数理解析研究所講究録 1435 巻 2005 年 39-43

CLASSIFICATION AND $\mathcal{Z}$-STABILITY

NATHANIAL P. BROWN

ABSTRACT. In this note we survey the current state of the classification program with special emphasis on the real rank zero case. We try to argue that a better understanding of $\mathcal{Z}$-stable algebras – those that tensorially absorb the Jiang-Su algebra – would lead to significant new results.

1. INTRODUCTION

In this short survey I will give a very biased and subjective view of the current state of Elliott's classification program. I will not prove any new results, but I think the spin may have some novelty. Perhaps the main point I wish to make is that recent counterexamples of Andrew Toms have convincingly demonstrated that, like it or not, *some sort of stabilization is required in the classification program.*

Ever since his remarkable paper [15] giving a definitive counterexample to Elliott's conjecture, Mikael Rørdam has been suggesting that we try to classify so-called $\mathcal{Z}$-stable algebras. That is, algebras such that

$$A \otimes \mathcal{Z} \cong A$$

where $\mathcal{Z}$ is the Jiang-Su algebra - the simple, unital, infinite dimensional, nuclear C*-algebra with Elliott invariant isomorphic to that of the complex numbers (cf. [3]). In [2] it was shown that A and $A \otimes \mathcal{Z}$ have isomorphic Elliott invariants if and only if $K_0(A)$ is weakly unperforated (i.e. $n \cdot x > 0$ implies $x > 0$ for all $x \in K_0(A)$). Hence, if A and $A \otimes \mathcal{Z}$ are classified by K-theory and if $K_0(A)$ is weakly unperforated then it necessarily follows that A is $\mathcal{Z}$-stable. Thus Rørdam's suggestion, that we simply assume $\mathcal{Z}$-stability and try to prove classification, is quite natural since $\mathcal{Z}$-stability is a necessary condition for classification (in the weakly unperforated case). However, many people (including myself) had psychological objections to assuming $\mathcal{Z}$-stability since we don't know when an algebra satisfies this condition and so it feels unnatural to assume it. On the other hand, Andrew Toms has now forced us to face reality: One *must* assume $\mathcal{Z}$-stability, in general, as there exists a simple, unital, AH algebra A with weakly unperforated $K_0(A)$ (even stable rank one!) but which is not $\mathcal{Z}$-stable (cf. [16],[17]). We don't have to like it but the truth is the truth and there is no hope of classifying the non-$\mathcal{Z}$-stable AH algebras by their Elliott invariants (and $\mathcal{Z}$-stability is not automatic even for simple AH algebras with stable rank one and weakly unperforated K_0-groups).

Given this unfortunate fact of life, my perspective of the classification program has shifted and I want to give my view of 'where we are and where we should go'. As I mentioned above, this is a *very* subjective survey and others in the classification program may disagree with the emphasis (or lack thereof) I put on certain problems and results. My goal is not to start arguments or offend but, rather, to highlight results and directions which strike me as important. (Not surprisingly, the problems I think are most interesting also turn out to be the most "important"! ;-) Moreover, I will concentrate mainly on the real rank zero case so there is no discussion of the existing 'higher rank' classification theorems.

Being lazy, I declare: *all C*-algebras in this note are assumed to be unital, separable, simple and nuclear.* Also, I need to thank Andrew Toms and Wilhelm Winter for sharing preliminary versions of their work with me and making helpful suggestions/corrections to the present article.

2. KIRCHBERG-PHILLIPS CLASSIFICATION

To set the stage, I want to quickly recall the two main steps in the Kirchberg-Phillips classification theorem (cf. [4], [5], [13]). Roughly speaking, the classification of purely infinite C*-algebras follows from two deep results.

Theorem 2.1 ($\mathcal{O}_\infty$-Stable Implies Classifiable). *Assume $A \otimes \mathcal{O}_\infty \cong A$, where $\mathcal{O}_\infty$ denotes the Cuntz algebra with infinitely many generators. Then A is classifiable.*[1]

This survey was written during a year-long visit to The University of Tokyo. I thank them - especially Yasuyuki Kawahigashi and Narutaka Ozawa - for their hospitality.

