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Quasi-variational inequalities for phase transitions

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Abstract. In this paper, we consider a quasi-variational problem for irreversible phase change with temperature. We propose a mathematical model of a class of irreversible phase change described by a system of PDEs including a quasi-variational inequality. One of our interests is an existence of solutions of this system. The existence of solutions is obtained as a limit of approximate solutions. Our approximate problems are formulated by using the Moreau-Yosida approximation. The convergence of approximate solutions is based on some uniform estimates and monotonicity techniques in the nonlinear operator theory.

1. Introduction

We consider the following system of PDEs:

$$\theta_t + w_t - \Delta\theta = h(t,x) \quad \text{in } Q := (0,T)\times\Omega, \tag{1.1}$$

$$w_t + \partial I_{\theta,N}(w_t) - \nu\Delta w \ni f(\theta,w) \quad \text{in } Q, \tag{1.2}$$

subject to the boundary conditions

$$\frac{\partial\theta}{\partial n} = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma := (0,T)\times\Gamma \tag{1.3}$$

and the initial conditions

$$\theta(0,\cdot) = \theta_0, \quad w(0,\cdot) = w_0 \quad \text{in } \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbf{R}^3$ with smooth boundary Γ, T is a finite positive number, ν is a positive constant, θ_t and w_t are the time derivatives of θ and w, Δ denotes the Laplace operator in space variable x and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on Γ; f is a given function on $\mathbf{R}^2$, h is a given function on Q; θ_0 and w_0 are the initial data of θ and w, respectively; $I_{\theta,N}(\cdot)$ is the indicator function of the interval $[g(\theta), g(\theta)+N]$ with a non-negative bounded smooth function g on $\mathbf{R}$ and a sufficiently large positive number N;

$$I_{\theta,N}(w_t) := \begin{cases} +\infty & \text{if } w_t < g(\theta) \text{ or } g(\theta)+N < w_t, \\ 0 & \text{if } g(\theta) \le w_t \le g(\theta)+N; \end{cases}$$

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$\partial I_{\theta,N}(w_t)$ is the subdifferential with respect to w_t, namely, it is a set-valued mapping defined by

$$\partial I_{\theta,N}(w_t) := \begin{cases} \emptyset & \text{if } w_t < g(\theta) \text{ or } g(\theta)+N < w_t, \\ (-\infty, 0] & \text{if } w_t = g(\theta), \\ \{0\} & \text{if } g(\theta) < w_t < g(\theta)+N, \\ [0,+\infty) & \text{if } w_t = g(\theta)+N. \end{cases}$$

For instance, in the context of solidification of multi-composite materials, the unknowns θ and w of the system $(P) := \{(1.1)\text{-}(1.4)\}$ are explored, respectively, as the temperature and the irreversible solidification parameter. (w is often called a phase change parameter or an order parameter.) Since the mapping $\partial I_{\theta,N}(\cdot)$ in (1.2) requires that w_t is within $(0 \le)\ g(\theta) \le w_t \le g(\theta)+N$, our system possibly describes the irreversibility effect. As for a mathematical treatment of irreversible phase change, there are some related works [3,6,7,8,13,15,16] and so on, however, in any case the restriction of w_t does not depend on the unknown functions θ and w. In our setting, $\partial I_{\theta,N}(\cdot)$ depends on the unknown function θ, which is one of new aspects of our work.

In this paper, we give an existence result for the system (P) under some assumptions on the data f, g, h, θ_0, w_0. Concerning the system (P) we have already discussed the case when Ω is a bounded domain in $\mathbf{R}^2$ (cf.[3]). In the paper of [3], we used the abstract quasi-variational evolution inequality established in [2] to get approximate solutions:

$$\partial\phi_{u(t)}(u'(t)) + \partial\psi(u(t)) \ni G(t,u(t)) \ \text{ in } \ X \ \text{ for a.e. } t \in (0,T),$$

where X is a real Hilbert space, ϕ_u is a proper lower semi-continuous convex function on X for each $u \in D(\psi) := \{z \in X; \psi(z) < +\infty\}$, ψ is a proper lower semi-continuous convex function on X, $\partial\phi_u$ and $\partial\psi$ are their subdifferentials in X, G is a single-valued operator from X into itself. Since we cannot apply such a procedure to (P), we shall employ a fixed point argument to construct approximate solutions of (P) and obtain a solution of (P) by showing their convergence.

Throughout this paper, H denotes the real Hilbert space $L^2(\Omega)$ with the usual inner product $(\cdot,\cdot)$ and V denotes the Sobolev space $H^1(\Omega)$ and it is a Hilbert space equipped with the following inner product:

$$(z,v)_V := (z,v) + a(z,v), \quad a(z,v) := \int_\Omega \nabla z(x)\cdot\nabla v(x)dx, \quad \forall z,v \in V$$

and norm $|z|_V := \sqrt{(z,z)_V}$. We use the notation Δ_0 to indicate the operator Δ with homogeneous Neumann boundary condition; note here that $-\Delta_0$ is linear, closed, nonnegative and self-adjoint in H; in fact, we have

$$D(-\Delta_0) = \left\{ z \in H^2(\Omega); \frac{\partial z}{\partial n} = 0 \ \text{ in } H^{\frac{1}{2}}(\Gamma) \right\}$$

and

$$-\Delta_0 z = -\Delta z \ \text{ in } \ H, \quad \forall z \in D(-\Delta_0).$$

Notation $|\cdot|_\infty$ stands for various L^∞-norms, for instance, $L^\infty(Q), L^\infty(\Omega)$ and so on.

Next we recall some basic properties on convex functions and their subdifferentials in a real Hilbert space; precisely see [4,5,9,12]. Let W be a real Hilbert space with inner product $(\cdot,\cdot)_W$ and norm $|\cdot|_W$. Let φ be a proper lower semi-continuous and convex function on W. The subset $D(\varphi) = \{z \in W; \varphi(z) < +\infty\}$ of W is called the effective domain of φ. The subdifferential $\partial\varphi$ of φ is a set-valued operator from W into itself defined by

$$z^* \in \partial\varphi(z) \underset{\text{def}}{\Longleftrightarrow} (z^*, v - z)_W \le \varphi(v) - \varphi(z), \quad \forall v \in W \tag{1.5}$$

and its domain is defined by $D(\partial\varphi) := \{z \in W; \partial\varphi(z) \neq \emptyset\}$. For each $\varepsilon > 0$, we define $J_\varepsilon^\varphi := (I + \varepsilon\partial\varphi)^{-1}$ which is called the resolvent of $\partial\varphi$, where I is the identity operator in W. The Moreau-Yosida approximation φ_ε of φ and its subdifferential $\partial\varphi_\varepsilon$ are defined by

$$\varphi_\varepsilon(z) = \inf_{v\in W}\left\{\frac{1}{2\varepsilon}|z - v|_W^2 + \varphi(v)\right\}, \quad \partial\varphi_\varepsilon(z) := \frac{z - J_\varepsilon^\varphi z}{\varepsilon}, \quad \forall z \in W. \tag{1.6}$$

Concerning the Moreau-Yosida approximation and the resolvent J_ε^φ, the following facts are often used in this paper:

$$\varphi_\varepsilon(z) = \varphi(J_\varepsilon^\varphi z) + \frac{1}{2\varepsilon}|z - J_\varepsilon^\varphi z|_W^2, \quad \forall\varepsilon > 0, \ \forall z \in W, \tag{1.7}$$

$$\varphi(J_\varepsilon^\varphi z) \le \varphi_\varepsilon(z) \le \varphi(z), \quad \lim_{\varepsilon\searrow 0}\varphi_\varepsilon(z) = \varphi(z), \quad \forall\varepsilon > 0, \ \forall z \in W, \tag{1.8}$$

$$|\partial\varphi_\varepsilon(z)|_W \le |\partial\varphi(z)| := \inf\{|w|_W; w \in \partial\varphi(z)\}, \quad \forall\varepsilon > 0, \ \forall z \in D(\partial\varphi). \tag{1.9}$$

Especially, in the case that Ω is a bounded domain with smooth boundary, for every $z \in H$, define

$$\varphi(z) = \begin{cases} \frac{1}{2}a(z,z) & \text{if } z \in V, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.10}$$

Then φ is a proper lower semi-continuous and convex function on H and $\partial\varphi = -\Delta_0$ in H (cf. [5]).

2. Main result

We make the following assumptions on the data:

(1) f is a Lipschitz continuous function from $\mathbf{R}^2$ into $\mathbf{R}$ and g is a non-negative function of C^2-class from $\mathbf{R}$ into itself such that the derivatives g' and g'' are bounded on $\mathbf{R}$.

(2) $h \in L^\infty(Q)$, $\theta_0 \in V \cap L^\infty(\Omega)$ and $w_0 \in D(-\Delta_0)$.

Now we give the definition of a solution of (P).

Definition 2.1. A pair of functions $\{\theta, w\}$ is called a solution of (P) if it satisfies the following conditions $(a1)$-$(a4)$:

(a1) $\theta,\ w \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$.

(a2) $\theta'(t) + w'(t) - \Delta_0\theta(t) = h(t)$ in H for a.e. $t \in (0,T)$.

(a3) There exists a function $\xi \in L^2(0,T;H)$ with $\xi \in \partial I_{\theta,N}(w')$ a.e. on Q such that

$$w'(t) + \xi(t) - \nu\Delta_0 w(t) = f(\theta(t), w(t)) \ \text{ in } H \ \text{ for a.e. } t \in (0,T).$$

(a4) $\theta(0) = \theta_0$ and $w(0) = w_0$ in H.

We denote the time-derivatives of θ and w by θ' and w', respectively.

Theorem 2.1. *Under the assumptions* (1) *and* (2), *problem* (P) *has at least one solution* $\{\theta, w\}$ *in the sense of Definition 2.1 such that* $\theta \in L^\infty(Q)$ *and* $w \in W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega))$.

The above existence result will be proved in the sections 3, 4 and 5.

