ \lambda_j(\eta'')$ is continuous. Let $T > 0$, and let $v \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\mathrm{supp}\ v \subset \{y \in \mathbb{R}^n;\ |y_1| \le T\}$. Then we have

$$|\hat{v}(\eta)|^2 = \left| \int_{-T}^{T} e^{-iy_1\eta_1} \tilde{v}(y_1, \eta'')\, dy_1 \right|^2 \le 2T \|\tilde{v}(y_1, \eta'')\|^2_{L^2(\mathbb{R}_{y_1})}, \tag{3.3}$$

where $\tilde{v}(y_1, \eta'') = \mathcal{F}_{y''}[v(y_1, y'')](\eta'')$. Let $\varepsilon > 0$, and let $\Lambda$ be a Lebesgue measurable set of $\mathbb{R}^n$ such that $\mu(\Lambda(\eta'')) \le \varepsilon$ for *a.e.* $\eta'' \in \mathbb{R}^{n-1}$, where $\Lambda(\eta'') := \{\eta_1 \in \mathbb{R};\ (\eta_1, \eta'') \in \Lambda\}$ and $\mu$ denotes the Lebesgue measure in $\mathbb{R}$. Then (3.3) yields

$$\|v\|^2 = (2\pi)^{-n} \int_\Lambda |\hat{v}(\eta)|^2\, d\eta + (2\pi)^{-n} \int_{\mathbb{R}^n \setminus \Lambda} |\hat{v}(\eta)|^2\, d\eta$$

$$
\begin{aligned}
&\le 2T(2\pi)^{-n}\int_{\mathbb{R}^{n-1}}\Big(\int_{\Lambda(\eta'')}\|\tilde{v}(y_1,\eta'')\|^2_{L^2(\mathbb{R}_{y_1})}\,d\eta_1\Big)\,d\eta''\\
&\quad+(2\pi)^{-n}\int_{\mathbb{R}^n\setminus\Lambda}|\hat{v}(\eta)|^2\,d\eta\\
&\le (T\varepsilon/\pi)(2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}}\|\tilde{v}(y_1,\eta'')\|^2_{L^2(\mathbb{R}_{y_1})}\,d\eta''+(2\pi)^{-n}\int_{\mathbb{R}^n\setminus\Lambda}|\hat{v}(\eta)|^2\,d\eta\\
&=(T\varepsilon/\pi)\|v\|^2+(2\pi)^{-n}\int_{\mathbb{R}^n\setminus\Lambda}|\hat{v}(\eta)|^2\,d\eta.
\end{aligned}
$$

Therefore, we have

$$\|v\|^2/2\le(2\pi)^{-n}\int_{\mathbb{R}^n\setminus\Lambda}|\hat{v}(\eta)|^2\,d\eta\quad\text{if } T\varepsilon/\pi\le 1/2. \tag{3.4}$$

Now we choose

$$\Lambda=\{\eta\in\mathbb{R}^n;\ |\eta_1-\mathrm{Re}\ \lambda_j(\eta'')|\le\varepsilon/(2m)\quad\text{for some } j\}. \tag{3.5}$$

Then $\Lambda$ is a Lebesgue measurable set of $\mathbb{R}^n$ and $\mu(\Lambda(\eta''))\le\varepsilon$ for each $\eta''\in\mathbb{R}^{n-1}$, since Re $\lambda_j(\eta'')$ is continuous. From (3.2), (3.4) and (3.5) we have

$$
\begin{aligned}
\|\widetilde{Q}(D)v\|^2&\ge(2\pi)^{-n}\int_{\mathbb{R}^n\setminus\Lambda}|\widetilde{Q}(\eta)\hat{v}(\eta)|^2\,d\eta\\
&\ge(\varepsilon/(2m))^{2m}(2\pi)^{-n}\int_{\mathbb{R}^n\setminus\Lambda}|\hat{v}(\eta)|^2\,d\eta\ge 2^{-2m-1}(\varepsilon/m)^{2m}\|v\|^2
\end{aligned}
$$

if $v\in\mathscr{S}(\mathbb{R}^n)$, supp $v\subset\{y\in\mathbb{R}^n;\ |y_1|\le T\}$ and $2T\varepsilon\le\pi$. This, together with (3.1), gives

$$
\begin{aligned}
\|{}^tQ(x,D)u\|^2&=\|\widetilde{Q}(D)\mathscr{T}u\|_X^2\ge 2^{-2m-1}(\varepsilon/m)^{2m}\|\mathscr{T}u\|_X^2\\
&=2^{-2m-1}(\varepsilon/m)^{2m}\|u\|^2
\end{aligned}\tag{3.6}
$$

if $u\in C_0^\infty(\mathbb{R}^n)$, supp $u\subset\{x\in\mathbb{R}^n;\ |x_1|\le T\}$ and $2T\varepsilon\le\pi$. Let $\gamma\in(\mathbb{Z}_+)^n$. Since $AD^\gamma=D^\gamma(A+i\gamma_n)$, we have $D^\gamma\,{}^tQ(x,D)u={}^tQ^\gamma(x,D)D^\gamma u$, where $Q^\gamma(x,D)=D_1^m+\sum_{|\alpha|\le m,\,\alpha_1<m}a_\alpha D^{\alpha'}(A+i\gamma_n)^{\alpha_n}$. (3.6) with $Q(x,D)$ replaced by $Q^\gamma(x,D)$ yields

$$\|D^\gamma u\|\le 2^{2m+1/2}(mT/\pi)^m\|D^\gamma\,{}^tQ(x,D)u\|$$

for $u\in C_0^\infty(\mathbb{R}^n)$ with supp $u\subset\{x\in\mathbb{R}^n;\ |x_1|\le T\}$. Therefore, for any $s\in\mathbb{Z}_+$ there is $C_s>0$ such taht

$$\|\langle D\rangle^s u\|\le C_sT^m\|\langle D\rangle^s\,{}^tQ(x,D)u\|$$

for $u\in C_0^\infty(\mathbb{R}^n)$ with supp $u\subset\{x\in\mathbb{R}^n;\ |x_1|\le T\}$. This, together with Proposition 2.1, proves Theorem 1.5.

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