[1]Recall my declaration on simplicity, nuclearity, etc. Also, see the papers referenced for the precise meaning of 'classifiable' as it depends whether one assumes the UCT or not.

Theorem 2.2 (Purely Infinite Implies $\mathcal{O}_\infty$-Stable). *Assume A is purely infinite. Then stability is automatic – i.e. $A \otimes \mathcal{O}_\infty \cong A$.*

Taken together these two theorems complete the purely infinite case of Elliott's conjecture (modulo the UCT). I have chosen to separate them, however, to help motivate my current view of the stably finite case; *we should concentrate on proving stably finite analogues of these two theorems.*

I can't pretend to know for sure what the 'right' analogues should be, but here are some very general versions. (I will give more tractable specializations later.)

Question 2.3. *Assume A is finite and $\mathcal{Z}$-stable. Is A classifiable?*

As I indicated, this problem has been posed by Mikael Rørdam both privately and publicly (cf. [14]). It is worth noting that even if one starts with a non-$\mathcal{Z}$-stable algebra A it is easy to get a well behaved algebra by replacing A with $A \otimes \mathcal{Z}$. (Jiang and Su showed that $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$ [3] and hence $\mathcal{Z}$-stability is easy to arrange.) I should also mention that Rørdam has shown that finite plus $\mathcal{Z}$-stable implies stable rank one and Blackadar's fundamental comparison property.[2] He can even characterize when such algebras have real rank zero [14]. (These statements depend on our blanket assumption of nuclearity and simplicity.)

Question 2.4. *Find abstract conditions which imply $\mathcal{Z}$-stability.*

Since Toms has shown that $\mathcal{Z}$-stability is not automatic, there is no hope of achieving the ultimate analogue of the 'purely infinite implies $\mathcal{O}_\infty$-stable' theorem above. Hence we have to settle for weaker versions and I will state a few possibilities later on.

3. The Unique Trace Case

In light of Huaxin Lin's celebrated classification theorem for tracially AF algebras (cf. [7]) our first analogues of Theorem 2.1 should be stated in the form 'When does stable imply tracially AF?' Indeed, most of what follows will be a survey of how much progress we have made on this question. This is both a strong indication that the classification of $\mathcal{Z}$-stable algebras (with real rank zero) is possible and will hopefully reinforce the idea that we desperately need to find good analogues of Theorem 2.2.

Let's restrict to the unique trace case and see what the classification program currently looks like. In my opinion the best result we have at the moment is due to Huaxin Lin [8] (though I have my own take on approximation properties of traces in [1]).

Theorem 3.1. *Up to $\mathcal{Z}$-stabilization, the real rank zero, unique trace, inductive limit of type I case of Elliott's program is complete. More precisely, if A is an inductive limit of type I C^*-algbras[3] with unique tracial state τ and for every $\epsilon > 0$ there is a projection $p \in A$ such that $\tau(p) < \epsilon$. Then $A \otimes \mathcal{Z}$ is tracially AF.*

The proof of this result depends on three facts: (a) the type I assumption implies that the unique trace satisfies a very strong approximation property (cf. [8], [1]) and tensoring with $\mathcal{Z}$ forces (b) Blackadar's fundamental comparison property and (c) real rank zero [14].

Here are a few non-trivial corollaries.

Corollary 3.2. *Let A be Villadsen's example of an AH algebra with stable rank greater than 1* [19]. *Then $A \otimes \mathcal{Z}$ is classifiable.*

See Proposition 11 in [19] for a proof that A has projections of arbitrarily small trace. It is admittedly irritating that one must tensor with $\mathcal{Z}$ in order to classify, but Truth is not concerned with human emotions. In other words, several experts believe that Villadsen's examples are not classifiable (by their Elliott invariants) and if this turns out to be correct then the theorem above will be best possible. We don't have to like it, but that is life.

In the absence of projections it is hard for me to imagine that we will prove very general classification results any time soon. For example, if A is an inductive limit of type I algebras with unique trace and few (or no) projections then I don't see any reasonable strategy for classifying $A \otimes \mathcal{Z}$ at the present time. On the other hand, if one allows a weaker stability then the classification program is already complete. Say that A is *rationally stable* if $A \cong A \otimes \mathcal{U}$, where $\mathcal{U}$ is the UHF algebra whose K_0-group is isomorphic to the rational numbers $\mathbb{Q}$.