3. Approximate problem

In this section, we consider the following approximate problem $(P_\varepsilon) := \{(3.1)\text{-}(3.3)\}$:

$$\theta_t + w_t - \Delta_0\theta = h \quad \text{in } Q, \tag{3.1}$$

$$w_t + \partial I_{\theta,N}(w_t) + \nu\partial\varphi_\varepsilon(w) \ni f(\theta, J_\varepsilon^\varphi w) \quad \text{in } Q, \tag{3.2}$$

$$\theta(0,\cdot) = \theta_0, \quad w(0,\cdot) = w_0 \ \text{ in } \Omega, \tag{3.3}$$

where φ_ε is the Moreau-Yosida approximation of φ defined by (1.10), $\partial\varphi_\varepsilon$ is the subdifferential of φ_ε in H and $J_\varepsilon^\varphi = (I + \varepsilon\partial\varphi)^{-1}$.

Definition 3.1. For every fixed $\varepsilon > 0$, a pair of functions $\{\theta_\varepsilon, w_\varepsilon\}$ is called a solution of (P_ε) if it satisfies the following conditions (b1)-(b4):

(b1) $\theta_\varepsilon, J_\varepsilon^\varphi w_\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$, $w_\varepsilon \in W^{1,2}(0,T;H)$.

(b2) $\theta_\varepsilon'(t) + w_\varepsilon'(t) - \Delta_0\theta_\varepsilon(t) = h(t)$ in H for a.e. $t \in (0,T)$.

(b3) There exists a function $\xi_\varepsilon \in L^2(0,T;H)$ with $\xi_\varepsilon \in \partial I_{\theta_\varepsilon,N}(w_\varepsilon')$ a.e. on Q such that

$$w_\varepsilon'(t) + \xi_\varepsilon(t) + \nu\partial\varphi_\varepsilon(w_\varepsilon(t)) = f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)) \ \text{ in } H \ \text{ for a.e. } t \in (0,T).$$

(b4) $\theta_\varepsilon(0) = \theta_0$ and $w_\varepsilon(0) = w_0$ in H.

Theorem 3.1. *Under the assumption* (1), *for any* $\theta_0, w_0 \in V, h \in L^2(0,T;H)$ *and for each* $\varepsilon > 0$, *there exists at least one solution* $\{\theta_\varepsilon, w_\varepsilon\}$ *of* (P_ε) *in the sense of Definition 3.1.*

First, we construct a local in time solution of (P_ε) by using the fixed point argument. To do so, prepare a set $X_T^\varepsilon(M_0)$ defined by

$$X_T^\varepsilon(M_0) := \left\{ (\bar\theta, \bar w) \;\middle|\; \begin{array}{l} \bar\theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), \ \ \bar\theta(0) = \theta_0, \\ J_\varepsilon^\varphi \bar w \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), \ \ \bar w(0) = w_0, \\ |\bar\theta'|^2_{L^2(0,T;H)} \le 1 + M_0, \ \ \sup_{t\in[0,T]} |\nabla\bar\theta(t)|_H^2 \le 1 + M_0, \\ |\bar w'|^2_{L^2(0,T;H)} \le 1 + M_0, \ \ \sup_{t\in[0,T]} |\nabla J_\varepsilon^\varphi \bar w(t)|_H^2 \le 1 + M_0 \end{array} \right\}, \quad (3.4)$$

where M_0 is a positive constant dependent on the norm of initial data and a fixed number ν, more precisely, $M_0 := |\theta_0|_V^2 + 2(1+\nu)|w_0|_V^2$. We see that $X_T^\varepsilon(M_0)$ is the convex and compact subset of the product space $C([0,T];H) \times C_w([0,T];H)$. We fix $\varepsilon > 0$ and take an element $(\bar\theta, \bar w)$ in $X_T^\varepsilon(M_0)$, and substitute $\bar\theta, \bar w$ for θ, w in the right side of (3.2), namely

$$w'(t) + \partial I_{\bar\theta(t),N}(w'(t)) + \nu\partial\varphi_\varepsilon(w(t)) \ni f(\bar\theta(t), J_\varepsilon^\varphi \bar w(t)) \quad \text{in } H \ \text{ for a.e. } t \in (0,T). \quad (3.5)$$

We denote by $(P_\varepsilon)_{(\bar\theta,\bar w)}$ the system (3.1), (3.5) for each $(\bar\theta, \bar w) \in X_T^\varepsilon(M_0)$ and (3.3). For every $(\bar\theta, \bar w) \in X_T^\varepsilon(M_0)$, (3.5) can be written in the form

$$w'(t) = \left(I + \partial I_{\bar\theta(t),N}\right)^{-1} \left(f(\bar\theta(t), J_\varepsilon^\varphi \bar w(t)) - \nu\partial\varphi_\varepsilon(w(t))\right),$$

where I is the identity in H. Noting that $(I + \partial I_{\bar\theta,N})^{-1}$ and $\partial\varphi_\varepsilon$ are Lipschitz continuous in H and the equation (3.1) is linear, we can find a unique solution $\{\theta, w\}$ of $(P_\varepsilon)_{(\bar\theta,\bar w)}$. Now, taking a number T_0 with $0 < T_0 \le T$, (determined later), we define a mapping S from $X_T^\varepsilon(M_0)$ into $W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ by the formula

$$S(\bar\theta, \bar w)(t) = \begin{cases} (\theta(t), w(t)) & \text{if } \ 0 \le t \le T_0, \\ (\theta(T_0), w(T_0)) & \text{if } T_0 \le t \le T, \end{cases}$$

where $\{\theta, w\}$ is the solution of $(P_\varepsilon)_{(\bar\theta,\bar w)}$. As to this mapping S, we see the following lemma:

Lemma 3.1. *There exists T_0 with $0 < T_0 \le T$ such that*

$$S(X_T^\varepsilon(M_0)) \subset X_T^\varepsilon(M_0).$$

Proof. Let $(\bar\theta, \bar w) \in X_T^\varepsilon(M_0)$ and $\{\theta, w\}$ be the solution of $(P_\varepsilon)_{(\bar\theta,\bar w)}$. From the assumption (1), without loss of generality, we may assume that g and g' are Lipschitz continuous on $\mathbf{R}$. Multiplying (3.5) by $w' - g(\bar\theta)$ in H and noting that

- $\left(w'(t), w'(t) - g(\bar\theta(t))\right) \ge \dfrac{3}{4}|w'(t)|_H^2 - |g(\bar\theta(t))|_H^2,$
- $\left(\xi(t), w'(t) - g(\bar\theta(t))\right) \ge 0, \quad \forall\xi(t) \in \partial I_{\bar\theta(t),N}(w'(t)),$
- $\left(\partial\varphi_\varepsilon(w(t)), w'(t) - g(\bar\theta(t))\right) = \dfrac{d}{dt}\varphi_\varepsilon(w(t)) - (\partial\varphi_\varepsilon(w(t)), g(\bar\theta(t))),$

- $\left(f(\bar\theta(t), J_\varepsilon^\varphi \bar w(t)), w'(t) - g(\bar\theta(t))\right)$

$$\begin{aligned}
&= \left(f(\bar\theta(t), J_\varepsilon^\varphi \bar w(t)), w'(t)\right) - \left(f(\bar\theta(t), J_\varepsilon^\varphi \bar w(t)), g(\bar\theta(t))\right) \\
&\le \frac{1}{4}|w'(t)|_H^2 + |f(\bar\theta(t), J_\varepsilon^\varphi \bar w(t))|_H^2 + \frac{1}{2}|f(\bar\theta(t), J_\varepsilon^\varphi \bar w(t))|_H^2 + \frac{1}{2}|g(\bar\theta(t))|_H^2 \\
&\le \frac{1}{4}|w'(t)|_H^2 + K_1\left(|\bar\theta(t)|_H^2 + |J_\varepsilon^\varphi \bar w(t)|_H^2 + 1\right),
\end{aligned}$$

we have

$$\frac{1}{2}|w'(t)|_H^2 + \nu\frac{d}{dt}\varphi_\varepsilon(w(t)) \le K_1\left(|\bar\theta(t)|_H^2 + |J_\varepsilon^\varphi \bar w(t)|_H^2 + 1\right) + \nu(\partial\varphi_\varepsilon(w(t)), g(\bar\theta(t)))$$

for a.e. $t \in (0,T)$, where K_1 is a positive constant depending only on Lipschitz constants of f and g, and norms $|f|_\infty$ and $|g|_\infty$. Combining the above inequality with the following inequalities:

$$\begin{aligned}
(\partial\varphi_\varepsilon(w(t)), g(\bar\theta(t))) &= \left(-\Delta_0 J_\varepsilon^\varphi w(t), g(\bar\theta(t))\right) \\
&= \left(\nabla J_\varepsilon^\varphi w(t), \nabla g(\bar\theta(t))\right) \\
&\le \frac{1}{2}|\nabla J_\varepsilon^\varphi w(t)|_H^2 + \frac{1}{2}|\nabla g(\bar\theta(t))|_H^2 \\
&\le \varphi_\varepsilon(w(t)) + \frac{|g'|_\infty^2(1+M_0)}{2},
\end{aligned}$$

$$\begin{aligned}
|\bar\theta(t)|_H^2 + |J_\varepsilon^\varphi \bar w(t)|_H^2 &\le |\bar\theta(t)|_H^2 + |\bar w(t)|_H^2 \\
&= \left|\bar\theta(0) + \int_0^t \bar\theta'(s)ds\right|_H^2 + \left|\bar w(0) + \int_0^t \bar w'(s)ds\right|_H^2 \\
&\le 2|\theta_0|_H^2 + 2\int_0^t |\bar\theta'(s)|_H^2 ds + 2|w_0|_H^2 + 2\int_0^t |\bar w'(s)|_H^2 ds \\
&\le 2|\theta_0|_H^2 + 2t|\bar\theta'|_{L^2(0,T;H)}^2 + 2|w_0|_H^2 + 2t|\bar w'|_{L^2(0,T;H)}^2 \\
&\le 2\left(|\theta_0|_H^2 + |w_0|_H^2\right) + 4T(1+M_0),
\end{aligned}$$

we see that

$$\frac{1}{2}|w'(t)|_H^2 + \nu\frac{d}{dt}\varphi_\varepsilon(w(t)) \le \nu\varphi_\varepsilon(w(t)) + K_2 \tag{3.6}$$

for a.e. $t \in (0,T)$, where K_2 is a positive constant depending only on the Lipschitz constants of f, g, norms $|f|_\infty, |g|_\infty, |\theta_0|_H, |w_0|_H$ and constants ν and M_0. Applying the Gronwall's lemma to (3.6), we obtain that

$$\nu\varphi_\varepsilon(w(t)) \le (\nu\varphi_\varepsilon(w_0) + K_2T)e^T \le (\nu|\nabla w_0|_H^2 + K_2T)e^T =: K_3, \quad \forall t \in [0,T].$$

