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Effect of Delayed Feedback Control for Chemostat Model

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1 Introduction

Feedback control for chemostat model appears first in [2]. They considered a dilution rate as a feedback control variable, keeping the input nutrient concentration to be constant, and showed that the coexistence of two organisms can be achieved under a single nutrient source in the form: a globally asymptotically stable equilibrium point in the interior of the non-negative orthant. The model is expreesed as follows:

$$\begin{cases} \dot{S}(t) = (1-S(t))D_t - \sum_{I=1}^{2} x_i(t) f_i(S), \\ \dot{x}_i(t) = x_i(t)\Big(f_i(S) - D_t\Big), \quad i = 1, 2, \end{cases} \tag{1.1}$$
$$\text{where } D_t = k_1 x_1(t) + k_2 x_2(t) + \varepsilon.$$

Here, S and x_i $(i = 1, 2)$ mean the concentartion of nutrient and organisms, respectively. D_t is dilution rate and depends affinaly on the state of organisims, in which parameters k_i, $i = 1, 2$ and ε are positive constants. The functions f_i are called uptake functions. We assume that they are monotonically increasing, continuously differentiable, concave for $S \geq 0$ and $f_i(0) = 0$ $(i = 1, 2)$. In addition, they have the following properties:

(**Hypothesis**) *The graphs of the functions f_1 and f_2 intersect once at S^*:*

$$f_1(S^*) = f_2(S^*) = D^*, \tag{1.2}$$

where $S^* \in (0, 1)$. For all $S \in (0, S^*)$ the inequality $f_1(S) > f_2(S)$ holds, while for all $S > S^*$, $f_1(S) < f_2(S)$ holds. Moreover $f_1'(S^*) > f_2'(S^*)$.

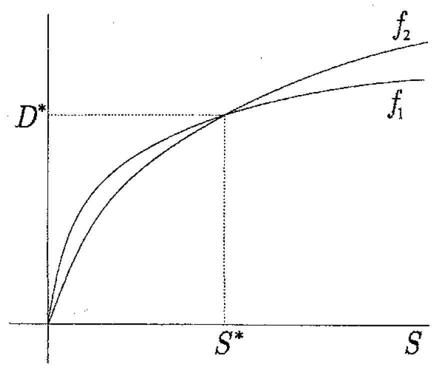

Figure 1: Illustrarion of Hypothesis of functions f_i

In (1.1), the dilution rate D_t depends on the state instantaneously. However the observer has a short time delay practically in changing the dilution rate after measuring the concentration of organisms. It may be natural to incorporate the time delay in studying the feedback control for chemostat. So, we consider an improvement of the system (1.1) as follows:

$$\begin{cases} \dot{S}(t) = (1 - S(t))D_{t-\tau} - \sum_{i=1}^{2} x_i(t) f_i(S), \\ \dot{x}_i(t) = x_i(t)\Big(f_i(S) - D_{t-\tau}\Big), \quad i = 1, 2, \end{cases} \tag{1.3}$$

$$\text{where } D_t = k_1 x_1(t-\tau) + k_2 x_2(t-\tau) + \varepsilon.$$

Here, initial conditions of (1.3) are given as:

$$S(0) > 0,\ x_i(\theta) \geq 0 \text{ for } {}^\forall\theta \in [-\tau, 0) \text{ and } x_i(0) > 0, i = 1, 2. \tag{1.4}$$

Throughout the remainder of this paper, we assume that $\varepsilon \in (0, D^*)$. Our subject is to study an effect on the coexistence of organisms by the time-delay. That is, we investigate whether parameters k_i, $i = 1, 2$ exist such that the system (1.3) is permanent. But, the system (1.3) is delay differential system, so it is not so easy to see the behavior of its solutions by the usual method. Then, by applying the method of "average Liapunov function" [1, 4] we can obtain the result of permanence problem for (1.3). Our results are following; if the time delay is very small so that two organisims can coexist permanently and moreover we can choose the parameters k_i, $i = 1, 2$ such that two oraganisims can coexist permanently for an arbitrary time-delay $\tau > 0$. Next section contains these results.

2 Main results

Our subject is to obtain a theorem which assure the permanent coexistence of organisms for (1.3). So, we apply the following theorem[4].

Theorem 2.1. ([4]) *Assume that* $(\mathbf{X}, d)$ *is compact metric space and that* $\mathbf{S}$ *is a compact subset of* $\mathbf{X}$ *with* $\mathring{\mathbf{S}} \neq \emptyset$. *Let* $\mathbf{S}$ *and* $\mathbf{X}\backslash\mathbf{S}$ *be forward invariant. Suppose that* $P : \mathbf{X} \to \mathbb{R}_+$ *is continuous and differentiable along orbit on* $\mathbf{X}$ *and that* $P^{-1}(0) = \mathbf{S}$. *And there is a continuous function* $\psi : \mathbf{X} \to \mathbb{R}$, *such that* $\dot{P}(\phi)/P(\phi) = \psi(\phi)$ *for all* $\phi \in \mathbf{X}$, *and for each* $\phi \in \mathbf{S}$

$$\sup_{T \geq 1} \frac{1}{T} \int_0^T \psi(\phi(t)) dt > 0. \tag{2.1}$$

Then S *is a repeller. It is also sufficient that* (2.1) *holds for all* $\phi \in \omega(\mathbf{S})$.