[2] This means that if p, q are projections and $\tau(p) < \tau(q)$ for all traces then p is equivalent to a subprojection of q.

[3] Weaker hypotheses like 'locally type I' or 'tracially type I' also suffice.

Corollary 3.3. *Up to rational stabilization, the inductive limit of type I, unique trace case of Elliott's program is complete. That is, if A is an inductive limit of type I algebras with unique trace then $A \otimes \mathcal{U}$ is tracially AF.*

Thanks to the deep work of Q. Lin and Phillips [10] we thus have the following corollary.

Corollary 3.4. *Let $h : M \to M$ be a minimal diffeomorphism of a compact manifold M and assume that h has a unique invariant measure. Then $(C(M) \rtimes_h \mathbb{Z}) \otimes \mathcal{U}$ is tracially AF.*

I know you don't want to tensor with a UHF algebra and I too would be thrilled if we only had to tensor with $\mathcal{Z}$. However, as I said above, it is hard for me to imagine how the proof would go so I am quite happy with the fact that, up to rational stabilization, this case of the program is finished. Of course, the results presented so far beg the following two questions.

Question 3.5. *Assume A is an inductive limit of type I algebras with unique trace and real rank zero. Is A automatically $\mathcal{Z}$-stable?*

Note that this question, if answered affirmatively, would imply that there are no Villadsen type examples with real rank zero and hence this is a non-trivial question. Indeed, even the AH case (rather than general type I) would be of significant interest.

Question 3.6. *Can one construct an inductive limit of type I algebras which has a unique trace and is $\mathcal{Z}$-stable but which is not classifiable?*

Doing this is probably quite hard as it would require, among other things, inventing a new invariant (all the usual suspects are well-behaved for $\mathcal{Z}$-stable algebras). If such counterexamples exist then it would show that the 'rational stabilization' theorem we already have is actually the best possible result in general. I am not suggesting that such counterexamples 'should' exist (indeed, I have no clue) I am just pointing out that it is possible our existing 'classification up to stabilization' theorems might be best possible for the general class of inductive limits of type I algebras with unique trace.

Finally, I will mention that one can formulate similar results under the weaker (but still quite restrictive) assumption that there are only countably many extreme traces. Passing from one trace to a countable number of extreme traces is just a technical argument (cf. [8]); the real challenge is passing to arbitrary tracial spaces.

4. General Tracial State Spaces

Now I want to discuss what can be said in the case of arbitrary tracial state spaces. The results are quite nice as are the remaining problems. In my opinion, there are two results which deserve special recognition. The first has to do with AH algebras while the second treats finite decomposition rank in the sense of Kirchberg-Winter.

If we combine the results of [9], [11] and [14] then we get a very satisfactory result in the case of AH algebras with real rank zero. I must emphasize that *there are no restrictions on dimension growth or the topology of the base spaces*[4] in the following theorem.

Theorem 4.1. *Up to $\mathcal{Z}$-stabilization, the real rank zero AH case of Elliott's program is complete. In other words, if A is an AH algebra*[5] *with real rank zero then $A \otimes \mathcal{Z}$ is tracially AF.*

Proof. Wilhelm Winter pointed out that there was a small gap in my original 'proof' of this result. Namely, I asserted that it followed immediately from the three papers cited above and he correctly pointed out that it does not! Oops! ;-) (Thanks Wilhelm.) Contrary to my lazy blanket assumption, algebras appearing in this proof are not necessarily simple.

Appealing to [11], we must show that $A \otimes \mathcal{Z}$ is locally approximated by subalgebras which (a) have Hausdorff spectrum and (b) are subhomogeneous. Since $\mathcal{Z}$ is an inductive limit of prime dimension drop algebras and A is locally approximated by homogeneous algebras, it suffices to establish two general facts: First we must show that prime dimension drop algebras have Hausdorff spectrum and, second, that the tensor product of two algebras with Hausdorff spectrum also has Hausdorff spectrum.