Hence by (3.6), the following holds:

$$\frac{1}{2}|w'(t)|_H^2 + \nu\frac{d}{dt}\varphi_\varepsilon(w(t)) \le K_2 + K_3 \quad \text{for a.e. } t \in (0,T).$$

Integrating the above in t over $[0,T']$ with $(0<T'\le T)$, we obtain that

$$\frac{1}{2}\int_0^{T'}|w'(t)|_H^2dt+\nu\varphi_\varepsilon(w(T'))\le\nu\varphi_\varepsilon(w_0)+T'(K_2+K_3),\quad \forall T'\in(0,T]. \tag{3.7}$$

Next multiplying θ' by (3.1) in H, we get that

$$\frac{1}{2}|\theta'(t)|_H^2+\frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|_H^2\le|w'(t)|_H^2+|h(t)|_H^2\quad\text{for a.e. } t\in(0,T).$$

Integrating the above in t over $[0,\tilde{T}]$ $(0<\tilde{T}\le T')$ and using (3.7), we see that

$$\begin{aligned}\frac{1}{2}\int_0^{\tilde{T}}|\theta'(t)|_H^2dt+\frac{1}{2}|\nabla\theta(\tilde{T})|_H^2 &\le \frac{1}{2}|\nabla\theta_0|_H^2+\int_0^{T'}|w'(t)|_H^2dt+\int_0^{T'}|h(t)|_H^2dt\\ &\le K_0+2T'(K_2+K_3)+\int_0^{T'}|h(t)|_H^2dt,\end{aligned}$$

where

$$K_0:=\frac{1}{2}|\nabla\theta_0|_H^2+\nu|\nabla w_0|_H^2\le\frac{M_0}{2}.$$

Taking T_0 with $T_0\le T$ such that

$$2T_0\left(1+\frac{1}{\nu}\right)(K_2+K_3)+\int_0^{T_0}|h(t)|_H^2dt\le\frac{1}{2},$$

we have the conclusion. $\diamondsuit$

Lemma 3.2. *For any $\varepsilon>0$ and any $\theta_0,w_0\in V,h\in L^2(0,T;H)$, there exists a solution $\{\theta,w\}$ of (P_ε) on the time-interval $[0,T_0]$ such that $\theta, J_\varepsilon^\varphi w\in W^{1,2}(0,T_0;H)\cap L^\infty(0,T_0;V)$ and $w\in W^{1,2}(0,T_0;H)$, where T_0 is a (small) positive number determined in Lemma 3.1.*

Proof. In order to get the conclusion of this lemma, we shall use the Schauder's fixed point theorem for the mapping S. First we show S is continuous in the topology of $C([0,T];H)\times C_w([0,T];H)$. We take a sequence $\{(\bar{\theta}_n,\bar{w}_n)\}\subset X_T^\varepsilon(M_0)$ converging to some element $(\bar{\theta},\bar{w})\in X_T^\varepsilon(M_0)$ in the topology of $C([0,T];H)\times C_w([0,T];H)$. Let $\{\theta_i,w_i\}$ be the solution of $(P_\varepsilon)_{(\bar{\theta}_i,\bar{w}_i)}$ each for $i\in\mathbf{N}$. Then the couple $\{\theta_i,w_i\}$ of functions satisfies

$$\theta_i'(t)+w_i'(t)-\Delta_0\theta_i(t)=h(t)\ \text{ in } H\ \text{ for a.e. } t\in(0,T_0), \tag{3.8}$$

$$w_i'(t)+\xi_i(t)+\nu\partial\varphi_\varepsilon(w_i(t))=f(\bar{\theta}_i(t),J_\varepsilon^\varphi\bar{w}_i(t))\ \text{ in } H\ \text{ for a.e. } t\in(0,T_0), \tag{3.9}$$

$$\theta_i(0)=\theta_0\ \text{ and }\ w_i(0)=w_0\ \text{ in } H, \tag{3.10}$$

where $\xi_i(t)\in\partial I_{\bar{\theta}_i(t),N}(w_i'(t))$ in H for a.e. $t\in(0,T_0)$. By (3.8) and (3.9), two solutions $\{\theta_i,w_i\}$, $i=m,n$, satisfy that

$$\theta_m'(t)-\theta_n'(t)+w_m'(t)-w_n'(t)-\Delta_0(\theta_m(t)-\theta_n(t))=0\ \text{ in } H \tag{3.11}$$

and

$$\begin{aligned}w_m'(t)-w_n'(t)+\xi_m(t)-\xi_n(t)+\nu\partial\varphi_\varepsilon(w_m(t))-\nu\partial\varphi_\varepsilon(w_n(t))&\\ =f(\bar{\theta}_m(t),J_\varepsilon^\varphi\bar{w}_m(t))-f(\bar{\theta}_n(t),J_\varepsilon^\varphi\bar{w}_n(t))\ \text{ in } H&\end{aligned} \tag{3.12}$$

for a.e. $t\in(0,T_0)$. For the sake of simplicity, we denote w_m-w_n, $\theta_m-\theta_n$, $\xi_m-\xi_n$, $\bar{\theta}_m-\bar{\theta}_n$ and $\bar{w}_m-\bar{w}_n$ by $\hat{w}$, $\hat{\theta}$, $\hat{\xi}$, $\hat{\bar{\theta}}$ and $\hat{\bar{w}}$, respectively. Multiplying (3.12) by $w_m'-g(\bar{\theta}_m)-(w_n'-g(\bar{\theta}_n))$ in H and noting that

- $\left(w_m'(t) - w_n'(t), w_m'(t) - g(\bar{\theta}_m(t)) - (w_n'(t) - g(\bar{\theta}_n(t)))\right)$

$$= |w_m'(t) - w_n'(t)|_H^2 - \left(w_m'(t) - w_n'(t), g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))\right)$$
$$\geq \frac{3}{4}|w_m'(t) - w_n'(t)|_H^2 - \left|g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))\right|_H^2$$
$$\geq \frac{3}{4}|\hat{w}'(t)|_H^2 - L_g^2|\hat{\bar{\theta}}(t)|_H^2,$$

- $\left(\xi_m(t) - \xi_n(t), w_m'(t) - g(\bar{\theta}_m(t)) - (w_n'(t) - g(\bar{\theta}_n(t)))\right) \geq 0,$
- $\left(\partial\varphi_\varepsilon(w_m(t)) - \partial\varphi_\varepsilon(w_n(t)), w_m'(t) - g(\bar{\theta}_m(t)) - (w_n'(t) - g(\bar{\theta}_n(t)))\right)$

$$= (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), w_m'(t) - w_n'(t)) - (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t)))$$
$$\geq \frac{d}{dt}\varphi_\varepsilon(w_m(t) - w_n(t)) - \frac{1}{2}|\partial\varphi_\varepsilon(w_m(t) - w_n(t))|_H^2 - \frac{1}{2}\left|g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))\right|_H^2$$
$$\geq \frac{d}{dt}\varphi_\varepsilon(\hat{w}(t)) - \frac{1}{2\varepsilon^2}|\hat{w}(t)|_H^2 - \frac{L_g^2}{2}|\hat{\bar{\theta}}(t)|_H^2,$$

- $\left(f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), w_m'(t) - g(\bar{\theta}_m(t)) - (w_n'(t) - g(\bar{\theta}_n(t)))\right)$

$$= \left(f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), w_m'(t) - w_n'(t)\right)$$
$$- \left(f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))\right)$$
$$\leq \frac{1}{4}|w_m'(t) - w_n'(t)|_H^2 + \left|f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t))\right|_H^2$$
$$+ \frac{1}{2}\left|f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t))\right|_H^2 + \frac{1}{2}\left|g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))\right|_H^2$$
$$\leq \frac{1}{4}|\hat{w}'(t)|_H^2 + 3L_f^2\left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2\right) + \frac{L_g^2}{2}|\hat{\bar{\theta}}(t)|_H^2,$$

we have that

$$\frac{1}{2}|\hat{w}'(t)|_H^2 + \nu\frac{d}{dt}\varphi_\varepsilon(\hat{w}(t)) \leq K_4\left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2 + |\hat{w}(t)|_H^2\right) \tag{3.13}$$

for a.e. $t \in (0, T_0)$, where K_4 is a positive constant dependent on $\varepsilon > 0$ and L_f and L_g are the Lipschitz constants of f and g, respectively. By the simple calculation, we have

$$\frac{d}{dt}|\hat{w}(t)|_H^2 = 2(\hat{w}'(t), \hat{w}(t)) \leq |\hat{w}'(t)|_H^2 + |\hat{w}(t)|_H^2. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\frac{d}{dt}\left\{\frac{1}{2}|\hat{w}(t)|_H^2 + \nu\varphi_\varepsilon(\hat{w}(t))\right\} \leq K_5\left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2\right) + K_6\left(\frac{1}{2}|\hat{w}(t)|_H^2 + \nu\varphi_\varepsilon(\hat{w}(t))\right)$$

for a.e. $t \in (0, T_0)$, where K_5 and K_6 are positive constants. Applying the Gronwall's lemma to the above inequality, we have

$$\frac{1}{2}|\hat{w}(t)|_H^2 + \nu\varphi_\varepsilon(\hat{w}(t)) \le e^{K_6 T_0}\left\{K_5 \int_0^{T_0} \left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2\right) dt\right\}, \quad \forall t \in [0, T_0].$$

This implies that

$$|\hat{w}(t)|_H^2 \le 2e^{K_6 T_0} K_5 \int_0^{T_0} \left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2\right) dt, \quad \forall t \in [0, T_0].$$

Then (3.13) gives

$$\frac{1}{2}|\hat{w}'(t)|_H^2 + \nu\frac{d}{dt}\varphi_\varepsilon(\hat{w}(t)) \le K_7\left\{|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2 + \int_0^{T_0}\left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2\right) dt\right\}$$

for a.e. $t \in (0, T_0)$, where K_7 is a positive constant. Integrating the above in t over $[0, T_0]$, then

$$\frac{1}{2}\int_0^{T_0} |\hat{w}'(t)|_H^2 dt + \nu\varphi_\varepsilon(\hat{w}(T_0)) \le K_7(1 + T_0)\int_0^{T_0}\left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2\right) dt.$$

