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The Elliott Program of Classification of Finite Simple Amenable C^*-Algebras

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1 Introduction

In this survey report we will discuss the classification of separable simple amenable C^*-algebras with tracial rank no more than one. This is a part of the Elliott program of classification of simple amenable C^*-algebras. The main question of interest is: Given two simple separable amenable C^*-algebras A and B, when they are isomorphic? The Elliott program is to find a complete K-theoretical isomorphic invariant for a class of simple C^*-algebras. Let us mention a couple of important developments in the program without attempting to give any historical account. The program started with the Elliott classification of inductive limits of circle algebras with real rank zero ([10]) which states that two unital inductive limits of circle algebras with real rank zero are isomorphic if the associated scaled ordered K-groups (K_0 together with K_1) are (order) isomorphic. A pre-program result of Elliott that classifies AF-algebras should also be mentioned ([9]). Another high light is the Elliott-Gong's ([12]) result of classification of simple AH-algebras with no dimension growth of real rank zero. For purely infinite simple C^*-algebras, there is no doubt that most satisfactory result is the Kirchberg and Phillips's classification of purely infinite simple separable amenable C^*-algebras which satisfy the Universal Coefficient Theorem (see [19] and [40], see also an earlier result of [43]). A more recent result of Elliott-Gong -Li ([13]) classifies simple AH-algebras with no-dimension growth opens the door to classify separable simple amenable C^*-algebras beyond the class of C^*-algebras with real rank zero.

In this report, we will only discuss stably finite C^*-algebras. We believe that, while the Elliott program is far from complete, it has sufficiently many fruitful results so that harvest of these results should also be on the top of agenda. For that, we mean the application of those results. For the application purpose, we will discuss how to classify simple C^*-algebras that are not assumed to be inductive limits of certain basic building blocks. One rich source of simple amenable C^*-algebras which satisfy the Universal Coefficient Theorem are crossed products of minimal homeomorphisms on some compact metric space. One of the question that we will discussed is: Does the Elliott program provide a classification for those C^*-algebras?

We will describe the classification of unital separable simple amenable C^*-algebras with tracial rank no more than one. The report is organized as follows. Section 2 gives the definition of C^*-algebras of tracial rank zero and introduce an isomorphism theorem for separable simple amenable C^*-algebras with tracial rank zero. Section 3 described certain C^*-algebras which have tracial rank zero. Section 4 discusses the question

when a crossed products arisen from minimal homeomorphisms on a compact metric space has tracial rank zero. We present a result in [36] which answers the question. Section 5 is devoted to the description of the proof of the classification result mentioned in Section 2. Starting Section 6, we discuss simple C^*-algebras with tracial rank one. We will attempt to describe the difficulties that one needs to overcome in order to give the classification theorem for simple amenable C^*-algebras with tracial rank greater than zero. In Section 7, we give some results concerning the unitary groups of simple C^*-algebras with tracial rank one. In Section 8, we present a version of uniqueness theorem that is required for the classification. In Section 9, we present an existence theorem. Finally, in Section 10, we present a classification theorem for unital separable simple amenable C^*-algebras with tracial rank no more than one.

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2 Classification of simple C^*-algebras with tracial rank zero

We start with C^*-algebras with "topological rank zero" and "topological rank one". We think that a " standard" C^*-algebra C with "topological rank zero" should have the form

$$C = \bigoplus_{i=1}^{m} M_{R(i)},$$

and a "standard" C^*-algebra C with "topological rank one" should have the form

$$C = \bigoplus_{i=1}^{m} M_{R(i)}(C(X_i)),$$

where each X_i is an one-dimensional finite CW complex.

To obtain more interesting C^*-algebras, one should consider the limits of C^*-algebras with "topological rank zero" and limits of C^*-algebras with "topological rank one." Thus all AF -algebras should have rank zero.

Let us consider only simple C^*-algebras.

Definition 2.1. ([28]) Let A be a unital simple C^*-algebra. Then A has tracial topological rank zero and we will write $TR(A) = 0$ if the following holds: For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ containing a nonzero element $a \in A_+$, there is a C^*-subalgebra C in A where $C = \oplus_{i=1}^{k} M_{n_i}$, such that $1_C = p$ satisfying the following:

(i) $\|px - xp\| < \varepsilon$ for $x \in \mathcal{F}$,

(ii) $pxp \in_\varepsilon C$ for $x \in \mathcal{F}$ and

(iii) $1 - p$ is equivalent to a projection in $\overline{aAa}$.

If p can be chosen to be 1, the above definition gives AF-algebras. The definition says that, in a unital simple C^*-algebra A with $TR(A) = 0$, the part that may not be approximated by finite dimensional C^*-algebras must have small "measure" (or rather small trace).

Theorem 2.2. ([28]) *Let A be a unital separable simple C^*-algebra with $TR(A) = 0$. Then*

- *A is quasidiagonal;*

- *A has real rank zero;*
- *A has stable rank one;*
- *$K_0(A)$ is weakly unperforated and with Riesz interpolation property;*
- *A has the fundamental comparison property:*
  if $p, q \in A$ are two projections and $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \sim q'$ with $q' \leq q$.

Theorem 2.2 suggests that the class of separable amenable simple C^*-algebras with tracial rank zero is a reasonable replacement for the class of separable amenable simple quasidiagonal C^*-algebras with real rank zero, stable rank one and with weakly unperforated K_0-groups.

Recall that a C^*-algebra A is AH, if

$$A = \lim_{n\to\infty} A_n,$$

where $A_n = \oplus_{i=1}^{k(n)} P_{(i,n)} M_{R(i,n)}(C(X_{(i,n)}))P_{(i,n)}$, and $P_{(i,n)} \in M_{R(i,n)}(C(X_n))$ is a projection and $X_{(i,n)}$ is a connected finite CW-complex.

