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Nonlinear Schrödinger equations with superposed delta-functions as initial data

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Abstract

We consider the initial value problem of the nonlinear Schrödinger equation with superposed δ-functions as initial data. We treat this problem case by case, i.e., the cases in which the initial data consists of single, double and triple δ-functions, respectively. In particular, when the initial data consists of double or triple δ-functions, the solution receives the generation of new modes which is visible only in the nonlinear problem (see section 3 and 4).

1 Introduction

We consider the initial value problem of the nonlinear Schrödinger equation like

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u = -\partial_x^2 u + \lambda \mathcal{N}(u), \\ u(0,x) = (\text{superposition of } \delta\text{-functions}). \end{cases}$$

where $(t,x) \in \mathbf{R} \times \mathbf{R}$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial_x$ and the unknown function $u = u(t,x)$ takes complex values. The gauge invariant power type nonlinearity $\mathcal{N}(u)$ is given by

$$\mathcal{N}(u) = |u|^{p-1}u \quad \text{with } 1 < p < 3.$$

The nonlinear coefficient λ takes arbitrary complex number. In particular, if $\mathrm{Im}\lambda < 0$, nonlinear term causes dissipative effect. We mainly treat this initial value problem by assuming that $u(0,x) = \mu_0\delta_0$, $u(0,x) = \mu_0\delta_0 + \mu_1\delta_a$ or $u(0,x) = \mu_{00}\delta_0 + \mu_{10}\delta_a + \mu_{01}\delta_b$, where δ_a denotes the well-known point mass measure supported at $x = a \in \mathbf{R}$ and μ_k, μ_{jk} $(j,k = 0,1)$ are any complex number.

From the physical point of view, the cubic nonlinearity (i.e. $p = 3$ which is excluded in our assumption for mathematical reason) frequently appears. For example, (NLS) with $\lambda \in \mathbf{R}$ and $p = 3$ is said to govern the motion of vortex filament in the ideal fluid [10]. In fact, letting $\kappa(t,x)$ be the curvature of the filament and $\tau(t,x)$ the tortion, we observe that $u(t,x) = \kappa(t,x)\exp(i\int_0^x \tau(t,y)\,dy)$ (which is called "Hasimoto transform" [10]) satisfies (NLS), where x stands for the position parameter along the filament. To our regret, our argument is slightly away from the cubic nonlinear case. However, if one allows us to treat the solution as a fine approximation of the physically important case, one can imagine the time evolution of vortex filament with the locally bended initial state, e.g., $\kappa(0,x) = \delta_a$.

The Cauchy problems with measures as initial data are extensively sutudied for various kinds of nonlinear evolution equations. As for the nonlinear parabolic equation,i.e., $\partial_t u - \partial_x^2 u + |u|^{p-1}u = 0$ with $u(0,x) = \delta_0$, Brezis-Friedman [2] specify the critical nonlinear power concerning the solvability. They prove that, if $3 \leq p$, there exists no solution continuously connected with the δ-function at $t = 0$ in the distribution sense and that, if $1 < p < 3$, it is posibble to construct a solution with a general measure as initial data. Their argunent relies on the comparison principle and the smothing property of the linear diffusion. For the KdV equation, Tsutsumi [23] constructs a solution by making use of Miura transformation [17] which deforms the original KdV equation into the modified one. Recently, Abe-Okazawa [1] have studied this kind of problem for the complex Ginzburg-Landau equation. The ideas to construct solutions in these known results are based on the strong smoothing effect of linear semi-group or the nonlinear transformation of unknown functions into the suitably handled equation. In the present case, however, the nonlinear Schrödinger equation have neither the useful smoothing properties like the heat equation nor the transformation of Miura type. Therefore, it is still open whether (NLS) is solvable when the initial data is arbitrary measure except for δ-functions.

We here remark Kenig-Ponce-Vega's work [15]. They proved the ill-posedness of the nonlinear Schrödinger equation with $u(0,x) = \delta_0$ and $3 \leq p$. The situation is very similar to the nonlinear heat case introduced above. They proved that (NLS) possesses either no solution or more than one in $C([0,T];\mathcal{S}'(\mathbf{R}))$, where $\mathcal{S}'(\mathbf{R})$ denotes the class of tempered distributions. In their work, the Gallilean invariance of (NLS) plays an important role, where the Gallilean invariance means the fact that, if $u(t,x)$ is a solution to (NLS), $u_N(t,x) = e^{-itN^2}e^{iNx}u(t,x-2tN)$ also satisfies (NLS). Then, the obvious identity $\delta_0 = e^{iNx}\delta_0$ determines the formula of u and the super critical power yields the divergence of the phase at $t = 0$. This rough sketch of their argument lets us expect that, for the subcritical case, it is pssible to construct a solution continuous at $t = 0$.