Taking $n, m \to +\infty$, we see that

$$K_7(1 + T_0)\int_0^{T_0}\left(|\hat{\bar{\theta}}(t)|_H^2 + |J_\varepsilon^\varphi \hat{\bar{w}}(t)|_H^2\right) dt \to 0.$$

This shows that $\{w_n'\}$ is a Cauchy sequence in $L^2(0, T_0; H)$. Hence there exists a function $w \in W^{1,2}(0, T_0; H)$ such that

$$w_n \to w \ \text{ in } C([0, T_0]; H) \ \text{ and } \ w_n' \to w' \ \text{ in } L^2(0, T_0; H) \ \text{ as } n \to +\infty. \tag{3.15}$$

For every fixed $\varepsilon > 0$, from (3.15) and the following inequality

$$\begin{aligned}\int_0^{T_0} |\partial\varphi_\varepsilon(w_n(t))|_H^2 dt &= \int_0^{T_0} |\partial\varphi_\varepsilon(w_n(t)) - \partial\varphi_\varepsilon(0)|_H^2 dt \\ &\le \frac{1}{\varepsilon^2}\int_0^{T_0} |w_n(t)|_H^2 dt\end{aligned}$$

we see that $\{\partial\varphi_\varepsilon(w_n)\}_{n=1}^{\infty}$ is bounded in $L^2(0, T_0; H)$. Putting

$$\xi_n(t) := -w_n'(t) - \nu\partial\varphi_\varepsilon(w_n(t)) + f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)) \quad \text{in } H \ \text{ for a.e. } t \in (0, T_0),$$

we see that $\xi_n(t) \in \partial I_{\bar{\theta}_n(t),N}(w_n'(t))$ in H for a.e. $t \in (0, T_0)$ and $\{\xi_n\}$ is bounded in $L^2(0, T_0; H)$. We may assume that for a subsequence $\{n_k\}$, $\{\xi_{n_k}\}$ converges weakly to ξ in $L^2(0, T_0; H)$ as $k \to +\infty$ and $\xi = -w' - \nu\partial\varphi_\varepsilon(w) + f(\bar{\theta}, J_\varepsilon^\varphi \bar{w})$, because $J_\varepsilon^\varphi \bar{w}_n \to J_\varepsilon^\varphi \bar{w}$ in $C([0, T_0]; H)$ as $n \to +\infty$. For simplicity, we use again n instead of n_k. Moreover, we can easily show that

$$\limsup_{n\to+\infty}\int_0^{T_0} (\xi_n(t), w_n'(t)) dt \le \int_0^{T_0} (\xi(t), w'(t)) dt, \tag{3.16}$$

because we see that

$$\begin{aligned}\int_0^{T_0}(\xi_n(t),w_n'(t))dt &= \int_0^{T_0}\Big(-w_n'(t)-\nu\partial\varphi_\varepsilon(w_n(t))+f(\bar\theta_n(t),J_\varepsilon^\varphi\bar w_n(t)),w_n'(t)\Big)\,dt\\ &= -\int_0^{T_0}|w_n'(t)|_H^2dt-\nu\varphi_\varepsilon(w_n(T_0))+\nu\varphi_\varepsilon(w_0)\\ &\quad+\int_0^{T_0}\Big(f(\bar\theta_n(t),J_\varepsilon^\varphi\bar w_n(t)),w_n'(t)\Big)\,dt,\end{aligned}$$

then

$$\begin{aligned}\limsup_{n\to+\infty}\int_0^{T_0}(\xi_n(t),w_n'(t))dt &\le -\liminf_{n\to+\infty}|w_n'|^2_{L^2(0,T_0;H)}-\nu\liminf_{n\to+\infty}\varphi_\varepsilon(w_n(T_0))+\nu\varphi_\varepsilon(w_0)\\ &\quad+\lim_{n\to+\infty}\int_0^{T_0}\Big(f(\bar\theta_n(t),J_\varepsilon^\varphi\bar w_n(t)),w_n'(t)\Big)\,dt\\ &\le -|w'|^2_{L^2(0,T_0;H)}-\nu\varphi_\varepsilon(w(T_0))+\nu\varphi_\varepsilon(w_0)\\ &\quad+\int_0^{T_0}\Big(f(\bar\theta(t),J_\varepsilon^\varphi\bar w(t)),w'(t)\Big)\,dt\\ &= \int_0^{T_0}\Big(-w'(t)-\nu\partial\varphi_\varepsilon(w(t))+f(\bar\theta(t),J_\varepsilon^\varphi\bar w(t)),w'(t)\Big)\,dt\\ &= \int_0^{T_0}(\xi(t),w'(t))dt.\end{aligned}$$

Since $I_{\bar\theta_n,N}(\cdot)\to I_{\bar\theta,N}(\cdot)$ on H in the sense of Mosco (cf.[4,12,17]) as $n\to+\infty$, by the usual monotonicity technique with the Mosco convergence and (3.16), we have the inclusion $\xi(t)\in\partial I_{\bar\theta(t),N}(w'(t))$ in H for a.e. $t\in(0,T_0)$. Finally, we have the following:

$$w'(t)+\xi(t)+\nu\partial\varphi_\varepsilon(w(t))=f(\bar\theta(t),J_\varepsilon^\varphi\bar w(t)),\quad \xi(t)\in\partial I_{\bar\theta(t),N}(w'(t))\ \text{ in } H \tag{3.17}$$

for a.e. $t\in(0,T_0)$. Multiplying (3.11) by $\hat\theta'$ in H with the following calculations

- $(\theta_m'(t)-\theta_n'(t),\theta_m'(t)-\theta_n'(t))=(\hat\theta'(t),\hat\theta'(t))=|\hat\theta'(t)|_H^2,$
- $(w_m'(t)-w_n'(t),\theta_m'(t)-\theta_n'(t))=(\hat w'(t),\hat\theta'(t))\ge-\frac{1}{2}|\hat\theta'(t)|_H^2-\frac{1}{2}|\hat w'(t)|_H^2,$
- $(-\Delta_0(\theta_m(t)-\theta_n(t)),\theta_m'(t)-\theta_n'(t))=(-\Delta_0\hat\theta(t),\hat\theta'(t))=\frac{1}{2}\frac{d}{dt}|\nabla\hat\theta(t)|_H^2,$

we have that

$$|\hat\theta'(t)|_H^2+\frac{d}{dt}|\nabla\hat\theta(t)|_H^2\le|\hat w'(t)|_H^2\quad\text{for a.e. } t\in(0,T_0). \tag{3.18}$$

Then on account of (3.15), the above inequality implies that $\{\theta_n\}$ is a Cauchy sequence in $W^{1,2}(0,T_0;H)\cap L^\infty(0,T_0;V)$. Therefore we may assume that there exists a function $\theta\in W^{1,2}(0,T_0;H)\cap L^\infty(0,T_0;V)$ such that

$$\theta_n\to\theta\ \text{ in } C([0,T_0];H)\ \text{ and }\ \theta_n'\to\theta'\ \text{ in } L^2(0,T_0;H)\ \text{ as } n\to+\infty. \tag{3.19}$$

It follows from (3.15) and (3.19) that

$$-\Delta_0\theta_n\to-\Delta_0\theta\ \text{ weakly in } L^2(0,T_0;H)\ \text{ as } n\to+\infty.$$

Hence the limit functions θ and w enjoy

$$\theta'(t) + w'(t) - \Delta_0\theta(t) = h(t) \text{ in } H \text{ for a.e. } t \in (0,T_0). \tag{3.20}$$

Therefore from (3.17) and (3.20), the pair of limit functions $\{\theta, w\}$ is the solution to $(P_\varepsilon)_{(\bar\theta,\bar w)}$ on $(0,T_0)$ with the regularities $\theta, J_\varepsilon^\varphi w \in W^{1,2}(0,T_0;H)\cap L^\infty(0,T_0;V)$ and $w \in W^{1,2}(0,T_0;H)$. Here, extend θ and w on $[0,T_0]$ onto the time interval $[0,T]$ by $\theta(T_0)$ and $w(T_0)$. Then $S(\bar\theta,\bar w) = (\theta, w)$ and S is continuous in the topology of $C([0,T];H) \times C_w([0,T];H)$. Hence we can apply the Schauder's fixed point theorem with respect to the mapping S in $X_T^\varepsilon(M_0)$ to find a fixed point (θ, w) of S which is a solution to (P_ε) on $[0,T_0]$. ◇

Lemma 3.3. *For every fixed $\varepsilon > 0$, the solution $\{\theta, w\}$ of (P_ε) is unique on any time interval $[0,T']$ $(0 < T' \le T)$.*

Proof. Let $\{\theta_m, w_m\}$ and $\{\theta_n, w_n\}$ be the solutions to (P_ε) on $[0,T']$ $(0 < T' \le T)$ with the same initial data, namely, they satisfy the following equations:

$$\theta_i'(t) + w_i'(t) - \Delta_0\theta_i(t) = h(t) \text{ in } H \text{ for a.e. } t \in (0,T'), \tag{3.21}$$

$$w_i'(t) + \xi_i(t) + \nu\partial\varphi_\varepsilon(w_i(t)) = f(\theta_i(t), J_\varepsilon^\varphi w_i(t)) \text{ in } H \text{ for a.e. } t \in (0,T'), \tag{3.22}$$

$$\theta_i(0) = \theta_0 \text{ and } w_i(0) = w_0 \text{ in } H, \tag{3.23}$$

where $\xi_i(t) \in \partial I_{\theta_i(t),N}(w_i'(t))$ in H for a.e. $t \in (0,T'), i = m,n$. By the above equations, two solutions $\{\theta_i, w_i\}$, $i = m,n$, satisfy that

$$\theta_m'(t) - \theta_n'(t) + w_m'(t) - w_n'(t) - \Delta_0(\theta_m(t) - \theta_n(t)) = 0 \text{ in } H \tag{3.24}$$

and

$$\begin{aligned} &w_m'(t) - w_n'(t) + \xi_m(t) - \xi_n(t) + \nu\partial\varphi_\varepsilon(w_m(t)) - \nu\partial\varphi_\varepsilon(w_n(t)) \\ &\qquad = f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)) \text{ in } H \end{aligned} \tag{3.25}$$

for a.e. $t \in (0,T')$. Then by the same calculations to get (3.13) and (3.18) as in Lemma 3.2: (3.24) $\times\ \hat\theta'$ and (3.25) $\times\ \{w_m' - g(\theta_m) - (w_n' - g(\theta_n))\}$,

