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ASYMPTOTIC EXPANSION OF THE BERGMAN KERNEL FOR TUBE DOMAINS OF FINITE TYPE

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1. Introduction

In the function theory of several complex variables, it is a very important thema to understand the boundary behavior of the Bergman kernel $B(z)$ (on the diagonal) and there are many interesting studies about this thema. In particular, the strongly pseudoconvex case is quite well understood. Let Ω be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Hörmander [11] and Diederich [4],[5] showed the limit of $B(z)d(z)^{n+2}$ at a boundary point z_0 equals the determinant of the Levi form at z_0 times $(n+1)!/4\pi^{n+1}$. Here $d(z)$ is the Euclidean distance from z to the boundary. Later C. Fefferman [9] obtained the following very strong result about the asymptotic expansion:

$$B(z)=\varphi(z)r(z)^{-n-2}+\psi(z)\log r(z), \tag{1.1}$$

where $-r$ is a defining function of Ω and φ,ψ are C^∞-functions on $\bar{\Omega}$.

Next let us consider the case of weakly pseudoconvex and of finite type domains. Many detailed results have been obtained in estimating the size of the Bergman kernel (see the reference in [1],[15], etc.). More precisely, Boas, Straube and Yu [1] (see also [7]) obtained a result about the *boundary limit* in the sense of Hörmander for some large class of domains of finite type. Indeed, they showed that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ and the boundary point z_0 is semiregular (which is also called h-extendible) with multitype $(1,2m_1,\ldots,2m_n)$, then the nontangential limit of $B(z)d(z)^{\sum_{j=1}^n 1/m_j+2}$ at z_0 equals some positive number which is determined by local model only. But, there seems very few study about asymptotic expansions like (1.1) in the weakly pseudoconvex case. The author [14] has computed an asymptotic expansion of the Bergman kernel for two-dimensional pseudoconvex tube domains of finite type. The purpose of this note is to announce a result about an asymptotic expansion of the Bergman kernel in general dimensional case.

Let us explain our analysis for the Bergman kernel. For tube domains, it is known in [16],[20],[17],[2],[8] that the Bergman kernel can be expressed by using relatively simple integrals. Our analysis is based on this integral expression. From this expression, the integral of the form:

$$F(x) = \int_{\mathbb{R}^n} e^{-2[f(w)-(x|w)]} dw \qquad (x \in \mathbb{R}^n)$$

appears and its analysis is important. Here the function f locally defines the base of the tube domain. The finite type condition implies that f can be locally approximated by a convex quasihomogeneous polynomial P. The tube domain defined by this polynomial P can be considered as an appropriate model and we analyze the singularity of the Bergman kernel for this model domain. On the other hand, the singularity of the Bergman kernel is completely determined by the local geometry of the boundary in our case. By using this localization, we can consider general domains as perturbations of model domains. Some computation implies that this perturbation relfects the many terms of the asymptotic expansion of the Bergman kernel. In the computation, the analysis of the integral F is necessary. Roughly speaking, we give some estimates for the derivatives of F by using F itself, which are necessary for the computation of the asymptotic expansion.

Last let us explain an important geometrical idea in our analysis. Let z_0 be a weakly pseudoconvex point on the boundary. Generally, the geometrical situation of the boundary around z_0 is complicated. Indeed, D'Angelo's variety type and Catlin's multitype are not always uniform around z_0. This situation gives an influence to the singularity of the Bergman kernel. It is a natural phenomenon that its behavior from the tangential direciton becomes complicated. But in the case of tube domains, domains can be approximated by quasihomogeneous domains whose boundaries have relatively simple stratification structures from the viewpoint of the multitype. (More generally, the class of semiregular domains has the same properties, see [6],[21].) From this geometrical property, we introduce new variables which induces a real blowing up at z_0. Roughly speaking, owing to these variables, the singularity can be stratified in clear form. We express the singularity from the vertical direction in the form of an asymptotic expansion. In the weakly peudoconvex case, several variables are necessary to express the singulary. In this respect, the weakly pseudoconvex case differs from the strongly pseudoconvex case.

2. Results

Let Ω be a domain in $\mathbb{C}^{n+1}$ and denote by $A^2(\Omega)$ the subspace of $L^2(\Omega)$ consisting of holomorphic functions. The Bergman kenrel $B(z)$ of Ω (on the diagonal) is defined by

$$B(z) = \sum_j |\varphi_j(z)|^2,$$

where $\{\varphi_j\}_j$ is a complete orthonormal basis of $A^2(\Omega)$. The above sum is uniformly convergent on any compact set in Ω. This implies that $B(z)$ is real analytic on Ω.