To apply Theorem 2.1, we should consider the dynamical system (1.3) on compact metric space $(\mathbf{X}, d)$. For the compactness of $\mathbf{X}$, next proposition is requaried.

Proposition 2.1. *All solutions* $\boldsymbol{x}(t) := (S(t), x_1(t), x_2(t))$ *of the system* (1.3) *with initial comdition* (1.4) *satisfy*

$$\lim_{t \to \infty} S(t) + x_1(t) + x_2(t) = 1 \tag{2.2}$$

Proof is omitted(cf. [2, Propsition 1.]). Proposition 2.1 implies that there is a constant $K > 1$ such that there exists a finite time $T(K) > 0$ and $S(t) + x_1(t) + x_2(t) \leq K$ for all $t > T(K)$. So, let be $Y := \{(S, x_1, x_2) \in \mathbb{R}_+^3 | S + x_1 + x_2 \leq K\}$. And we define the following set as $\mathbf{X}$ at Theorem 2.1:

$$\mathbf{X} := \left\{ \phi \in C([-\tau, 0], Y) \,|\, |\phi(u) - \phi(v)| \leq K'|u - v| \text{ for } {}^{\forall}u, v \in [-\tau, 0] \right\}.$$

Here, $\phi = (S(\theta), x_1(\theta), x_2(\theta))$ and K' is positive constant. Now, $\mathbf{S}$ is defined as follows:

$$\mathbf{S} := \mathbf{S}_1 \cap \mathbf{S}_2,$$

where $\mathbf{S}_i := \left\{ \phi \in \mathbf{X} | S(\theta) \geq 0,\ x_i(\theta) \geq 0,\ x_j(\theta) \geq 0 \text{ for } {}^{\forall}\theta \in [-\tau, 0), S(0) > 0\ , x_i(0) > 0, x_j(0) = 0 \right\}$,

$$(i, j) = (1, 2), (2, 1).$$

Let construct continuous function $P : \mathbf{X} \to \mathbb{R}_+$ as follows:

$$P(\phi) = x_1(0)x_2(0).$$

By direct verification it is found from (1.3) that

$$\dot{P}(\phi)/P(\phi) = f_1(S_t(0)) + f_2(S_t(0)) - 2k_1x_{1t}(-\tau) - 2k_2x_{2t}(-\tau) - 2\varepsilon := \psi(\phi(t)).$$

By theorem 2.1, we can find that the system (1.3) is permanent, if $\sup_{T \geq 1} \int_0^T \psi(\phi(t))dt > 0$ with respect to its solutions starting in $\mathbf{S}$. So, it is necessary to investigate the behavior of the solutions. It is stated in the following lemmas.

Lemma 2.1. *All solutions of* (1.3) *starting in* $\mathbf{S}_i$ $(i = 1, 2)$ *staisfy*

$$e^{-k_i\tau}(1 - \lambda_i - \underline{\epsilon_i}) \leq x_i(t) \leq \min\{e^{k_i\tau}(1 - \lambda_i + \overline{\epsilon_i}), 1\}$$

for a sufficient large time. Here, λ_i $(i = 1, 2)$ *are defined as the unique root of equation* $f_i(\lambda_i) = k_i(1 - \lambda_i) + \varepsilon$ *and* $\underline{\epsilon_i}, \overline{\epsilon_i}$ *are some positive constant depended on* f_i *and* k_i.

We show that the outline of the proof of Lemma 2.1. By $\lim_{t\to\infty} S(t) + x_i(t) = 1$ for all solutions of (1.3) starting in $\mathbf{S}_i$, for $\epsilon > 0$ there exist a finite time $T(\epsilon) > 0$ such that $x_i(t)$ is the solution of

$$\dot{x}_i(t) = x_i(t)(f_i(1 - x_i \pm \epsilon) - k_ix_i(t - \tau) - \varepsilon) \tag{2.3}$$

for $t \geq T(\epsilon)$. Let us consider a continuous function $W : \mathbf{X} \to \mathbb{R}$ defined by $W(\phi) = \phi(0)e^{-k_i \int_{-\tau}^{0} \phi(s)ds}$. We can obtain the statement of Lemma 2.1 by estimating properly the derivate of $W(\phi)$ along the solution of (2.3).

Lemma 2.2. *Equilibria* E_i *of* (1.3) *are globally asymptotically stable with respect to the solution starting in* $\mathbf{S}_i$, *if the following equalities either (i) or (ii) hold:*

(i) $\tau < \dfrac{k_i + f_i'(\beta_i)}{k_i(k_i + f_i'(\alpha_i))}$.