- $(w_m'(t) - w_n'(t), w_m'(t) - g(\theta_m(t)) - (w_n'(t) - g(\theta_n(t))))$

$$\begin{aligned} &= |w_m'(t) - w_n'(t)|_H^2 - (w_m'(t) - w_n'(t), g(\theta_m(t)) - g(\theta_n(t))) \\ &\ge \frac{3}{4}|w_m'(t) - w_n'(t)|_H^2 - |g(\theta_m(t)) - g(\theta_n(t))|_H^2 \\ &\ge \frac{3}{4}|\hat w'(t)|_H^2 - L_g^2|\hat\theta(t)|_H^2, \end{aligned}$$

- $(\xi_m(t) - \xi_n(t), w_m'(t) - g(\theta_m(t)) - (w_n'(t) - g(\theta_n(t)))) \ge 0,$
- $\frac{1}{2}|\nabla J_\varepsilon^\varphi(\hat w(t))|_H^2 = \varphi(J_\varepsilon^\varphi \hat w(t)) \le \varphi_\varepsilon(\hat w(t)),$

- $\frac{1}{2}|\nabla g(\theta_m(t)) - \nabla g(\theta_n(t))|_H^2 = \frac{1}{2}|g'(\theta_m(t))\nabla\theta_m(t) - g'(\theta_n(t))\nabla\theta_n(t)|_H^2$
$$\le |g'(\theta_m(t))\nabla\theta_m(t) - g'(\theta_m(t))\nabla\theta_n(t)|_H^2 + |g'(\theta_m(t))\nabla\theta_n(t) - g'(\theta_n(t))\nabla\theta_n(t)|_H^2$$
$$\le |g'(\theta_m(t))|_H^2|\nabla\theta_m(t) - \nabla\theta_n(t)|_H^2 + |\nabla\theta_n(t)|_H^2|g'(\theta_m(t)) - g'(\theta_n(t))|_H^2$$
$$\le |g'(\theta_m(t))|_H^2|\nabla\hat\theta(t)|_H^2 + L_{g'}^2|\nabla\theta_n(t)|_H^2|\hat\theta(t)|_H^2$$
$$\le K_8\left(|\nabla\hat\theta(t)|_H^2 + |\hat\theta(t)|_H^2\right),$$
- $(\partial\varphi_\varepsilon(w_m(t)) - \partial\varphi_\varepsilon(w_n(t)), w_m'(t) - g(\theta_m(t)) - (w_n'(t) - g(\theta_n(t))))$
$$= (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), w_m'(t) - w_n'(t)) - (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), g(\theta_m(t)) - g(\theta_n(t)))$$
$$\ge \frac{d}{dt}\varphi_\varepsilon(w_m(t) - w_n(t)) - \frac{1}{2}|\nabla J_\varepsilon^\varphi(w_m(t) - w_n(t))|_H^2 - \frac{1}{2}|\nabla g(\theta_m(t)) - \nabla g(\theta_n(t))|_H^2$$
$$\ge \frac{d}{dt}\varphi_\varepsilon(\hat w(t)) - \varphi_\varepsilon(\hat w(t)) - K_8\left(|\nabla\hat\theta(t)|_H^2 + |\hat\theta(t)|_H^2\right),$$
- $(f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)), w_m'(t) - g(\theta_m(t)) - (w_n'(t) - g(\theta_n(t))))$
$$= (f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)), w_m'(t) - w_n'(t))$$
$$- (f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)), g(\theta_m(t)) - g(\theta_n(t)))$$
$$\le \frac{1}{4}|w_m'(t) - w_n'(t)|_H^2 + |f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t))|_H^2$$
$$+ \frac{1}{2}|f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t))|_H^2 + \frac{1}{2}|g(\theta_m(t)) - g(\theta_n(t))|_H^2$$
$$\le \frac{1}{4}|\hat w'(t)|_H^2 + 3L_f^2\left(|\hat\theta(t)|_H^2 + |J_\varepsilon^\varphi \hat w(t)|_H^2\right) + \frac{L_g^2}{2}|\hat\theta(t)|_H^2$$
$$\le \frac{1}{4}|\hat w'(t)|_H^2 + K_9\left(|\hat w(t)|_H^2 + |\hat\theta(t)|_H^2\right),$$

we deduce that

$$\frac{1}{2}|\hat w'(t)|_H^2 + \nu\frac{d}{dt}\varphi_\varepsilon(\hat w(t)) \le \nu\varphi_\varepsilon(\hat w(t)) + K_{10}\left(|\hat\theta(t)|_H^2 + |\nabla\hat\theta(t)|_H^2 + |\hat w(t)|_H^2\right) \qquad (3.26)$$

and

$$|\hat\theta'(t)|_H^2 + \frac{d}{dt}|\nabla\hat\theta(t)|_H^2 \le |\hat w'(t)|_H^2 \qquad (3.27)$$

for a.e. $t \in (0, T')$, where $\hat\theta = \theta_m - \theta_n$, $\hat w = w_m - w_n$ and K_8, K_9 and K_{10} are positive constants independent of $\varepsilon > 0$ and $L_{g'}$ is a Lipschitz constant of g'. Computing (3.26) + (3.27) $\times \frac{1}{4}$, we have that

$$\frac{1}{4}|\hat w'(t)|_H^2 + \frac{1}{4}|\hat\theta'(t)|_H^2 + \frac{d}{dt}\left\{\nu\varphi_\varepsilon(\hat w(t)) + \frac{1}{4}|\nabla\hat\theta(t)|_H^2\right\} \le \nu\varphi_\varepsilon(\hat w(t)) + K_{10}\left(|\hat\theta(t)|_V^2 + |\hat w(t)|_H^2\right)$$

for a.e. $t \in (0,T')$. Making use of (3.14) for both w and θ, we have the following

$$\frac{d}{dt}\left\{\frac{1}{4}|\hat{\theta}(t)|_V^2 + \frac{1}{4}|\hat{w}(t)|_H^2 + \nu\varphi_\varepsilon(\hat{w}(t))\right\} \le K_{11}\left(\frac{1}{4}|\hat{\theta}(t)|_V^2 + \frac{1}{4}|\hat{w}(t)|_H^2 + \nu\varphi_\varepsilon(\hat{w}(t))\right)$$

for a.e. $t \in (0,T')$, where K_{11} is a positive constant. Applying the Gronwall's lemma to the above inequality, the uniqueness follows at once. ◇

Lemma 3.4. *For every fixed* $\varepsilon > 0$, *the solution* $\{\theta, w\}$ *of* (P_ε) *can be extended in time to the interval* $[0,T]$.

Proof. Let T^* be the supremum of all $T_0 \in [0,T]$ such that (P_ε) has a (unique) solution $\{\theta, w\}$ on $[0,T_0]$. By Lemma 3.3, $\{\theta, w\}$ is uniquely determined on the interval $[0,T^*)$. Let T_0 be any number such that $0 < T_0 < T^*$. The solution $\{\theta, w\}$ satisfies that:

$$\theta'(t) + w'(t) - \Delta_0\theta(t) = h(t) \quad \text{in } H, \tag{3.28}$$

$$w'(t) + \xi(t) + \nu\partial\varphi_\varepsilon(w(t)) = f(\theta(t), J_\varepsilon^\varphi w(t)) \quad \text{in } H, \tag{3.29}$$

$$\theta(0) = \theta_0 \quad \text{and} \quad w(0) = w_0 \quad \text{in } H, \tag{3.30}$$

where $\xi(t) \in \partial I_{\theta(t),N}(w'(t))$ in H. Multiplying (3.29) by $w' - g(\theta)$ and (3.28) by θ' in H with the following calculations:

- $(w'(t), w'(t) - g(\theta(t))) \ge \frac{3}{4}|w'(t)|_H^2 - |g(\theta(t))|_H^2$,
- $(\xi(t), w'(t) - g(\theta(t))) \ge 0, \quad \forall\xi(t) \in \partial I_{\theta(t),N}(w'(t))$,
- $(\partial\varphi_\varepsilon(w(t)), w'(t) - g(\theta(t))) = \frac{d}{dt}\varphi_\varepsilon(w(t)) - (\partial\varphi_\varepsilon(w(t)), g(\theta(t)))$,
- $(\partial\varphi_\varepsilon(w(t)), g(\theta(t))) \le \varphi_\varepsilon(w(t)) + \frac{|g'|_\infty^2}{2}|\nabla\theta(t)|_H^2$,
- $(f(\theta(t), J_\varepsilon^\varphi w(t)), w'(t) - g(\theta(t)))$

$$\begin{aligned}
&= (f(\theta(t), J_\varepsilon^\varphi w(t)), w'(t)) - (f(\theta(t), J_\varepsilon^\varphi w(t)), g(\theta(t))) \\
&\le \frac{1}{4}|w'(t)|_H^2 + |f(\theta(t), J_\varepsilon^\varphi w(t))|_H^2 + \frac{1}{2}|f(\theta(t), J_\varepsilon^\varphi w(t))|_H^2 + \frac{1}{2}|g(\theta(t))|_H^2 \\
&\le \frac{1}{4}|w'(t)|_H^2 + K_{12}\left(|\theta(t)|_H^2 + |J_\varepsilon^\varphi w(t)|_H^2 + 1\right),
\end{aligned}$$

we have that

$$\frac{1}{2}|w'(t)|_H^2 + \nu\frac{d}{dt}\varphi_\varepsilon(w(t)) \le K_{13}\left(|\theta(t)|_V^2 + |w(t)|_H^2 + \nu\varphi_\varepsilon(w(t)) + 1\right) \tag{3.31}$$

and

$$\frac{1}{2}|\theta'(t)|_H^2 + \frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|_H^2 \le |w'(t)|_H^2 + |h(t)|_H^2 \tag{3.32}$$