If A is simple, we say A has *slow dimension growth* if

$$\lim_{n\to\infty} \max_i \frac{\dim X_{(i,n)}}{1 + \mathrm{rank} P_{(i,n)}} = 0.$$

A is said to have *no dimension growth*, if there is an integer $m > 0$ such that

$$\dim X_{(i,n)} \leq m$$

for all i and n.

Elliott and Gong ([12]) showed that every simple AH-algebra with no dimension growth and real rank zero has tracial rank zero.

Theorem 2.3. ([12]) *Let A and B be two unital simple AH-algebras with no dimension growth and with real rank zero. Then $A \cong B$ if and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Moreover, for any weakly unperforated ordered group G_0 with the Riesz interpolation property, an order unit $e \in G_0$, and for any countable abelian group G_1, there exists a unital simple AH-algebra A with no dimension growth and with real rank zero such that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (G_0, (G_0)_+, e, G_1).$$

Later M. Dadarlat ([6]) and G. Gong ([15] and [16]) showed that simple AH-algebras with slow dimension growth and with real rank zero have no dimension growth.

For unital simple separable C^*-algebras with tracial rank zero, one has

Theorem 2.4. ([30]) *Let A and B be two unital separable amenable simple C^*-algebras with $TR(A) = TR(B) = 0$ which satisfy the UCT.*

Then $A \cong B$ f and only if

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Some further references: [26], [27], [7], [29] and [37],

3 Which C^*-algebras have tracial rank zero ?

In this section we will discuss the problem when the converse of Theorem 2.2 holds.

We begin with simple AH-algebras.

Theorem 3.1. ([33]) *For a unital simple AH-algebra, the following are equivalent:*

- $TR(A) = 0$;
- *A has real rank zero and has the fundamental comparison property;*
- *A has real rank zero and slow dimension growth;*
- *A has real rank zero, stable rank one and has weakly unperforated $K_0(A)$.*

As observed by N. Brown (see, for example,[3]) that some additional condition on the structure of traces are required to obtain a converse of Theorem 2.2. We described below.

Definition 3.2. Let A be a C^*-algebra. Denote by ${}_n\hat{A}$ the subset of $\hat{A}$ consisting of irreducible representations with finite dimension less than or equal to n. Put $\hat{A}_n = {}_n\hat{A} \setminus {}_{n-1}\hat{A}$. It is known that ${}_n\hat{A}$ is always closed and $\hat{A}_n$ is a Hausdorff space in its relative topology.

The following proposition also serves as a definition.

Proposition 3.3. *Let A be a C^*-algebra and τ be a tracial state. Then the following are equivalent:*

1) *τ is AC;*

2) *There is a sequence $\{a_n\}$ of nonnegative numbers with $\sum_{n=1}^{\infty} a_n = 1$ and a sequence of positive regular probability Borel measures μ_n on $\hat{A}_n$ such that*

$$\mu = \sum_{n=1}^{\infty} a_n \mu_n;$$

3) $\|\tau|_{I_n}\| \to 0$, where

$$I_n = \{a \in A : \pi(a) = 0 \text{ for } \pi \in {}_n\hat{A}\}.$$

Recall that a separable C^*-algebra A is said to be RFD, if for any $a \in A$, there is a finite dimensional irreducible representation π such that $\pi(a) \neq 0$.

The tracial state space of a unital simple C^*-algebras with $TR(A) = 0$ has the following special feature.

Theorem 3.4. *Let A be a unital separable simple C^*-algebra with $TR(A) = 0$. Then there is an increasing sequence of RFD C^*-algebras A_n such that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ and $\tau|_{A_n}$ is AC for each tracial state τ of A. In other words, $T(A)$ is a set of approximately AC tracial states.*

Theorem 3.5. ([32]) *Let A be a unital separable simple C^*-algebra with countably many extremal tracial states. Then $TR(A) = 0$ if and only if A has real rank zero, stable rank one, weakly unperforated $K_0(A)$ and $T(A)$ is approximately AC.*

For many cases, $T(A)$ is always approximately AC.

Proposition 3.6. *If A is a unital C^*-algebra such that A is an inductive limit of type I C^*-algebras, then $T(A)$, the tracial state space of A, is approximate AC.*

Combining 3.5 and 3.6, we have the following.

Corollary 3.7. ([32]) *Let A be a unital simple C^*-algebra which is an inductive limit of type I C^*-algebras. Suppose that $T(A)$ has countably many extremal points. Then $TR(A) = 0$ if and only if A has real rank zero, stable rank one and weakly unperforated $K_0(A)$.*

It perhaps worth to point out that the class of inductive limits of type I C^*-algebras includes C^*-algebras with the following forms:

(1) $A = \lim_{n\to\infty} A_n$, each A_n has continuous trace;

(2) $A = \lim_{n\to\infty} A_n$, each A_n is sub-homogeneous;

(3) $A = \lim_{n\to\infty} A_n$, each A_n has only finite dimensional irreducible representation;

We would like to leave the following question:

Question: Let A be a unital separable amenable quasidiagonal simple C^*-algebra with unique tracial state. Suppose that A has real rank zero, stable rank one and weakly unperforated $K_0(A)$. Does A have tracial rank zero?

Some further references: [3] and [5] .

4 Simple Crossed Products

Definition 4.1. Let X be a compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism. Let $C(X) \rtimes_\alpha \mathbb{Z}$ be the transformation group C^*-algebra (the crossed product).

If X has infinitely many points, then $C(X) \rtimes_\alpha \mathbb{Z}$ is simple. Note that $C(X) \rtimes_\alpha \mathbb{Z}$ satisfies the Universal Coefficient Theorem (UCT).

Question: When $TR(C(X) \rtimes_\alpha \mathbb{Z}) = 0$?