There are large amount of articles concerning the local or global well-posedness for the nonlinear Schrödinger equations in the $L^2(\mathbf{R})$ or $H^s(\mathbf{R})$ $(s > 0)$ frame work (see [5, 6, 8, 11, 12, 13, 18, 19, 21, 22] and references therein). Roughly speaking, this is because these function spaces works well via the conservation laws, energy estimates and Strichartz' estimates [20, 24]. On the other hand, since the present situation is away from the well-known frame, we require another method to construct a solution. Our idea to solve (NLS) is based on the reduction of the original problem into the ordinary differential equation (ODE) system as in the following sections.

We prove that the solution is explicitly obtained when the initial data consists of single δ-function (see section 2). Furthermore, we observe that, when the initial data consists of double (or more) δ-functions, the superposition of infinitely many linear solutions immediately appers in the solution to (NLS) (see section 3 and 4). In this paper, we call this feature "the generalization of new modes". Let us state our main results case by case.

2 The case $u(0,x) = \mu_0\delta_0$

This case simply gives an explicit solution. Namely, the solution to (NLS) is given by

$$u(t,x) = A(t)\exp(it\partial_x^2)\delta_0, \tag{2.1}$$

where $\exp(it\partial_x^2)\delta_0 = (4\pi i t)^{-1/2}\exp(ix^2/4t)$ and the modified amplitude $A(t)$ is

$$A(t) = \begin{cases} \mu_0 \exp\left(\dfrac{2\lambda|\mu_0|^{p-1}}{i(3-p)}|4\pi t|^{-(p-1)/2}t\right) & \text{if } \mathrm{Im}\lambda = 0, \\ \mu_0\left(1 - \dfrac{2(p-1)\mathrm{Im}\lambda|\mu_0|^{p-1}}{3-p}|4\pi t|^{-(p-1)/2}t\right)^{\frac{i\lambda}{(p-1)\mathrm{Im}\lambda}} & \text{if } \mathrm{Im}\lambda \neq 0. \end{cases} \tag{2.2}$$

In fact, by substituting (2.1) into (NLS), we have the ordinary differential equation (ODE) of $A(t)$:

$$\begin{cases} i\dfrac{dA}{dt} = \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(A), \\ A(0) = \mu_0. \end{cases} \tag{2.3}$$

To solve (2.3), we first multiply $\overline{A(t)}$ on both hand sides of (2.3). Then, we have $\dfrac{d}{dt}|A|^2 = 2|4\pi t|^{-(p-1)/2}\mathrm{Im}\lambda|A|^{p+1}$ and so

$$|A(t)| = \left(|\mu_0|^{-(p-1)} - (p-1)\mathrm{Im}\lambda\int_0^t |4\pi\tau|^{-(p-1)/2}\,d\tau\right)^{-1/(p-1)} \tag{2.4}$$

The integral in the parenthesis of (2.4) makes a sense since $p < 3$. Substituting (2.4) into (2.3) and solving the simple ODE, we obtain (2.2). Note that $\mathrm{Im}\lambda > 0$ implies blowing-up of $A(t)$ in positive finite time.

3 The case $u(0,x) = \mu_0\delta_0 + \mu_1\delta_a$

In this section, we observe that the superposition of δ-functions causes "the mode generation" for $t \neq 0$. Before stating our results, we introduce several notations. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$

where $\mathbf{Z}$ stands for the set of integers. Throughout this section, the Lebesgue space $L^q(= L^q(\mathbf{T}))$ denotes the class of mesureble functions on $\mathbf{T}$ with $\|f\|_{L^q}^q \equiv \int^{2\pi} f(\theta)\, d\theta < \infty$. Also, the Sobolev space $H^s(= H^s(\mathbf{T}))$ is defined by

$$H^s = \{f(\theta) \in L^2;\ \|f\|_{H^s}^2 < \infty\},$$

where $\|f\|_{H^s}^2 = \sum_{k\in\mathbf{Z}}(1+|k|)^{2s}|C_k|^2$ with $C_k = (2\pi)^{-1}\int f(\theta)e^{-ik\theta}\, d\theta$. Let ℓ_α^2 be the weighted sequence space defined by

$$\ell_\alpha^2 = \{\{A_k\}_{k\in\mathbf{Z}};\ \|\{A_k\}_{k\in\mathbf{Z}}\|_{\ell_\alpha^2}^2 = \sum_{k\in\mathbf{Z}}(1+|k|)^{2\alpha}|A_k|^2 < \infty\}.$$