for a.e. $t \in (0, T_0)$, respectively, where K_{12} and K_{13} are positive constants. Computing (3.31) + (3.32) $\times \frac{1}{4}$, we have that

$$\frac{1}{4}|w'(t)|_H^2 + \frac{1}{8}|\theta'(t)|_H^2 + \frac{d}{dt}\left\{\frac{1}{8}|\nabla\theta(t)|_H^2 + \nu\varphi_\varepsilon(w(t))\right\}$$

$$\leq K_{14}\left(|\theta(t)|_V^2 + |w(t)|_H^2 + \nu\varphi_\varepsilon(w(t)) + |h(t)|_H^2 + 1\right) \tag{3.33}$$

for a.e. $t \in (0, T_0)$, where K_{14} is a positive constant. Using (3.14) for both θ and w with the suitable arrangement, we have the following:

$$\frac{d}{dt}E(t) \leq K_{15}\left(E(t) + |h(t)|_H^2 + 1\right) \quad \text{for a.e. } t \in (0, T_0), \tag{3.34}$$

where

$$E(t) := \frac{1}{8}|\theta(t)|_V^2 + \frac{1}{4}|w(t)|_H^2 + \nu\varphi_\varepsilon(w(t)) \quad \text{for a.e. } t \in (0, T_0)$$

and K_{15} is a positive constant. Applying the Gronwall's lemma to (3.34), we obtain that

$$E(t) \leq \left(E(0) + K_{15}\int_0^{T_0} |h(t)|_H^2 dt + K_{15}T_0\right) e^{K_{15}T_0}, \quad \forall t \in [0, T_0]. \tag{3.35}$$

Integrating (3.33) in t over $[0, T_0]$, then by (3.35) we have that

$$\frac{1}{4}\int_0^{T_0} |w'(t)|_H^2 dt + \frac{1}{8}\int_0^{T_0} |\theta'(t)|_H^2 dt + \frac{1}{8}|\nabla\theta(T_0)|_H^2 + \nu\varphi_\varepsilon(w(T_0))$$

$$\leq K_{16}\left(\varphi_\varepsilon(w_0) + |\theta_0|_V^2 + \int_0^T |h(t)|_H^2 dt + 1\right), \tag{3.36}$$

where K_{16} is a positive constant. Noting that (3.36) is valid for any $T_0 \in [0, T^*)$ because the value of right hand side of (3.36) is independent of T_0, and $|(J_\varepsilon^\varphi w)'|_{L^2(0,T_0;H)} \leq |w'|_{L^2(0,T_0;H)}$, we obtain that

$$\theta, J_\varepsilon^\varphi w \in W^{1,2}(0, T^*; H) \cap L^\infty(0, T^*; V) \quad \text{and} \quad w \in W^{1,2}(0, T^*; H).$$

Therefore the following limits exist:

$$\lim_{t \nearrow T^*} \theta(t) =: \theta^* \quad \text{and} \quad \lim_{t \nearrow T^*} w(t) =: w^* \quad \text{in } H.$$

Hence by the local existence result again we see that $\{\theta, w\}$ can be extended to the time beyond T^*. It contradicts the hypothesis of T^*. Finally, we obtain that $T = T^*$. $\diamondsuit$

Proof of Theorem 3.1: It follows immediately from Lemmas 3.2-3.4. $\diamondsuit$

4. Convergence of approximate solutions

In this section we discuss the convergence of approximate solutions. Let $\{\theta_\varepsilon, w_\varepsilon\}$ be the solution of (P_ε) obtained in Theorem 3.1, namely, it satisfies that

$$\theta_\varepsilon'(t) + w_\varepsilon'(t) - \Delta_0\theta_\varepsilon(t) = h(t) \text{ in } H \text{ a.e. } t \in (0,T), \tag{4.1}$$

$$w_\varepsilon'(t) + \partial I_{\theta_\varepsilon(t),N}(w_\varepsilon'(t)) + \nu\partial\varphi_\varepsilon(w_\varepsilon(t)) \ni f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)) \text{ in } H \text{ a.e. } t \in (0,T), \tag{4.2}$$

$$\theta_\varepsilon(0) = \theta_0 \text{ in } H \text{ and } w_\varepsilon(0) = w_0 \text{ in } H. \tag{4.3}$$

We need some uniform estimates of approximate solutions $\{\theta_\varepsilon, w_\varepsilon\}$ to discuss the convergence.

Lemma 4.1. *Any approximate solution $\{\theta_\varepsilon, w_\varepsilon\}$ satisfies*

$$|\theta_\varepsilon|_\infty, |w_\varepsilon'|_\infty \le M_1 + M_1 T,$$

where $M_1 = |\theta_0|_\infty + |h|_\infty + |g|_\infty + N$.

Proof. Define a function p on $[0,T]$ by

$$p(t) := M_1 + M_1 t.$$

Noting that $|w_\varepsilon'|_\infty \le |g|_\infty + N$ holds for any $\varepsilon > 0$ by the definition of a solution of (P_ε), we observe that

$$(\theta_\varepsilon - p)' - \Delta_0(\theta_\varepsilon - p) = h - w_\varepsilon' - M_1 \le 0 \quad \text{in } Q. \tag{4.4}$$

Multiplying (4.4) by $[\theta_\varepsilon - p]^+$ in H, we have that

$$\frac{1}{2}\frac{d}{dt}|[\theta_\varepsilon(t) - p(t)]^+|_H^2 + |\nabla[\theta_\varepsilon(t) - p(t)]^+|_H^2 \le 0 \quad \text{for a.e. } t \in (0,T).$$

Integrating the above inequality in t, we see that

$$|[\theta_\varepsilon(t) - p(t)]^+|_H^2 \le |[\theta_0 - p(0)]^+|_H^2 = 0, \quad \forall t \in [0,T].$$

This implies that $\theta_\varepsilon \le p \le M_1 + M_1 T$. On the other hand,

$$(-\theta_\varepsilon - p)' - \Delta_0(-\theta_\varepsilon - p) = -h + w_\varepsilon' - M_1 \le 0 \quad \text{in } Q. \tag{4.5}$$

Multiplying (4.5) by $[-\theta_\varepsilon - p]^+$ in H, we have that

$$\frac{1}{2}\frac{d}{dt}|[-\theta_\varepsilon(t) - p(t)]^+|_H^2 + |\nabla[-\theta_\varepsilon(t) - p(t)]^+|_H^2 \le 0 \quad \text{for a.e. } t \in (0,T).$$

Integrating the above in t, we see that

$$|[-\theta_\varepsilon(t) - p(t)]^+|_H^2 \le |[-\theta_0 - p(0)]^+|_H^2 = 0, \quad \forall t \in [0,T],$$

which gives $\theta_\varepsilon(t) \geq -p(t) \geq -M_1 - M_1 T$. Hence we complete the proof. $\diamondsuit$

Lemma 4.2. *There exists a positive constant R_1 independent of $\varepsilon > 0$ such that*

$$|w_\varepsilon'|^2_{L^2(0,T;H)} + |\theta_\varepsilon'|^2_{L^2(0,T;H)} + |\Delta_0\theta_\varepsilon|^2_{L^2(0,T;H)} \leq R_1$$

and

$$\sup_{t\in[0,T]} |\nabla\theta_\varepsilon(t)|^2_H + \sup_{t\in[0,T]} |\nabla J_\varepsilon^\varphi w_\varepsilon(t)|^2_H \leq R_1.$$

Proof. Multiplying $w_\varepsilon' - g(\theta_\varepsilon)$ by (4.2) in H and noting that

$$(\xi_\varepsilon(t), w_\varepsilon'(t) - g(\theta_\varepsilon)) \geq 0, \quad \forall \xi_\varepsilon(t) \in \partial I_{\theta_\varepsilon(t),N}(w_\varepsilon'(t)) \text{ in } H \text{ for a.e. } t \in (0,T),$$

we have with the similar calculation to obtain (3.31)

$$\frac{1}{2}|w_\varepsilon'(t)|^2_H + \nu\frac{d}{dt}\varphi_\varepsilon(w_\varepsilon(t)) \leq N_1\left(|\theta_\varepsilon(t)|^2_H + |w_\varepsilon(t)|^2_H + 1\right) + \nu(\partial\varphi_\varepsilon(w_\varepsilon(t)), g(\theta_\varepsilon(t))) \quad (4.6)$$

for a.e. $t \in (0,T)$, where N_1 is a positive constant independent of $\varepsilon > 0$. Noting that $\partial\varphi_\varepsilon(w_\varepsilon(t)) = -\Delta_0 J_\varepsilon^\varphi w_\varepsilon(t)$ in H and the boundedness of g, we have

$$\begin{aligned}(\partial\varphi_\varepsilon(w_\varepsilon(t)), g(\theta_\varepsilon(t))) &= (\nabla J_\varepsilon^\varphi w_\varepsilon(t), \nabla g(\theta_\varepsilon(t)))\\ &\leq \frac{1}{2}|\nabla J_\varepsilon^\varphi w_\varepsilon(t)|^2_H + \frac{1}{2}|\nabla g(\theta_\varepsilon(t))|^2_H\\ &\leq \varphi_\varepsilon(w_\varepsilon(t)) + \frac{|g'|^2_\infty}{2}|\nabla\theta_\varepsilon(t)|^2_H\end{aligned}$$

for a.e. $t \in (0,T)$. From the above inequality and (4.6) we observe that

$$\frac{1}{2}|w_\varepsilon'(t)|^2_H + \nu\frac{d}{dt}\varphi_\varepsilon(w_\varepsilon(t)) \leq N_2\left(|\theta_\varepsilon(t)|^2_H + |\nabla\theta_\varepsilon(t)|^2_H + |w_\varepsilon(t)|^2_H + 1\right) + \nu\varphi_\varepsilon(w_\varepsilon(t)) \quad (4.7)$$

for a.e. $t \in (0,T)$, where N_2 is a positive constant independent of $\varepsilon > 0$. Next, multiplying (4.1) by θ_ε' and $-\Delta_0\theta_\varepsilon$ in H, we have

$$\frac{1}{2}|\theta_\varepsilon'(t)|^2_H + \frac{1}{2}\frac{d}{dt}|\nabla\theta_\varepsilon(t)|^2_H \leq |w_\varepsilon'(t)|^2_H + |h(t)|^2_H \quad (4.8)$$