Definition 4.2. Let A be a unital stably finite C^*-algebra and let $\mathrm{Aff}(T(A))$ be the space of all real affine continuous functions on the compact convex set $T(A)$. Let

$$\rho : K_0(A) \to Aff(T(A))$$

be the positive homomorphism induced by

$$\rho([p])(\tau) = \tau(p)$$

for projection $p \in A$.

If A is a unital simple C^*-algebra with real rank zero and stable rank one, then $\rho(K_0(A))$ is dense in $Aff(T(A))$. (see [2])

Furthermore, a result of N. C. Phillips (1.10 of [41]) states that if X is an infinite compact metric space and $\alpha : X \to X$ is a minimal homeomorphism, and if $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ has real rank zero, then $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha))$.

A recent result shows that, under the assumption that X has finite dimensional, the converse of the above and much more are true.

Theorem 4.3. ([36])
Let X be an infinite compact metric space with finite covering dimension and let $\alpha : X \to X$ be a minimal homeomorphism. Denote $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$. Then $TR(A_\alpha) = 0$ if and only if $\rho(A_\alpha)$ is dense in $Aff(T(A_\alpha))$.

Let us consider the case that (X, α) is uniquely ergodic.

Let X be a connected compact metric space, let $\alpha\colon X \to X$ be a homeomorphism, and let μ be an α-invariant Borel measure on X. Then the rotation number ρ_α^μ associated with α and μ is a homomorphism with domain the kernel of the homomorphism

$$\mathrm{id} - (\alpha^{-1})^* \colon K^1(X) \to K^1(X)$$

and codomain $\mathbb{R}/\mathbb{Z}$.

It is defined as follows. As usual, let $\phi_\alpha : C(X) \to C(X)$ be the automorphism $\phi_\alpha(f) = f \circ \alpha^{-1}$.

Let $u \in U(M_n(C(X)))$ satisfy $(\mathrm{id} - (\alpha^{-1})^*)([u]) = 0$. Let $v = \phi_\alpha(u^*)u$. Then $[v] = 0$ in $K_1(C(X))$. Increasing the matrix size and replacing u by $\mathrm{diag}(u, 1)$, we may assume that $v \in U_0(M_n(C(X)))$. Then there exist $a_1, a_2, \ldots, a_m \in M_n(C(X))_{s.a.}$ such that $\prod_{k=1}^m e^{ia_k} = v$.

Now define

$$\rho_\alpha^\mu([u]) = \mathbb{Z} + \frac{1}{2\pi} \int_X \sum_{k=1}^m \mathrm{Tr}(a_k(x))\, d\mu(x)$$

(see [14]).

Corollary 4.4. ([36]) *Let X be a finite connected CW complex and let $\alpha : X \to X$ be a minimal homeomorphism. Suppose that α is unique ergodic. Then $TR(C(X) \rtimes_\alpha \mathbb{Z}) = 0$ if and only if the associated rotation number of h has irrational value.*

The following recovers a theorem of Elliott and Evans ([EE]): Every irrational rotation algebra is an AT-algebra of real rank zero.

Theorem 4.5. *Let X be a compact metric space with finite covering dimension, let $\alpha : X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(C(X) \rtimes_\alpha \mathbb{Z}))$ is dense in $AffT(C(X) \rtimes_\alpha \mathbb{Z}))$. Then $C(X) \rtimes_\alpha \mathbb{Z}$ is a simple AH-algebra.*

The proof of the above theorem is an application of 4.3 and the classification theorem 2.4.

We end this section with the following theorem which is also a combination of Theorem 4.3 and 2.4.

Theorem 4.6. ([36]) *Let X be a compact metric space with finite covering dimension, let $\alpha, \beta : X \to X$ be a minimal homeomorphism and let $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ and $A_\beta = C(X) \rtimes_\beta \mathbb{Z}$. Suppose that $\rho(K_0(A_\alpha))$ and $\rho(K_0(A_\beta))$ are dense in $AffT(A_\alpha))$ and $AffT(A_\beta))$, respectively.*

Then

$$A_\alpha \cong A_\beta$$

if and only if

$$(K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}], K_1(A_\alpha)) \cong (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}], K_1(A_\beta)).$$

5 Proof of Theorem 2.4

Let A and B be two C^*-algebras and let $\phi : A \to B$ be a contractive completely positive linear map. Let $\mathcal{G} \subset A$ be a finite subset and let $\delta > 0$. We say that ϕ is $\mathcal{G}$-δ-multiplicative if

$$\|\phi(a)\phi(b) - \phi(ab)\| < \delta$$

for all $a, b \in \mathcal{F}$.

Homomorphisms from A to B are always $\mathcal{G}$-δ -multiplicative for any finite subset $\mathcal{G}$ and $\delta > 0$. In general a $\mathcal{G}$-δ-multiplicative may not close to any homomorphisms.

Let $L_1, L_2 : A \to B$ be two maps We write

$$L_1 \approx_\varepsilon L_2 \text{ on } \mathcal{F}$$

if

$$\|L_1(a) - L_2(a)\| < \varepsilon, \text{ for all } a \in \mathcal{F}.$$

Let A be a C^*-algebra. Denote by $\mathbf{P}(A)$ the set of all projections and unitaries in $M_\infty(\widetilde{A \otimes C_n})$, $n = 1, 2, ...$, where C_n is an abelian C^*-algebra so that

$$K_i(A \otimes C_n) = K_*(A; \mathbf{Z}/n\mathbf{Z}).$$

One also has the following exact sequence

$$\begin{array}{ccccc} K_0(A) & \to & K_0(A, \mathbf{Z}/k\mathbf{Z}) & \to & K_1(A) \\ \uparrow_{\mathbf{k}} & & & & \downarrow_{\mathbf{k}} \\ K_0(A) & \leftarrow & K_1(A, \mathbf{Z}/k\mathbf{Z}) & \leftarrow & K_1(A) \end{array}$$

(see [44]).