Recall that if p and q are relatively prime integers then the associated prime dimension drop algebra is defined as

$$\mathcal{I}(p, q) = \{f \in C([0, 1], M_p(\mathbb{C}) \otimes M_q(\mathbb{C})) : f(0) \in M_p(\mathbb{C}) \otimes 1 \text{ and } f(1) \in 1 \otimes M_q(\mathbb{C})\}.$$

[4] In other words, the building blocks are corners of algebras of the form $M_n(C(X))$ where X is any compact metric space – possibly even infinite dimensional!

[5] Even 'locally AH' is sufficient – see [11].

A moments thought reveals that the center of $\mathcal{I}(p,q)$ is isomorphic to $C([0,1])$. A bit more contemplation and one realizes that for each primitive ideal $J \lhd C([0,1])$, the ideal $J\mathcal{I}(p,q) \lhd \mathcal{I}(p,q)$ is also primitive and, moreover, this gives a bijective map $\mathrm{Prim}(C([0,1])) \to \mathrm{Prim}(\mathcal{I}(p,q))$ of primitive ideal spaces. (Surjectivity follows from the fact that $\mathcal{I}(p,q)/J\mathcal{I}(p,q)$ is a full matrix algebra.) That this map is a homeomorphism follows easily from the definition of the closure of a set of primitive ideals. Hence the spectrum of a prime dimension drop algebra is homeomorphic to $[0,1]$.

To show that Hausdorff spectrum is preserved under taking tensor products it suffices to recall the general fact that if A and B are unital type I C*-algebras then

$$\widehat{A \otimes B} \cong \hat{A} \times \hat{B}.$$

This result can be found in [21]; for convenience we sketch a proof of the case we are interested in.

Assume both A and B have Hausdorff spectrum. Since $\widehat{A \otimes B}$ is compact (our algebras are unital) and $\hat{A} \times \hat{B}$ is Hausdorff it suffices to show the existence of a continuous bijection

$$\widehat{A \otimes B} \to \hat{A} \times \hat{B}.$$

If $\pi : A \otimes B \to B(H)$ is an irreducible representation then both $\pi(A \otimes 1)''$ and $\pi(1 \otimes B)''$ must be factors; since A and B are type I, they must be full matrix algebras or $B(K)$ for some infinite dimensional Hilbert space K. In any case it follows that π is unitarily equivalent to the tensor product of two irreducible representations and this evidently yields a bijective map $\widehat{A \otimes B} \to \hat{A} \times \hat{B}$. The definition of convergence of nets of irreducible representations readily implies continuity of this map.[6] □

I should mention that one need not tensor with $\mathcal{Z}$ if A is known to satisfy Blackadar's fundamental comparison property. Hence the following questions seem quite natural.

Question 4.2. *Does every AH algebra with real rank zero satisfy Blackadar's fundamental comparison property?*

Question 4.3. *Is every AH algebra with real rank zero automatically $\mathcal{Z}$-stable?*

Affirmative answers to these questions would, of course, be major contributions to the classification program. For example, it would follow that real rank zero AH algebras always have weakly unperforated K_0-groups and stable rank one (without assumptions on dimension growth or topology of base spaces!). I hope you will now agree that we have significant motivation to launch a full scale attack on finite analogues of Theorem 2.2. I am not suggesting this is a simple problem, but I am saying that major cases of the classification program will be complete as soon as we do.

In another possible direction, it would be very nice to relax the AH assumption above to allow for general subhomogeneous algebras or, better yet, type I.

Question 4.4. *If A is an inductive limit of type I algebras (or just subhomogeneous algebras) and has real rank zero then does it follow that $A \otimes \mathcal{Z}$ is tracially AF?*

Finally, I want to state a theorem which was recently proved by Wilhelm Winter [20]. For the definition of *decomposition rank* we refer to the original paper of Kirchberg-Winter [6]. For our purposes it suffices to say that this is a generalization of classical covering dimension. However, the point to emphasize is that the definition *does not assume any sort of inductive limit decomposition or tracial approximation by 'tractable' subalgebras.* It is an abstract hypothesis which is closely related to quasidiagonality. As such, I find the following result particularly attractive. Moreover, it completes the classification of inductive limits of recursive subhomogeneous algebras with no dimension growth, at least up to $\mathcal{Z}$-stabilization (cf. [12, Conjecture 4.6]).