and

$$\frac{1}{2}\frac{d}{dt}|\nabla\theta_\varepsilon(t)|^2_H + \frac{1}{2}|\Delta_0\theta_\varepsilon(t)|^2_H \leq |w_\varepsilon'(t)|^2_H + |h(t)|^2_H \quad (4.9)$$

for a.e. $t \in (0,T)$, respectively. Computing (4.7) + (4.8) $\times \frac{1}{8}$ + (4.9) $\times \frac{1}{8}$, we infer that

$$\begin{aligned}&\frac{1}{16}|\theta_\varepsilon'(t)|^2_H + \frac{1}{4}|w_\varepsilon'(t)|^2_H + \frac{1}{16}|\Delta_0\theta_\varepsilon(t)|^2_H + \frac{d}{dt}\left\{\frac{1}{8}|\nabla\theta_\varepsilon(t)|^2_H + \nu\varphi_\varepsilon(w_\varepsilon(t))\right\}\\ &\leq N_3\left(|\theta_\varepsilon(t)|^2_H + |\nabla\theta_\varepsilon(t)|^2_H + |w_\varepsilon(t)|^2_H + 1\right) + \nu\varphi_\varepsilon(w_\varepsilon(t)) \quad (4.10)\end{aligned}$$

for a.e. $t \in (0,T)$, where N_3 is a positive constant independent of $\varepsilon > 0$. By (3.14) with some suitable arrangements in (4.10), we deduce that

$$\frac{d}{dt}E(t) \leq N_4(E(t) + 1) \quad \text{for a.e. } t \in (0,T), \quad (4.11)$$

where N_4 is a positive constant independent of $\varepsilon > 0$ and

$$E(t) := \frac{1}{16}|\theta_\varepsilon(t)|_H^2 + \frac{1}{8}|\nabla\theta_\varepsilon(t)|_H^2 + \frac{1}{4}|w_\varepsilon(t)|_H^2 + \nu\varphi_\varepsilon(w_\varepsilon(t)), \quad \forall t \in [0,T]. \tag{4.12}$$

Applying the Gronwall's lemma to (4.11), we have the following:

$$E(t) \le (E(0) + N_4 T)e^{N_4 T}, \quad \forall t \in [0,T]. \tag{4.13}$$

Combining (4.12) with (4.13), we can find a positive constant N_5 independent of $\varepsilon > 0$ such that

$$\sup_{t\in[0,T]} |\nabla\theta_\varepsilon(t)|_H^2 + \sup_{t\in[0,T]} |\nabla J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 \le N_5. \tag{4.14}$$

By (4.10) and (4.13), we can find a positive constant N_6 independent of $\varepsilon > 0$ such that

$$|w_\varepsilon'|_{L^2(0,T;H)}^2 + |\theta_\varepsilon'|_{L^2(0,T;H)}^2 + |\Delta_0\theta_\varepsilon|_{L^2(0,T;H)}^2 \le N_6. \tag{4.15}$$

By (4.14) and (4.15), put $R_1 := N_5 + N_6$ to get the conclusion. ◇

Lemma 4.3. *There exists a positive constant R_2 independent of $\varepsilon > 0$ such that*

$$|\Delta_0 g(\theta_\varepsilon(t))|_H^2 \le R_2\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1\right) \quad \textit{for a.e. } t \in (0,T).$$

Proof. From the fact that

$$\Delta_0 g(\theta_\varepsilon(t)) = g'(\theta_\varepsilon(t))\Delta_0\theta_\varepsilon(t) + g''(\theta_\varepsilon(t))|\nabla\theta_\varepsilon(t)|^2,$$

it follows that

$$|\Delta_0 g(\theta_\varepsilon(t))|_H^2 \le N_7\left(|\nabla\theta_\varepsilon(t)|_{L^4(\Omega)}^4 + |\Delta_0\theta_\varepsilon(t)|_H^2\right) \tag{4.16}$$

for a.e. $t \in (0,T)$, where $N_7 := 2\max\{|g'|_\infty, |g''|_\infty\}$. By the Gagliardo-Nirenberg interpolation inequality (cf.[18]):

$$|\nabla z|_{L^4(\Omega)} \le C|z|_{H^2(\Omega)}^{\frac{1}{2}}\,|z|_\infty^{\frac{1}{2}}, \quad \forall z \in L^\infty(\Omega)\cap H^2(\Omega)$$

and Lemma 4.1 and 4.2, the following inequalities hold:

$$\begin{aligned} |\nabla\theta_\varepsilon(t)|_{L^4(\Omega)}^4 &\le C^4|\theta_\varepsilon(t)|_{H^2(\Omega)}^2|\theta_\varepsilon(t)|_\infty^2 \\ &\le C^4\left(|\theta_\varepsilon(t)|_H^2 + |\nabla\theta_\varepsilon(t)|_H^2 + |\Delta_0\theta_\varepsilon(t)|_H^2\right)|\theta_\varepsilon(t)|_\infty^2 \\ &\le N_8\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1\right) \quad \text{for a.e. } t \in (0,T), \end{aligned}$$

where C and N_8 are positive constants independent of $\varepsilon > 0$. In virtue of (4.16) we can find the desired constant R_2. ◇

Remark 4.1. In the case that $\Omega \subset \mathbf{R}^2$, we see that the above constant R_2 is independent of both parameters ε and N. By the Gagliardo-Nirenberg interpolation inequality for 2-dimensional case

$$|z|_{L^4(\Omega)} \le \tilde{C}|z|_H^{\frac{1}{2}}\,|z|_V^{\frac{1}{2}}, \quad \forall z \in V,$$

and Lemma 4.2 we can get the following inequalities without the L^∞-estimate of θ_ε obtained in Lemma 4.1:

$$\begin{aligned} |\Delta_0 g(\theta_\varepsilon(t))|_H^2 &\le 2|g'(\theta_\varepsilon(t))\Delta_0\theta_\varepsilon(t)|_H^2 + 2|g''(\theta_\varepsilon(t))|\nabla\theta_\varepsilon(t)|^2|_H^2 \\ &\le N_7\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + |\nabla\theta_\varepsilon(t)|_{L^4(\Omega)}^4\right) \\ &\le N_7\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + \tilde{C}^4|\nabla\theta_\varepsilon(t)|_H^2|\nabla\theta_\varepsilon(t)|_V^2\right) \\ &\le \tilde{C}^4 N_7 R_1^4 + N_7(1+\tilde{C}^4R_1^2)|\Delta_0\theta_\varepsilon(t)|_H^2 \\ &\le N_9\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1\right) \quad \text{for a.e. } t \in (0,T), \end{aligned}$$

where $\tilde{C}$ and N_9 are positive constants independent of ε and N.

Lemma 4.4. *There exists a positive constant R_3 independent of $\varepsilon > 0$ such that*

$$|\nabla J_\varepsilon^\varphi w_\varepsilon'|_{L^2(0,T;H)}^2 + \sup_{t\in[0,T]} |\Delta_0 J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 \le R_3.$$

Proof. Multiplying (4.2) by $\partial\varphi_\varepsilon(w_\varepsilon' - g(\theta_\varepsilon))$ in H and note that

- $(w_\varepsilon'(t), \partial\varphi_\varepsilon(w_\varepsilon'(t) - g(\theta_\varepsilon(t)))) = (w_\varepsilon'(t), \partial\varphi_\varepsilon(w_\varepsilon'(t))) - (w_\varepsilon'(t), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t))))$,
- $(w_\varepsilon'(t), \partial\varphi_\varepsilon(w_\varepsilon'(t))) = (w_\varepsilon'(t) - J_\varepsilon^\varphi w_\varepsilon'(t), \partial\varphi_\varepsilon(w_\varepsilon'(t))) + (J_\varepsilon^\varphi w_\varepsilon'(t), \partial\varphi_\varepsilon(w_\varepsilon'(t)))$

  $= \varepsilon|\partial\varphi_\varepsilon(w_\varepsilon'(t))|_H^2 + |\nabla J_\varepsilon^\varepsilon w_\varepsilon'(t)|_H^2$

  $\ge |\nabla J_\varepsilon^\varepsilon w_\varepsilon'(t)|_H^2$,
- $(w_\varepsilon'(t), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t)))) = (\nabla J_\varepsilon^\varphi w_\varepsilon'(t), \nabla g(\theta_\varepsilon(t)))$

  $\le \dfrac{1}{4}|\nabla J_\varepsilon^\varphi w_\varepsilon'(t)|_H^2 + |\nabla g(\theta_\varepsilon(t))|_H^2$

  $\le \dfrac{1}{4}|\nabla J_\varepsilon^\varphi w_\varepsilon'(t)|_H^2 + |g'|_\infty^2|\nabla\theta_\varepsilon(t)|_H^2$

  $\le \dfrac{1}{4}|\nabla J_\varepsilon^\varphi w_\varepsilon'(t)|_H^2 + N_{10}$,
- $(\xi_\varepsilon(t), \partial\varphi_\varepsilon(w_\varepsilon'(t) - g(\theta_\varepsilon(t)))) \ge 0, \quad \forall \xi_\varepsilon(t) \in \partial I_{\theta_\varepsilon(t),N}(w_\varepsilon'(t))$,
- $(\partial\varphi_\varepsilon(w_\varepsilon(t)), \partial\varphi_\varepsilon(w_\varepsilon'(t) - g(\theta_\varepsilon(t))))$

  $= (\partial\varphi_\varepsilon(w_\varepsilon(t)), \partial\varphi_\varepsilon(w_\varepsilon'(t))) - (\partial\varphi_\varepsilon(w_\varepsilon(t)), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t))))$

  $= \dfrac{1}{2}\dfrac{d}{dt}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 - (\partial\varphi_\varepsilon(w_\varepsilon(t)), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t))))$,
- $(\partial\varphi_\varepsilon(w_\varepsilon(t)), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t))))$

  $\le \dfrac{1}{2}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \dfrac{1}{2}|\partial\varphi_\varepsilon(g(\theta_\varepsilon(t)))|_H^2$

$$\leq \frac{1}{2}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \frac{1}{2}|\partial\varphi(g(\theta_\varepsilon(t)))|_H^2$$
$$= \frac{1}{2}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \frac{1}{2}|-\Delta_0 g(\theta_\varepsilon(t)))|_H^2$$
$$\leq \frac{1}{2}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \frac{R_2}{2}\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1\right),$$