In this paper, we consider the following class of domains. Given a domain ω in $\mathbb{R}^{n+1}$. The *tube domain* over the base ω is defined by

$$\Omega = \mathbb{R}^{n+1} + i\omega = \{z = x + iy \in \mathbb{C}^{n+1}; x \in \mathbb{R}^{n+1}, y \in \omega\}.$$

Here we set $z = (z', z_{n+1}) = (z_1, \dots, z_n, z_{n+1}) \in \mathbb{C}^{n+1}$ with $z_j = x_j + iy_j$, $x = (x', x_{n+1}) = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ and $y = (y', y_{n+1}) = (y_1, \dots, y_n, y_{n+1}) \in \mathbb{R}^{n+1}$. A projection Π from $\mathbb{C}^{n+1}$ to $\mathbb{R}^{n+1}$ is defined by $\Pi(z) = \Im(z) = y$. It is well known that the pseudoconvexity of $\Omega = \mathbb{R}^{n+1} + i\omega$ is equivalent to the convexity of the base ω.

2.1. Appropriate coordinates.

We assume that ω is a convex domain in $\mathbb{R}^{n+1}$ with C^∞-smooth boundary. Let y_0 be a boundary point of ω. We can choose a coordinate in $\mathbb{R}^{n+1}$, where the base is contained, so that:

(1) The point y_0 is the origin.
(2) The $y_1, \dots, y_n$ directions give the tangent plane to $\partial\omega$ at y_0.
(3) The y_{n+1} direciton gives the normal (in the case of a bounded ω) or it gives a half line which is contained in $\overline{\omega}$ (in the case of an unbounded ω).

We remark that the convexity of ω implies that a convex cone (or a half line) is contained in $\overline{\omega}$ in the unbounded case, so we can take the above coordinates. For an unbounded ω, there are a domain A in $\mathbb{R}^n$ (possibly, $A = \mathbb{R}^n$) containing the origin and a C^∞-function f on A such that $f(0) = |\nabla f(0)| = 0$ and

$$\omega = \{y \in \mathbb{R}^{n+1}; y_{n+1} > f(y_1, \dots, y_n) = f(y') \quad (y' \in A)\}.$$

For a bounded domain ω, there are a domain A in $\mathbb{R}^n$ containing the origin and C^∞-functions f and $\tilde{f}$ on A such that

$$\omega = \{y \in \mathbb{R}^{n+1}; f(y') < y_{n+1} < \tilde{f}(y') \quad (y' \in A)\}.$$

Let z_0 be a boundary point of $\Omega = \mathbb{R}^{n+1}+i\omega$. We assume that z_0 is of finite type, in the sense of D'Angelo, and Catlin's multitype of z_0 is $(m_1(\partial\Omega, z_0), \ldots, m_{n+1}(\partial\Omega, z_0))$. Then Lemma 3.1, below, implies that $y_0 = \Pi(z_0) \in \partial\omega$ is of finite type in the sense of Bruna-Nagel-Wainger [3].

According to the following result of Schulz [19], the finite type condition implies that the function $f(y')$ can be decomposed into a quasihomogeneous convex polynomial and a remainder term as follows.

Lemma 2.1 ([19]). *There exists a rotation L in $\mathbb{R}^n$ so that the function $f(y')$ can be expressed near the origin as follows:*

$$f(Ly') = P(y') + R(y').$$

Here P and R satisfy the following properties. Set $m_j = m_j(\partial\Omega, P)/2$ $(j = 1, \ldots, n)$.

(i) *$P(y')$ is a convex polynomial having the quasihomogeneity:*

$$P(t^{1/2m_1}y_1, \ldots, t^{1/2m_n}y_n) = tP(y_1, \ldots, y_n) \quad \text{for all } t > 0$$

and $P(y') > 0$ if $y' \neq 0$.

(ii) *There exist cnstants $C > 0$ and $\gamma \in (0, 1]$ such that $|R(y')| \leq C\sigma(y')^{1+\gamma}$, where $\sigma(y') := \sum_{j=1}^n y_j^{2m_j}$ near the origin.*

From the above lemma, we will consider the domain $\omega_P = \{y \in \mathbb{R}^{n+1}; y_{n+1} > P(y')\}$ as an appropriate model for the analysis on the domain ω near the origin. Hereafter we choose the coordinates $y' = (y_1, \ldots, y_n)$ so that f is divided as in the above lemma.

2.2. **Real blowing up.** Next let us introduce a mapping "real blowing up" at $y_0 \in \partial\omega$.

For $\delta > 0$, let $\omega_\delta = \{y \in \omega; y_{n+1} < \delta\}$. Let m be the least common multiplicity of $m_1, \ldots, m_n$ and let $l_j = m/m_j$. Let $\tilde{\pi}$ be a mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ such that $\tilde{\pi}(\tau_1, \ldots, \tau_n, \rho) = \tilde{\pi}(\tau, \rho) = (y_1, \ldots, y_n, y_{n+1})$, where

$$\begin{cases} y_j = \tau_j \rho^{l_j} & (j = 1, \ldots, n) \\ y_{n+1} = \rho^{2m}. \end{cases} \tag{2.1}$$

We set $U = \tilde{\pi}^{-1}(\omega)$ and $U_\delta = \tilde{\pi}^{-1}(\omega_\delta)$. It is easy to see $\tilde{\pi}^{-1}(\omega_P) = \Delta_P \times (0, \infty)$, where

$$\Delta_P = \{\tau \in \mathbb{R}^n; P(\tau) < 1\}.$$

Let π be the restriction of the mapping $\tilde{\pi}$ on the set $\bar{U}$. Note that π is a diffeomorphic mapping from U to ω and $\pi^{-1}(0) = \overline{\Delta_P} \times \{0\}$. This fact means that π is a *real blowing up* at 0.