Following Dadarlat and Loring ([8]), we use the notation

$$\underline{K}(A) = \oplus_{i=0,1,n\in\mathbf{Z}_+} K_i(A; \mathbf{Z}/n\mathbf{Z}).$$

By $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-called Bockstein operations. Denote by $Hom_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$ those $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ with the property that $\alpha(K_0(A)_+ \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$. If A satisfies the Universal Coefficient Theorem, then $Hom_\Lambda(\underline{K}(A), \underline{K}(B)) \cong KL(A, B)$.

Moreover, one has the following short exact sequence,

$$0 \to Pext(K_*(A), K_*(B)) \to KK(A,B) \to KL(A,B) \to 0$$

Let $L : A \to B$ be a contractive completely positive linear map. We also use L for the extension from $A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ as well as maps from $\widetilde{A \otimes C_n} \to \widetilde{B \otimes C_n}$ for all n.

Given a projection $p \in \mathbf{P}(A)$, if $L : A \to B$ is an $\mathcal{F}$-δ-multiplicative contractive completely positive linear map with sufficiently large $\mathcal{F}$ and sufficiently small δ,

$$\|L(p) - q'\| < 1/4$$

for some projection q'. Define $[L](p) = [q']$ in $\underline{K}(B)$. It is easy to see this is well defined. Suppose that q is also in $\mathbf{P}(A)$ with $[q] = k[p]$ for some integer k. By adding sufficiently many elements (partial isometries) in $\mathcal{F}$, we can assume that $[L](q) = k[L](p)$.

Similarly, one can do the same for unitaries. Let $\mathcal{P} \subset \mathbf{P}(A)$ be a finite subset. We say $[L]|_{\mathcal{P}}$ is well defined if $[L](p)$ is well defined for every $p \in \mathcal{P}$and if $[p'] = [p]$ and $p' \in \mathcal{P}$, then $[L](p') = [L](p)$. This always occurs if $\mathcal{F}$ is sufficiently large and δ is sufficiently small. In what follows we write $[L]|_{\mathcal{P}}$ when $[L]$ is well defined on $\mathcal{P}$.

Given two separable amenable simple C^*-algebras A and B as described in Theorem 2.4. To prove that $A \cong B$, we deploy a strategy of Elliott, called approximate intertwining.

We first to construct a map $\phi : A \to B$ from the order isomorphism from $K_*(A)$ to $K_*(B)$ and a map $\psi : B \to A$ from the order isomorphism from $K_*(B)$ to $K_*(A)$, respectively. A theorem provides ϕ and ψ is called "existence theorem". If there were a unitary $u_1 \in A$ and there were a unitary $u_2 \in B$ such that $\mathrm{ad}\, u_1 \circ (\psi \circ \phi) = \mathrm{id}_A$ and $\mathrm{ad}\, u_2 \circ (\phi \circ \psi) = \mathrm{id}_B$, then one would immediately obtain the desired isomorphism. However, the best possible uniqueness theorem can only assure that $\psi \circ \phi$ is approximately unitarily equivalent to id_A and $\phi \circ \psi$ is approximately unitarily equivalent to id_B. Nevertheless, the Elliott argument of approximately intertwining will then provide the desired isomorphism.

It turns out, however, without assuming that C^*-algebras A and B are inductive limits of certain building block, the existence theorem is difficult to established. In fact, prior to the proof of Theorem 2.4, one could only provide maps that are not homomorphisms each of which carries only a partial K-theoretical information given by the order isomorphism on K-theory. This adds further difficulty to the uniqueness theorem. In other words, a uniqueness theorem should deal with maps which are not even homomorphisms. A search for a uniqueness theorem for amenable C^*-algebras which are not assumed to be inductive limits of basic building blocks leads us to the following.

Theorem 5.1. *Let A be a separable unital amenable C^*-algebra and let B a unital C^*-algebra. Suppose that $h_1, h_2 : A \to B$ are two unital homomorphisms such that*

$$[h_1] = [h_2] \text{ in } KL(A,B).$$

Suppose that $h_0 : A \to B$ is a full unital monomorphism. Then, for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there is an integer n and a unitary $W \in U(M_{n+1}(B))$ such that

$$\|W^* \mathrm{diag}(h_1(a), h_0(a), \cdots, h_0(a))W - \mathrm{diag}(h_2(a), h_0(a), \cdots, h_0(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$.

The original version of this first appeared in an earlier version of [25]. A better version later stated in [7]. The above statement is taken from [34]. After we found (an earlier version) of the above, it becomes clear to us that a uniqueness theorem can be established for simple amenable C^*-algebras with a property that we called "TAF" which is equivalent to what we called now "tracial rank zero".

Theorem 5.2. *Let A be a separable unital amenable simple C^*-algebra with $TR(A) = 0$ satisfying the UCT. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following:*

for any unital C^*-algebra B with $TR(B) = 0$, and any two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $L_1, L_2 : A \to B$ with

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}$$

there exists a unitary $U \in B$ such that

$$\mathrm{ad}U \circ L_1 \approx_\varepsilon L_2 \text{ on } \mathcal{F}.$$

Combing the above uniqueness theorem with the following existence theorem, by applying the Elliott approximate intertwining argument, we establish 2.4.

Theorem 5.3. *Let A and B be two unital separable simple amenable C^*-algebras with tracial rank zero which satisfy the Universal Coefficient Theorem. Then, for any $z \in KL(A, B)$ which gives an order unit preserving order isomorphism from $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ to $(K_0(B), K_0(B)_+, [1_B], K_1(B))$, there exists a sequence of contractive completely positive linear maps $\phi_n : A \to B$ such that*

$$\lim_{n\to\infty} \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| = 0$$

for all $a, b \in A$ and $\{\phi_n\}$ induces z.