For the simplicity of description, we often use $\{A_k\}$ in place of $\{A_k\}_{k\in\mathbf{Z}}$. Then, our results are

Theorem 3.1 (local result) *For some $T > 0$, there exists a unique solution to (NLS) described as*

$$u(t,x) = \sum_{k\in\mathbf{Z}} A_k(t)\exp(it\partial_x^2)\delta_{ka}, \tag{3.1}$$

where $\{A_k(t)\} \in C([0,T];\ell_1^2)\cap C^1((0,T];\ell_1^2)$ with $A_0(0) = \mu_0$, $A_1(0) = \mu_1$ and $A_k(0) = 0$ $(k \neq 0,1)$.

Remark 3.1. Let us call $A_k(t)\exp(it\partial_x^2)\delta_{ka}$ the k-th mode. Then, (3.1) suggests that new modes away from 0-th and first ones appear in the solution while the initial data contains only the two modes. This special property is visible only in the nonlinear problem.

Remark 3.2. Reading the proof of Theorem 3.1, we see that it is possible to generalize the initial data. Namely, (NLS) is solvable even when point masses are distributed on a line at equal intervals, i.e., the initial data is given by $u(0,x) = \sum_{k\in\mathbf{Z}}\mu_k\delta_{ka}$, where $\{\mu_k\} \in \ell_1^2$. In this case, the solution is described similarly to (3.1) but $\{A_k(0)\} = \{\mu_k\}$ for $k \in \mathbf{Z}$. The decay condition on the coefficients is required to estimate the nonlinearity. This is because we use the inequality like $\|\mathcal{N}(v)\|_{L^2} \le C\|v\|_{L^\infty}^{\frac{p-1}{2}}\|v\|_{L^2}$ where $v = v(t,\theta) = \sum_k A_k e^{-ik\theta}e^{i(ka)^2/4t}$ and $\theta \in [0,2\pi]$ (see Lemma 3.4 below). Accordingly, to estimate $\|v\|_{L^\infty}$, we require the decay condition of $\{A_k\}$.

Remark 3.3. The infinite summation of (3.1) converges in $L^\infty_{loc}((0,T];L^\infty(\mathbf{R}))$, since, for any $\tau \in (0,T)$,

$$\begin{aligned}\sup_{\tau\le t\le T}\|u(t,\cdot)\|_{L^\infty(\mathbf{R})} &\le (4\pi\tau)^{-1/2}\sup_{\tau\le t\le T}\sum_k|A_k(t)| \\ &\le C(4\pi\tau)^{-1/2}\|\{A_k(t)\}\|_{L^\infty([0,T];\ell_1^2)} \\ &< \infty.\end{aligned}$$

This implies that the nonlinearity $\mathcal{N}(u(t,x))$ makes a sense as a function for $t \neq 0$. We also note that $u(t,x) \in C([0,T]; \mathcal{S}'(\mathbf{R}))$.

Remark 3.4. The representation (3.1) is derived by the following rough consideration. Since the nonlinear solution is first well-approximated by the linear solution $u_1(t,x) = \exp(it\partial_x^2)(\mu_0\delta_0 + \mu_1\delta_a)$ around $t = 0$, the second approximation $u_2(t,x)$ is given by solving

$$\begin{aligned} (i\partial_t + \partial_x^2)u_2 &= \mathcal{N}(u_1) \\ &= \mathcal{N}((2\pi)^{-1/2}e^{ix^2/4t}D(\mu_0 + \mu_1 e^{-iax}e^{ia^2/4t})) \\ &= |4\pi t|^{-(p-1)/2}(2\pi)^{-1/2}e^{ix^2/4t}D\mathcal{N}(1 + e^{-iax}e^{ia^2/4t}), \end{aligned} \tag{3.2}$$

where we have used $u_1 = e^{ix^2/4t}D\mathcal{F}e^{ix^2/4t}u(0,x)$, $Df(t,x) = (2it)^{-n/2}f(t,x/2t)$ and $\mathcal{F}$ denotes the Fourier transform. Let us replace ax by θ. Then, the nonlinearity in (3.2) is regarded as a 2π-periodic function of θ, and hence the Fourier series expansion yields

$$\begin{aligned} (\text{the right hand side of (3.2)}) &= |4\pi t|^{-(p-1)/2}(2\pi)^{-1/2}e^{ix^2/4t}D\sum_{k\in\mathbf{Z}}\tilde{B}_k(t)e^{i(ka)^2/4t}e^{-ik\theta} \\ &= |4\pi t|^{-(p-1)/2}\sum_{k\in\mathbf{Z}}B_k(t)\exp(it\partial_x^2)\delta_{ka}, \end{aligned}$$

where $B_k(t)e^{i(ka)^2/4t}$ is the Fourier coefficient. By the Duhamel principle, one can imagine that the solution to (NLS) has the description as in (3.1).