2.3. **Asymptotic expansion.** Let D be a set in $\mathbb{R}^p$, not necessarily open. We say that f is a C^∞-function on D if f is C^∞-smooth in the interior of D and all partial derivatives of f can be continuously extended to the boundary. For $\delta > 0$, we define $\Gamma_\delta = \{(\tau, \rho) \in \Delta_P \times [0, \delta); P(\tau) + C\rho^{2m\gamma}\sigma(\tau)^{1+\gamma} < 1\}$, where C, γ are positive numbers as in Lemma 2.1. The following is a main result of this paper.

Theorem 2.2. *The Bergman kernel $B(z)$ of a tube domain $\Omega = \mathbb{R}^{n+1} + i\omega$ has the form near $z_0 \in \partial\Omega$:*

$$B(z) = \frac{\Phi(\tau,\rho)}{\rho^{2m(\nu+2)}} + \tilde{\Phi}(\tau,\rho)\log\rho, \tag{2.2}$$

where $\nu = \sum_{j=1}^n 1/m_j$ and $\Phi(\tau,\rho)$, $\tilde{\Phi}(\tau,\rho)$ are C^∞-functions on the set U_δ, with some small positive number δ, satisfying the following properties.

(i) *$\Phi(\tau,\rho)$ can be extended to be a C^∞-function on $U_\delta \cup (\Delta_P \times \{0\})$. More precisely, $\Phi(\tau,\rho)$ admits the following asymptotic expansion with respect to ρ: for any $N \in \mathbb{N}$,*

$$\Phi(\tau,\rho) = \sum_{k=0}^{N} \Phi_k(\tau)\rho^k + R_N(\tau,\rho)\rho^{N+1} + \tilde{\tilde{\Phi}}(\tau,\rho)\rho^{2m(\nu+2)},$$

where each coefficients $\Phi_k(\tau)$ are C^∞-functions on Δ_P, $R_N(\tau,\rho)$ is continuous on Γ_δ and $\tilde{\tilde{\Phi}}(\tau,\rho)$ is a C^∞-function on $\overline{U_\delta}$. In particular, the first coefficient $\Phi(\tau, 0) = \Phi_0(\tau)$ is

$$\Phi(\tau) = \frac{2}{(2\pi)^{n+1}} \int_0^\infty e^{-2s} \left[\int_{\mathbb{R}^n} e^{2\sum_{j=1}^n s^{1/(2m_j)}\tau_j\zeta_j} \frac{1}{E(\zeta)} d\zeta \right] s^{\nu+1} ds,$$

which is always positive on Δ_P and is unbounded as τ approaches the boundary of Δ_P.

(ii) *$\tilde{\Phi}(\tau,\rho)$ can be extended to be a C^∞-function on $\overline{U_\delta}$.*

Remark 2.3. From the theorem, we obtain a result about the boundary limit as in the Introduction. The nontangential limit of $B(z)\rho^{2m(\nu+2)}$, as $z \to z_0 \in \partial\Omega$, equals

$$\Phi(0) = \frac{1}{2^{n+\nu+2}\pi^{n+1}} \int_{\mathbb{R}^n} \frac{d\zeta}{\int_{\mathbb{R}^n} e^{-2[P(\mu)-(\zeta|\mu)]} d\mu}.$$

This value is determined by the function P only. More precisely, if we restrict the Bergman kernel on the set $\{y \in \mathbb{R}^{n+1}; y_{n+1} > P(y')^{1+\epsilon}\}$ $(\epsilon > 0)$, then the coefficient of ρ^k equals the constant $\Phi_k(0)$ for $k \geq 0$.

Remark 2.4. In this paper, we do not discuss about the singularity of the coefficient functions $\Phi_k(\tau)$ at $\partial\Delta_P$ in detail. Roughly speaking, the singularity with respect to τ concerns with the singularity from the tangential direction. In two-dimensional case, their singularities are computed in [14]. But, the boundary geometrical situation around z_0 is very complicated in general dimensional case. Therefore the singularity also becomes complicated and it must be expressed by using several variables. The singularity from the vartical direction is essentially important and it can be seen as in the theorem.

Remark 2.5. Our asymptotic expansion, with respect to ρ, has a similar form to (1.1) in the strongly pseudoconvex case. The essential difference appears in the expansion variable. That is, in the strongly or the weakly pseudoconvex case, the asymptotic expansion has the Taylor type or the Puiseux type, respectively. In [15], a similar asymptotic expansion is computed for another class of domains of semiregular. From these observation, we may conjecture that the Bergman kernel always admits an asymptotic expansion like (2.2) for the class of pseudoconvex domains of semiregular. But this type of expansion can not be generalized to the general domains of finite type (see [10],[15]).

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