Some further references: [27],[7], [29] and [18].

6 Tracial rank one

Now we turn to C^*-algebras with tracial rank one.

Definition 6.1. Let A be a unital simple C^*-algebra. Then A has tracial topological rank no more than one and we will write $TR(A) \leq 1$ if the following holds: For any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset A$ containing a nonzero element $a \in A_+$, there is a C^*-subalgebra C in A where $C = \oplus_{i=1}^k M_{n_i}(C(X_i))$, where each X_i is a finite CW complex with dimension no more than one such that $1_C = p$ satisfying the following:

(i) $\|px - xp\| < \varepsilon$ for $x \in \mathcal{F}$,

(ii) $pxp \in_\varepsilon C$ for $x \in \mathcal{F}$ and

(iii) $1 - p$ is equivalent to a projection in $\overline{aAa}$.

In the above definition, if C can always be chosen to be a finite dimensional C^*-subalgebra then $TR(A) = 0$. If $TR(A) \leq 1$ but $TR(A) \neq 0$ then we will write $TR(A) = 1$. The definition requires that the part of C^*-algebra A which can not be approximated by C^*-algebras with the form C described above has small "measure" (or small trace). It is clear that if A is an inductive limit of C^*-algebras A_n with the form $A_n = \oplus_{j=1}^{k(n)} M_{r(i,n)}(X_{i,n})$, where each $X_{i,n}$ is a finite CW complex with dimension 1 then $TR(A) \leq 1$.

Guihua Gong proves the following theorem.

Theorem 6.2. (G. Gong–[17]) *Every simple AH-algebra with no dimension growth has tracial rank one or zero.*

For unital simple separable C^*-algebras with tracial rank no more than one, we have the following.

Theorem 6.3. ([28]) *Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Then*

- *A is quasidiagonal;*
- *A has real rank zero or one;*
- *A has stable rank one;*
- *$K_0(A)$ is weakly unperforated and with Riesz interpolation property;*
- *A has the fundamental comparison property:*
  if $p, q \in A$ are two projections and $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \sim q'$ with $q' \leq q$.

Theorem 6.4. *If $TR(A) = 1$ and A has real rank zero, then $TR(A) = 0$.*

In the definition of 6.1, C has the form $\bigoplus_{i=1}^{k} M_{R(i)}(C(X_i))$, where each X_i is a one-dimensional finite CW complex. In fact, it is equivalent to require that C has the form $\bigoplus_{i=1}^{k} M_{R(i)}(C([0,1]))$.

For simple AH-algebras with no dimension growth, we have the following classification theorem.

Theorem 6.5. (Elliott, Gong and Li–[13]) *Let A and B be two unital simple AH-algebras with no dimension growth. Then $A \cong B$ if and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)).$$

Definition 6.6. By

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)),$$

we mean

- there is an order isomorphism $\gamma_0 : K_0(A) \to K_0(B)$ with $\gamma_0([1_A]) = [1_B]$,
- there is an isomorphism $\gamma_1 : K_1(A) \to K_1(B)$ and
- there is an affine homeomorphism $\gamma_2 : T(A) \to T(B)$ such that $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for all $\tau \in T(A)$ and $x \in K_0(A)$.

Tracial topological rank can be defined for C^*-algebras which are not simple (see [32]). In particular, a unital commutative C^*-algebra $A = C(X)$, where X is a compact metric space has tracial rank k if and only if the covering dimension of X is k. In [35], it is shown that any crossed product of a unital separable simple C^*-algebra with tracial rank one by an action on $\mathbb{Z}$ which has tracial cyclic Rohlin property has tracial rank one or zero.

7 Unitary group $U(A)$

Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A and denote by $U_0(A)$ the path connected component containing the identity.

For C^*-algebras with real rank zero, one has the following.

Theorem 7.1. [24] *Let A be a unital C^*-algebra with real rank zero. Then every unitary $u \in U_0(A)$ can be approximated in norm by unitaries with finite spectrum.*

Let A be a unital C^*-algebra and let $u \in U_0(A)$. Suppose that

$$h \in C([0,1], U_0(A)) \ \ h(0) = u \quad \text{and} \quad h(1) = 1_A.$$

Put

$$\mathrm{cel}(h) = \sup\{\sum_{i=1}^{k} \|h(t_i) - h(t_{i-1})\| : t_0 = 0 < t_1 < \cdots < t_k = 1\}.$$

Define

$$\mathrm{cel}(u) = \inf\{\mathrm{cel}(h) : h(t) \in C([0,1], U_0(A)) \ \ h(0) = u \quad \text{and} \quad h(1) = 1_A\}.$$

Corollary 7.2. *Let A be a unital C^*-algebra with real rank zero. Then*

$$\mathrm{cel}(u) \leq \pi$$

for all $u \in U_0(A)$.

This is no longer true for C^*-algebras with tracial rank one. In fact, if $A = C([0,1])$, for any $L > 0$, then there are $u \in C([0,1])$ such that

$$\mathrm{cel}(u) > L.$$

It turns out that the unboundedness of exponential length for unitaries in a unital simple C^*-algebras with tracial rank one causes tremendous amount of trouble, in particular, when C^*-algebras are not assumed to be inductive limits of certain basic building blocks. It effects both so-called uniqueness theorem and existence theorem.

Definition 7.3. Let A be a unital C^*-algebra. Let $CU(A)$ be the *closure* of the commutator subgroup of $U(A)$. Clearly that the commutator subgroup forms a normal subgroup of $U(A)$. It follows that $CU(A)$ is a normal subgroup of $U(A)$. It should be noted that $U(A)/CU(A)$ is commutative.