(ii) $\dfrac{f_i(1 - \alpha_i) - f_i(\lambda_i)}{1 - \alpha_i - \lambda_i} > k_i$ *for* ${}^{\forall}\tau > 0$.

Here, equlubria $E_1 := (\lambda_1, 1 - \lambda_1, 0)$ *and* $E_2 := (\lambda_2, 0, 1 - \lambda_2)$ *always exist and* λ_i, $(i = 1, 2)$ *satisfy* $f_i(\lambda_i) = k_i(1 - \lambda_i) + \varepsilon$. *And* $\alpha_i = e^{-k_i\tau}(1 - \lambda_i - \underline{\epsilon_i})$, $\beta_i = \min\{e^{k_i\tau}(1 - \lambda_i + \overline{\epsilon_i}), 1\}$.

Lemma 2.2 can be obtined by Liapunov functional method[3]. Condition (i) is obtained by constructing the following functional:

$$V_{(i)}(\phi) = x_i(0) - (1-\lambda_i) - (1-\lambda_i)\log\frac{x_i(0)}{(1-\lambda_i)} + \frac{k_i f_i'(\alpha_i)}{2}\int_{-\tau}^{0}\int_{u}^{0}|x_i(s)-(1-\lambda_i)|^2 ds du$$
$$+\frac{k_i^2}{2}\int_{-\tau}^{0}\left\{\int_{u}^{0}|x_i(s-\tau)-(1-\lambda_i)|^2 ds + \tau|x_i(u)-x_i^\dagger|^2\right\}du.$$

The other (ii) is obtained by construcrting the following functional:

$$V_{(ii)}(\phi) = x_i(0) - (1-\lambda_i) - (1-\lambda_i)\log\frac{x_i(0)}{(1-\lambda_i)} + \mu\int_{-\tau}^{0}(x_i(s)-(1-\lambda_i))^2 ds,$$

where $\mu = \frac{f_i(1-\alpha_i)-f_i(\lambda_i)}{2(1-\alpha_i-\lambda_i)}$.

By Lemma 2.2, we can find the condition of $\omega(\mathbf{S}_i) = E_i$. Next, let us show the estimation of $\sup_{T\geq 1}\frac{1}{T}\int_0^T \psi(\phi(t))dt$ for all $\phi \in \omega(\mathbf{S})$. First, we state it in case $i=1$.

$$\begin{aligned}\sup_{T\geq 1}\frac{1}{T}\int_0^T \psi(\phi(t))dt =& \sup_{T\geq 1}\frac{1}{T}\int_0^T (f_1(\lambda_1)+f_2(\lambda_1)-2k_1(1-\lambda_1)-2\varepsilon)\,dt\\ =& f_2(\lambda_1)-k_1(1-\lambda_1)-\varepsilon\\ =& f_2(\lambda_1)-f_1(\lambda_1).\end{aligned}$$

$f_2(\lambda_1)-f_1(\lambda_1)>0$ is equivalent to $k_2<k^*<k_1$ which is the condition to assure an existence and global asymptotic stability of interior equilibrium E_3 in case $\tau=0$. Here, $k^* = \frac{D^*-\varepsilon}{1-S^*}$. Therefore, if $k_2<k^*<k_1$ hold, $\sup_{T\geq 1}\frac{1}{T}\int_0^T \psi(\phi(t))dt > 0$ for all $\phi\in\omega(\mathbf{S}_1)$. In the other case $i=2$, we can find similarly that if $k_2<k^*<k_1$ hold, $\sup_{T\geq 1}\frac{1}{T}\int_0^T \psi(\phi(t))dt > 0$ for all $\phi\in\omega(\mathbf{S}_2)$. So, if $k_2<k^*<k_1$ and the inequalities in Lemma 2.2 holds, $\sup_{T\geq 1}\frac{1}{T}\int_0^T \psi(\phi(t))dt > 0$ for all $\phi\in\omega(\mathbf{S})$.

The above argument can be summerized in the following our main result.

Theorem 2.2. *If $k_2<k^*<k_1$ and the condition (i) or (ii) in Lemma 2.2 holds,* (1.3) *is permanent.*

3 Conclusions

In [2], it is shown that the global asymptotic stability of an interior equilibrium expresses the form of the coexistence of organisms for chmostat model with instantaneously feedback control. While in this paper, though the form is not known, permanece problem is solved for chemostat model with delayed feedback control.

By Theorem 2.2, we can say that even if the tim delay arising from feedback control is considered, there are conditions of parameters k_i and time delay τ such that two organism can coexist. But in view of control, the conditions of Theorem 2.2 is not so good. Because, one of conditions, which is $k_2<k^*<k_1$ and (i) in Lemma 2.2, has a fault that time delay must be very short. The time delay rising from feedback control may be large. And the other condtion, which is $k_2<k^*<k_1$ and (ii) in Lemma 2.2, has a fault that we may not be able to choose parameter k_1

staifying both inequalities. These probrems should be improved. We will try to improve it in the near future.

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