- $(f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(w_\varepsilon'(t) - g(\theta_\varepsilon(t))))$

$$= (f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(w_\varepsilon'(t))) - (f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t)))),$$

- $(f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(w_\varepsilon'(t)))$

$$\leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'(t)|_H^2 + |\nabla f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t))|_H^2$$
$$\leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'(t)|_H^2 + N_{11}\left(|\nabla\theta_\varepsilon(t)|_H^2 + |\nabla J_\varepsilon^\varphi(w_\varepsilon(t))|_H^2\right)$$
$$\leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'(t)|_H^2 + N_{11}\left(|\nabla\theta_\varepsilon(t)|_H^2 + \varphi_\varepsilon(w_\varepsilon(t))\right)$$
$$\leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'(t)|_H^2 + N_{12},$$

- $(f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t))))$

$$\leq \frac{1}{2}|f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t))|_H^2 + \frac{1}{2}|\partial\varphi_\varepsilon(g(\theta_\varepsilon(t)))|_H^2$$
$$\leq N_{13}\left(|\theta_\varepsilon(t)|_H^2 + |J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 + |\Delta_0\theta_\varepsilon(t)|_H^2 + 1\right)$$
$$\leq N_{14}\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1\right),$$

we have with the help of Lemmas 4.2 and 4.3 that

$$\frac{1}{2}|\nabla J_\varepsilon^\varphi w_\varepsilon'(t)|_H^2 + \frac{\nu}{2}\frac{d}{dt}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 \leq N_{15}\left(|\Delta_0\theta_\varepsilon(t)|_H^2 + |\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + 1\right) \tag{4.17}$$

for a.e. $t \in (0,T)$, where N_i $(i = 10, 11, 12, 13, 14, 15)$ are positive constants independent of $\varepsilon > 0$. Moreover, multiply (4.1) by $-\Delta_0\theta_\varepsilon$ to have

$$\frac{1}{2}\frac{d}{dt}|\nabla\theta_\varepsilon(t)|_H^2 + \frac{1}{2}|\Delta_0\theta_\varepsilon(t)|_H^2 \leq |w_\varepsilon'(t)|_H^2 + |h(t)|_H^2 \tag{4.18}$$

for a.e. $t \in (0,T)$. Integrating (4.18) in t over $[0,T]$ and using Lemma 4.2, we have that

$$\int_0^T |\Delta_0\theta_\varepsilon(t)|_H^2 dt \leq |\theta_0|_V^2 + 2R_1^2 + 2\int_0^T |h(t)|_H^2 dt =: N_{16}. \tag{4.19}$$

Applying the Gronwall's lemma to (4.17) and using (4.19), we obtain that

$$\sup_{t\in[0,T]} |\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 \leq \left(\frac{2(N_{15}N_{16} + N_{15}T)}{\nu} + |w_0|_{H^2(\Omega)}^2\right)\exp\left\{\frac{2N_{15}T}{\nu}\right\} =: N_{17}. \tag{4.20}$$

Again by (4.17), we have with the above estimate (4.20) that

$$\int_0^T |\nabla J_\varepsilon^\varphi w_\varepsilon'(t)|_H^2 dt \le \nu |w_0|_{H^2(\Omega)}^2 + 2N_{15}(N_{16} + N_{17}T + T) =: N_{18}.$$

Putting $R_3 := N_{17} + N_{18}$, we complete the proof. $\diamondsuit$

5. Proof of Theorem 2.1

We are now in a position to give a proof of Theorem 2.1.

Proof of Theorem 2.1 Let $\{\theta_\varepsilon, w_\varepsilon\}$ be the solution of (P_ε). By the uniform estimates in Lemmas 4.1-4.4 we can choose a sequence $\{\varepsilon_n\}$ tending to 0 as $n \to +\infty$ with functions $\theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ and $w \in W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega))$ such that

$$\theta_n := \theta_{\varepsilon_n} \to \theta \ \text{ in } C([0,T];H) \ \text{ as } n \to +\infty \tag{5.1}$$

and

$$J_{\varepsilon_n}^\varphi w_n := J_{\varepsilon_n}^\varphi w_{\varepsilon_n} \to w \ \text{ in } C([0,T];V) \ \text{ as } n \to +\infty. \tag{5.2}$$

It follows from Lemma 4.4 and (1.6) that

$$|w_n(t) - J_{\varepsilon_n}^\varphi w_n(t)|_H \le \varepsilon_n |\partial\varphi_{\varepsilon_n}(w_n(t))|_H \le \varepsilon_n R_3, \quad \forall t \in [0,T].$$

This implies that

$$w_n \to w \ \text{ in } C([0,T];H) \ \text{ as } n \to +\infty. \tag{5.3}$$

Now, let $n \to +\infty$. Then it follows that

$$\partial\varphi_{\varepsilon_n}(w_n) \to \partial\varphi(w) = -\Delta_0 w \ \text{ weakly in } L^2(0,T;H) \ \text{ as } n \to +\infty.$$

Here, we observe that the function

$$\xi_n(t) := -w_n'(t) - \nu\partial\varphi_{\varepsilon_n}(w_n(t)) + f(\theta_n(t), J_{\varepsilon_n}^\varphi w_n(t)) \ \text{ in } H \ \text{ for a.e. } t \in (0,T).$$

satisfies that $\xi_n(t) \in \partial I_{\theta_n(t),N}(w_n'(t))$ in H for a.e. $t \in (0,T)$ and $\{\xi_n\}$ is bounded in $L^2(0,T;H)$. Therefore there exist a subsequence $\{n_k\}$ of and a function $\xi \in L^2(0,T;H)$ such that

$$\xi_{n_k} \to \xi \ \text{ weakly in } L^2(0,T;H) \ \text{ as } k \to +\infty.$$

Clearly $\xi = -w' + \nu\Delta_0 w + f(\theta, w)$ in $L^2(0,T;H)$. Now let us show the inclusion $\xi(t) \in \partial I_{\theta(t),N}(w'(t))$ in H for a.e. $t \in (0,T)$. In order to do so, we employ the usual monotonicity technique. Since $I_{\theta_n,N}(\cdot) \to I_{\theta,N}(\cdot)$ on H in the sense of Mosco, we have only to show

$$\limsup_{k \to +\infty} \int_0^T (\xi_{n_k}(t), w_{n_k}'(t)) dt \le \int_0^T (\xi(t), w'(t)) dt. \tag{5.4}$$

This can be proved as follows:

$$\begin{aligned}
&\limsup_{k\to+\infty}\int_0^T(\xi_{n_k}(t),w'_{n_k}(t))dt\\
&\le -\liminf_{k\to+\infty}|w'_{n_k}|^2_{L^2(0,T;H)}+\nu\lim_{k\to+\infty}\varphi_{\varepsilon_{n_k}}(w_0)-\nu\liminf_{k\to+\infty}\varphi_{\varepsilon_{n_k}}(w_{n_k}(T))\\
&\quad+\lim_{k\to+\infty}\int_0^T(f(\theta_{n_k}(t),J^{\varphi}_{\varepsilon_{n_k}}w_{n_k}(t)),w'_{n_k}(t))dt\\
&\le -|w'|^2_{L^2(0,T;H)}+\frac{\nu}{2}|\nabla w_0|^2_H-\frac{\nu}{2}|\nabla w(T)|^2_H+\int_0^T(f(\theta(t),w(t)),w'(t))dt\\
&=\int_0^T(\xi(t),w'(t))dt.
\end{aligned}$$

Therefore we obtain $\xi(t)\in\partial I_{\theta(t),N}(w'(t))$ in H for a.e. $t\in(0,T)$ and

$$w'(t)+\xi(t)-\nu\Delta_0 w(t)=f(\theta(t),w(t))\quad\text{in } H\quad\text{for a.e. } t\in(0,T).$$

By using regularity $w\in L^\infty(0,T;H^2(\Omega))$ and Lemma 4.1, we obtain $w\in W^{1,\infty}(Q)$. Since the equation (4.1) is linear, it is easy to discuss the convergence for (4.1) and obtain that

$$\theta'(t)+w'(t)-\Delta_0\theta(t)=h(t)\quad\text{in } H\quad\text{for a.e. } t\in(0,T).$$

This completes the proof of Theorem 2.1. ◇

Remark 5.1. In the case of $\Omega\subset\mathbf{R}^2$, from Remark 4.1 it is possible to take $N\to+\infty$ in $I_{\theta(t),N}(\cdot)$ and hence the solutions obtained in the above satisfy the following system $(P)'$:

$$(P)'\quad\begin{cases}\theta_t+w_t-\Delta_0\theta=h(t,x) & \text{in } Q,\\ w_t+\partial I_\theta(w_t)-\nu\Delta_0 w\ni f(\theta,w) & \text{in } Q,\\ \theta(0,\cdot)=\theta_0,\quad w(0,\cdot)=w_0 & \text{in } \Omega,\end{cases}$$

where

$$I_\theta(w_t):=\begin{cases}+\infty & \text{if } w_t<g(\theta),\\ 0 & \text{if } w_t\ge g(\theta);\end{cases}$$

and $\partial I_\theta(w_t)$ is its subdifferential:

$$\partial I_\theta(w_t):=\begin{cases}\emptyset & \text{if } w_t<g(\theta),\\ (-\infty,0] & \text{if } w_t=g(\theta),\\ \{0\} & \text{if } w_t\ge g(\theta).\end{cases}$$

$(P)'$ is the same system that we have already discussed in [3]. We note here that if N is a sufficiently large positive number, $I_{\theta,N}(\cdot)$ is close to $I_\theta(\cdot)$. Therefore we can consider that the problem (P) is one of approximate problems of $(P)'$, because the indicator function $I_{\theta,N}(\cdot)$ can be regarded as an approximation of $I_\theta(\cdot)$.

Remark 5.2. In the case of $\Omega\subset\mathbf{R}^3$, we are very interested in the situation when the fixed large number N goes to $+\infty$. In that case we can not obtain the uniform estimate in Lemma 4.1. This unables us to discuss the convergence of approximate solutions.

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