Definition 7.4. If $\bar{u}, \bar{v} \in U(A)/CU(A)$ define

$$\mathrm{dist}(\bar{u}, \bar{v}) = \inf\{\|x - y\| : x, y \in U(A) \ \ \bar{x} = \bar{u}, \ \bar{y} = \bar{v}\}.$$

If $u, v \in U(A)$ then

$$\mathrm{dist}(\bar{u}, \bar{v}) = \inf\{\|uv^* - x\| : x \in CU(A)\}.$$

We have the following:

Lemma 7.5. *Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Let $u \in U_0(A)$. Then, for any $\varepsilon > 0$, there are unitaries $u_1, u_2 \in A$ such that u_1 has exponential length no more than 2π, u_2 is an exponential and*

$$\|u - u_1u_2\| < \varepsilon.$$

The following is very useful in establishing both uniqueness theorem and existence theorem.

Lemma 7.6. *Let A be a unital C^*-algebra.*

(1) *$U_0(A)/CU(A)$ is divisible.*

(2) *If $u \in U(A)$ such that $u^k \in U_0(A)$, then there is $v \in U_0(A)$ such that*

$$\bar{v}^k = \bar{u}^k \text{ in } U(A)/CU(A).$$

(3) *Suppose that $K_1(A) = U(A)/U_0(A)$ and $G \subset U(A)/CU(A)$ is finitely generated subgroup. Then one has $G = G \cap (U_0(A)/CU(A)) \oplus \kappa(G)$, where*

$$\kappa : U(A)/CU(A) \to U(A)/U_0(A)$$

is the quotient map.

Theorem 7.7. ([31]) *Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and let $u \in CU(A)$. Then $u \in U_0(A)$ and for any $\varepsilon > 0$, $\mathrm{cel}(u) \leq 8\pi + \varepsilon$.*

Theorem 7.8. ([31]) *Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Let $u, v \in U(A)$ such that $[u] = [v]$ in $K_1(A)$ and*

$$u^k, v^k \in U_0(A) \quad \text{and} \quad \mathrm{cel}((u^k)^*v^k) < L.$$

Then for any $\varepsilon > 0$,

$$cel(u^*v) \leq 8\pi + L/k + \varepsilon.$$

Moreover, there is $y \in U_0(A)$ with

$$\mathrm{cel}(y) < L/k + \varepsilon$$

such that $\overline{u^*v} = \bar{y}$ in $U(A)/CU(A)$.

Theorem 7.9. *Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$ and $u \in U_0(A)$. Suppose that $u^k \in CU(A)$ for some integer $k > 0$, then $u \in CU(A)$. In particular, $U_0(A)/CU(A)$ is torsion free.*

Some further references: [45], [46], [38] and [39].

8 A uniqueness theorem

An easily neglected fact used to obtain Theorem 5.2 from 5.1 is the following well-known fact.

Proposition 8.1. *Let F be a finite dimensional C^*-algebra and B be a unital C^*-algebra of stable rank one. If $\phi_1, \phi_2 : F \to B$ are two unital monomorphisms such that*

$$(\phi_1)_* = (\phi_2)_*$$

as homomorphisms from $K_0(F)$ to $K_0(B)$, then ϕ_1 and ϕ_2 are unitarily equivalent.

This is no long true if we replace F by, say $C([0,1])$ or $M_k(C([0,1])$, and even if we also replace unitary equivalence by approximate unitary equivalence. Obviously, in order to establish a uniqueness theorem for simple C^*-algebras with tracial rank one, one has to deal with this problem. Given two positive elements $a_1, a_2 \in B$ with $sp(a_1) = sp(a_2) = [0,1]$, when they are approximately unitarily equivalent? In general, this is hopeless.

But we have the following:

Lemma 8.2. *Let $B = \oplus_{i=1}^k B_i$ be a unital C^*-algebra with $B_i = M_{R(i)}(C(X_i))$, where $X_i = [0,1]$ or X_i is a point .*

For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset B$ and any integer $L > 0$, there exist a finite subset $\mathcal{G} \subset B$ depending on ε and $\mathcal{F}$ but not L, and $\delta = 1/4L$ such that the following holds.

If A is a unital separable nuclear simple C^*-algebra with $TR(A) \leq 1$ and $\phi_i : B \to A$ are two homomorphisms satisfying the following:

(i) *there are $a_{g,i}, b_{g,j} \in A, i, j \leq L$ with*

$$\|\sum_i a_{g,i}^* \phi_1(g) a_{g,i} - 1_A\| < 1/16 \quad \text{and}$$

$$\|\sum_j b_{g,j}^* \phi_2(g) b_{g,j} - 1_A\| < 1/16$$

for all $g \in \mathcal{G}$;

(ii) $(\phi_1)_* = (\phi_2)_*$ *on $K_0(B)$; and,*

(iii) *if $\|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)\| < \delta$ for all $g \in \mathcal{G}$,*

then there exists a unitary $u \in A$ such that

$$\|\phi_1(f) - u^* \phi_2(f) u\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

From the above we obtain the following theorem which is an approximate version of 8.1

Theorem 8.3. *Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and C be a C^*-subalgebra of A with the form*
$C = \oplus M_{R(i)}(C(X_i))$, where $X_i = [0,1]$, or X is a point. Then for any finite subset $\mathcal{F} \subset C$ and $\varepsilon > 0$, there exist $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following:

if $L_1, L_2 : A \to B$ are two unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps, where B is a unital simple C^*-algebra with $TR(B) \leq 1$, with $(L_1|_C)_* = (L_2|_C)_*$ on $K_0(C)$ and

$$|\tau(L_1(g)) - \tau \circ L_2(g)| < \sigma$$

for all $g \in \mathcal{G}$ and for all $\tau \in T(B)$, then there is a unitary $u \in A$ such that

$$\|L_1(f) - u^* L_2(f) u\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

An easy version of 8.2 is the following.