Theorem 4.5. *Assume A has real rank zero and finite decomposition rank. Then $A \otimes \mathcal{Z}$ is tracially AF.*

As before, this result dares us to tackle the following question.

Question 4.6. *Is every algebra with real rank zero and finite decomposition rank automatically $\mathcal{Z}$-stable?*

[6] Recall that $\pi_\lambda \to \pi$ in $\hat{C}$ if and only if $\liminf \|\pi_\lambda(c)\| \geq \|\pi(c)\|$ for all $c \in C$.

5. Summary

Please forgive the repetition, but I will summarize.

Unfortunate Fact: Andrew Toms has conclusively demonstrated that it is necessary to settle for classification theorems 'up to stabilization'. This is the best we can ever hope for since he has constructed an AH algebra with weakly unperforated K_0-group but which is not $\mathcal{Z}$-stable.

Fortunate Fact: Combining the work of many hands, most notably Huaxin Lin, we have now completed a number of interesting cases of Elliott's program 'up to $\mathcal{Z}$-stabilization'. The remaining problems are probably not easy, but they have been isolated and we know where to focus.

Future Fact(?): Somebody will prove a wonderful theorem which shows that $\mathcal{Z}$-stability is automatic for large classes of (real rank zero) algebras thus completing a vast swath of the classification program.

See [18, Sections 2 and 3] for general results which may help with this future fact.

References

1. N.P. Brown, *Invariant means and finite representation theory of C^*-algebras*, preprint.
2. G. Gong, X. Jiang and H. Su, *Obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras* Canad. Math. Bull. **43** (2000), 418–426.
3. X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, Amer. J. Math. **121** (1999), 359–413.
4. E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theory*, preprint.
5. E. Kirchberg and N.C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine Angew. Math. **525** (2000), 17–53.
6. E. Kirchberg and W. Winter, *Covering dimension and quasidiagonality*, Internat. J. Math. **15** (2004), 63–85.
7. H. Lin, *Classification of simple C^*-algebras of tracial topological rank zero*, Duke Math. J. **125** (2004), 91–119.
8. ———, *Traces and simple C^*-algebras with tracial topological rank zero*, J. Reine Angew. Math. **568** (2004), 99 – 137.
9. ———, *Simple AH-algebras of real rank zero*, Proc. Amer. Math. Soc. **131** (2003), 3813–3819.
10. Q. Lin and N.C. Phillips, *The structure of C^*-algebras of minimal diffeomorphisms*, in preparation.
11. P.W. Ng, *Simple real rank zero algebras with locally Hausdorff spectrum*, preprint.
12. N.C. Phillips, *Real rank and property (SP) for direct limits of recursive subhomogeneous algebras*, preprint.
13. ———, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114.
14. M. Rørdam, *The stable and real rank of $\mathcal{Z}$-absorbing C^*-algebras*, International J. Math. **15** (2004), 1065-1084.
15. ———, *A simple C^*-algebra with a finite and an infinite projection*, Acta Math. **191** (2003), 109–142.
16. A.S. Toms, *On the independence of K-theory and stable rank for simple C^*-algebras*, J. Reine Angew. Math. (to appear).
17. ———, *On the Classification problem for nuclear C^*-algebras*, preprint.
18. A.S. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*, preprint.
19. J. Villadsen, *On the stable rank of simple C^*-algebras*, J. Amer. Math. Soc. **12** (1999), 1091–1102.
20. W. Winter, *On the classification of simple $\mathcal{Z}$-stable algebras with real rank zero and finite decomposition rank*, preprint.
21. A. Wolfsohn, *The primitive spectrum of a tensor product of C^*-algebras*, Proc. Amer. Math. Soc. **19** (1968), 1094–1096.

Department of Mathematics, Penn State University, State College, PA 16802. Currently visiting The University of Tokyo.

E-mail address: nbrown@math.psu.edu or nbrown@ms.u-tokyo.ac.jp