Theorem 8.4. *Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and $B = \oplus M_{R(i)}(C(X_i))$ with $X_i = [0,1]$, or X_i is a point. Let $\phi_i : B \to A$ be two monomorphisms such that*

$$(\phi_1)_* = (\phi_2)_* : K_0(B) \to K_0(A) \text{ and}$$

$$\tau \circ \phi_1 = \tau \circ \phi_2$$

for all $\tau \in T(A)$.

Then there is a sequence of unitaries $u_n \in A$ *such that*

$$\lim_{n\to\infty} u_n^* \phi_1(x) u_n = \phi_2(x) \text{ for all } x \in B.$$

For the uniqueness theorem, we begin with the following. The proof of it depends on an approximate version of 5.1 and results in section 7 such as 7.6 (see also [18]).

Theorem 8.5. *Let* A *be a unital separable simple amenable* C^**-algebra which satisfies the Universal Coefficient Theorem and* $\mathbf{L} : U(M_\infty(A)) \to \mathbf{R}_+$ *be a map. For any* $\varepsilon > 0$ *and any finite subset* $\mathcal{F} \subset A$ *there exist a positive number* $\delta > 0$, *a finite subset* $\mathcal{G} \subset A$, *a finite subset* $\mathcal{P} \subset \mathbf{P}(A)$ *and an integer* $n > 0$ *satisfying the following: for any unital simple* C^**-algebra* B *with* $TR(B) \le 1$, *if* ϕ, ψ, $\sigma : A \to C$ *are three* $\mathcal{G}$*-*δ*-multiplicative contractive completely positive linear maps with*

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},$$

$$\mathrm{cel}(\tilde{\phi}(u)^*\tilde{\psi}(u)) \le \mathbf{L}(u)$$

for all $u \in U(A) \cap \mathcal{P}$ *and* σ *is unital,*

then there is a unitary $u \in M_{n+1}(B)$ *such that*

$$u^* \mathrm{diag}(\phi(a), \sigma(a), \cdots, \sigma(a)) u \approx_\varepsilon \mathrm{diag}(\psi(a), \sigma(a), \cdots, \sigma(a))$$

for all $a \in \mathcal{F}$, *where* $\sigma(a)$ *is repeated* n *times.*

If $TR(A) \le 1$, one can absorb the map σ. To control $\mathrm{cel}(\tilde{\phi}(u)^*\tilde{\psi}(u))$, on the other hand, is entirely a different matter. We found when $K_1(A)$ has no infinite cyclic part, a uniqueness theorem could be easily stated and not too difficult to obtain from the above and Theorem 8.3. Note that tracial information becomes a part of invariant.

Theorem 8.6. *Let* A *be a unital separable simple* C^**-algebra with* $TR(A) \le 1$ *and with torsion* $K_1(A)$. *For any* $\varepsilon > 0$ *and any finite subset* $\mathcal{F} \subset A$ *there exist* $\delta > 0$, $\sigma > 0$, *a finite subset* $\mathcal{P} \subset \mathbf{P}(A)$ *and a finite subset* $\mathcal{G} \subset A$ *satisfying the following:*

for any unital simple C^**-algebra* B *with* $TR(B) \le 1$, *any two* $\mathcal{G}$*-*δ*-multiplicative completely positive linear contractions* $L_1, L_2 : A \to B$ *with*

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}$$

and

$$\sup_{\tau \in T(B)} \{|\tau \circ L_1(g) - \tau \circ L_2(g)|\} < \sigma$$

for all $g \in \mathcal{G}$, *there exists a unitary* $U \in B$ *such that*

$$ad(U) \circ L_1 \approx_\varepsilon L_2 \text{ on } \mathcal{F}.$$

An immediate consequence of the above is the following.

Theorem 8.7. *Let* A *be a unital amenable simple* C^**-algebra with* $TR(A) \le 1$ *and with torsion* $K_1(A)$ *which satisfies the UCT. Then an automorphism* $\alpha : A \to A$ *is approximately inner if and only if* $[\alpha] = [\mathrm{id}_A]$ *in* $KL(A, A)$ *and* $\tau \circ \alpha(x) = \tau(x)$ *for all* $x \in A$ *and* $\tau \in T(A)$.

Some further references: [22] and [23]

9 An existence theorem

Since in 8.6, tracial information is needed in the uniqueness theorem, in the statement of existence theorem, one also needs to match the required tracial information. Theorem 9.3 is the first step in that direction.

Definition 9.1. Let A and B be two unital stably finite C^*-algebras and let $\alpha : K_0(A) \to K_0(B)$ be a positive homomorphism and $\Lambda : T(B) \to T(A)$ be a continuous affine map. We say Λ is compatible to α if $\Lambda(\tau)(x) = \tau(\alpha(x))$ for all $x \in K_0(A)$, where we view τ as a state on $K_0(A)$.

Let S be a compact convex set. Denote by $Aff(S)$ the set of all (real) continuous affine functions on S. Let $\Lambda : S \to T$ be a continuous affine map from S to another compact convex set T. We denote by $\Lambda_\natural : Aff(T) \to Aff(S)$ the unital positive linear continuous map defined by $\Lambda_\natural(f)(s) = f(\Lambda(s))$ for $f \in Aff(T)$.

Definition 9.2. A positive linear map $\xi : AffT(A) \to AffT(B)$ is said to be compatible to α if $\xi(\hat{p})(\tau) = \tau(\alpha(p))$ for all $\tau \in T(B)$ and any projection $p \in M_\infty(A)$.

Let A be a unital C^*-algebra (with at least one normalized trace). Define $Q : A_{sa} \to AffT(A)$ by $Q(a)(\tau) = \tau(a)$ for $a \in A$. Then Q is a unital positive linear map.

Theorem 9.3. *Let $A = M_k(C([0,1]))$, let B be a unital separable nuclear simple C^*-algebra with $TR(B) \le 1$, let $\gamma : K_0(A) \to K_0(B)$ be a positive homomorphism and let $\Lambda : T(B) \to T(A)$ be an affine continuous map which is compatible to γ.*

Then, for any $\sigma > 0$ and any finite subset $\mathcal{G} \subset A$, there exists a unital monomorphism $\phi : A \to B$ such that

$$\sup_{\tau \in T(B)} \{|\tau \circ \phi(g) - \Lambda(\tau)(g)|\} < \sigma$$

for all $g \in \mathcal{G}$ and $\phi_* = \gamma$.

To construct a map with given "KK-data," we use the known result for simple C^*-algebras with tracial rank zero. The strategy is first to map the given unital simple C^*-algebra A with $TR(A) \le 1$ to a unital simple C^*-algebra C with $TR(C) = 0$ whose scaled order K-groups are the same as that of B. We then maps C to B. To this end, we begin with the following.

Proposition 9.4. *Let B be a unital separable amenable simple C^*-algebra with $TR(B) \le 1$. Then there exists a unital separable amenable simple C^*-algebra C with $TR(C) = 0$ such that*

$$(K_0(C), K_0(C)_+, [1_C], K_1(C)) = (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

We then to establish the following.

Lemma 9.5. *Let A and B be unital separable nuclear simples C^*-algebra with $TR(A) \le 1$ and $TR(B) \le 1$ satisfying the UCT such that*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)). \tag{e1}$$

Suppose that there exists a unital separable nuclear simple C^*-algebra C with $TR(C) = 0$ satisfying UCT and the following:

$$(K_0(C), K_0(C)_+, [1_C], K_1(C)) = (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Then there exists a sequence of contractive completely positive linear maps $\Phi_n : A \to C$ such that (i) $\lim_{n\to\infty} \|\Phi_n(ab) - \Phi_n(a)\Phi_n(b)\| = 0$ *for $a, b \in A$,*

(ii) *For each finite subset $\mathcal{P} \subset \mathbf{P}(A)$ there exists an integer $N > 0$ such that*

$$[\Phi_n]|_{\mathcal{P}} = [\alpha]|_{\mathcal{P}}$$

for all $n \geq N$, where $\alpha \in KL(A, B)$ which gives an identification in (e 1) above.

We then combine 9.5, with 9.4 and 9.3 to prove the following.

Theorem 9.6. *Let A and B be two unital separable amenable simple C^*-algebras with $TR(A) \leq 1$ and $TR(B) \leq 1$ satisfying AUCT such that*

$$(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)) = (K_0(A), K_0(A), [1_A], K_1(A), T(A)). \tag{e 2}$$

Then there is a sequence of contractive completely positive linear maps $\{\Psi_n\}$ from A to B such that

(i) $\lim_{n\to\infty} \|\Psi_n(ab) - \Psi_n(a)\Psi_n(b)\| = 0$ *for all $a, b \in A$,*

(ii) *for any finite subset set $\mathcal{P} \subset \mathbf{P}(A)$,*

$$[\Psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}},$$

for all sufficiently large n, where $\alpha \in KL(A, B)$ gives the identification on K-theory in (e 2) *and*

(iii)

$$\lim_{n\to\infty} \sup_{\tau\in T(B)} \{|\tau \circ \Psi_n(a) - \xi(Q(a))(\tau)|\} = 0$$

for all $a \in A_{s.a}$, where $\xi : AffT(A) \to AffT(B)$ is the affine isometry given above (in (e 2)*).*

Some further references: [13] and [46]

10 The classification theorem

Now, by applying, again, the Elliott approximate intertwining argument, and by combing the uniqueness theorem (8.6) and the existence theorem (9.6), we establish the following classification theorem.

Theorem 10.1. *Let A and B be two unital separable simple amenable C^*-algebras with $TR(A) \leq 1$ and $TR(B) \leq 1$ which satisfy the UCT. Suppose that $K_1(A)$ and $K_1(B)$ are torsion. Then $A \cong B$ if and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)).$$

For general case, there is no difficulty to state a right uniqueness theorem which can be easily derived from Theorem 8.5. Let A be a unital separable simple amenable C^*-algebra with infinite cyclic elements in $K_1(A)$. Then tracial information provides only a portion of the information about the group $U(A)/CU(A)$.

Inevitably, controlling exponential length of homomorphisms becomes difficult. Since maps that provided by 9.3 are not even multiplicative, controlling the exponential length becomes even messy.

Given a unitary $u \in U(A) \setminus U_0(A)$, there is nothing to measure the "length" of u or $\phi(u)$, where $\phi : A \to B$ is a map provided by 9.3 since they do not connect to the identity. So one cannot choose ϕ to meet the requirement of controlling exponential length. The length issue comes when we have the second map $\psi : B \to A$. At that point, we need to control $\mathrm{cel}(\psi \circ \phi(u)u^*)$. If $K_1(A)$ (and $K_1(B)$) is a torsion group, with the tracial information together with Theorem 7.8, $\mathrm{cel}(\psi \circ \phi(u)u^*)$ is already under control. However, in general, there is nothing one can say about $\mathrm{cel}(\psi \circ \phi(u)u^*)$. What we need is another type of existence theorem which can alter the known length of $(\psi \circ \phi(u)u^*)$ so that it can be bounded by a per-determined bound. The results in section 7 helps but not sufficient. In the actual proof of Theorem 10.2 below we will map A into an AH-algebra and control the exponential length there. A few things have to be done before this could be made possible. While the structure of $U(A)/CU(A)$ is heavily used in the proof of the following theorem, it should be noted that $U(A)/CU(A)$ is not used as part of the isomorphic invariant in the statement.

Theorem 10.2. ([31]) *Let A and B be two unital separable simple amenable C^*-algebras with $TR(A) \leq 1$ and $TR(B) \leq 1$ which satisfy the UCT. Then $A \cong B$ if and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)).$$

Some further references: [47] and [48].

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