Our next interest is to see the global solbavility of (NLS). The sign of Imλ determines the blow-up or global existence.

Theorem 3.2 (blowing up or global result) *(1) Let Im$\lambda > 0$. Then, the solution as in Theorem 3.1 blows up in positive finite time. Precisely speaking, the ℓ_0^2-norm of $\{A_k(t)\}$ tends to infinity at some positive time.*

(2) Let Im$\lambda \leq 0$. Then, there exists a unique global solution as in Theorem 3.1 with $\{A_k(t)\} \in C([0,\infty); \ell_1^2) \cap C^1((0,\infty); \ell_1^2)$.

Let us present the proof of Theorem 3.1 and 3.2. The idea is based on the reduction of (NLS) into the ODE system of $\{A_k(t)\}_{k\in\mathbf{Z}}$. The next key lemma gives the representation formula of $\mathcal{N}(\sum_k A_k \exp(it\partial_x^2)\delta_{ka})$.

Lemma 3.3 *Let $\{A_k(t)\} \in C([0,T]; \ell_1^2)$. Then, we have*

$$\mathcal{N}(\sum_{k\in\mathbf{Z}} A_k(t)\exp(it\partial)\delta_{ka}) = |4\pi t|^{-(p-1)/2}\sum_{k\in\mathbf{Z}}\tilde{A}_k(t)\exp(it\partial)\delta_{ka}, \tag{3.3}$$

where $\widetilde{A}_k(t) = (2\pi)^{-1}e^{-i(ka)^2/4t}\langle\mathcal{N}(v), e^{-ik\theta}\rangle_\theta$ *with* $v = v(t,\theta) = \sum_j A_j(t)e^{-ij\theta}e^{i(ja)^2/4t}$ *and* $\langle f, g\rangle_\theta = \int_0^{2\pi} f(\theta)\overline{g(\theta)}d\theta$.

Proof of Lemma 3.3. Note that the linear Schrödinger group is factorized as follows.

$$\begin{aligned}\exp(it\partial_x^2)f &= (4\pi it)^{-1/2}\int \exp(i|x-y|^2/4t)f(y)dy \\ &= MD\mathcal{F}Mf,\end{aligned}$$

where

$$\begin{aligned}Mg(t,x) &= e^{ix^2/4t}g(x), \\ Dg(t,x) &= (2it)^{-1/2}g(x/2t), \\ \mathcal{F}g(\xi) &= (2\pi)^{-1/2}\int e^{-i\xi x}g(x)dx \quad \text{(Fourier transform of } g).\end{aligned}$$

Then, we see that

$$\begin{aligned}&\mathcal{N}(\sum_j A_j(t)\exp(it\partial_x^2)\delta_{ja}) \\ &= \mathcal{N}((2\pi)^{-1/2}MD\sum_j A_j(t)e^{-ijax+i(ja)^2/4t}) \\ &= |4\pi t|^{-(p-1)/2}(2\pi)^{-1/2}MD\mathcal{N}(\sum_j A_j(t)e^{-ijax+i(ja)^2/4t}).\end{aligned} \tag{3.4}$$

Note that, to show the last equality in (3.4), we make use of the gauge invariance of the nonlinearity. Replacing ax by θ, we can regard $\mathcal{N}(\sum_j A_j(t)e^{-ij\theta+i(ja)^2/4t})$ as a 2π-periodic function of θ. Therefore, by the Fourier series expansion,

$$\begin{aligned}\mathcal{N}(\sum_j A_j(t)e^{-ij\theta+i(ja)^2/4t}) &= \sum_k C_k(t)e^{-ik\theta} \\ &= \sum_k \widetilde{A}_k(t)e^{i(ka)^2/4t}e^{-ik\theta} \\ &= (2\pi)^{1/2}\sum_k \widetilde{A}_k(t)\mathcal{F}M\delta_{ka},\end{aligned}$$

where we let $C_k(t) = (2\pi)^{-1}\langle\mathcal{N}(v), e^{-ik\theta}\rangle_\theta$ and rewrote $C_k(t) = \widetilde{A}_k(t)e^{i(ka)^2/4t}$. Plugging this into (3.4), we obtain Lemma 3.3. □

We now explain how to reduce (NLS) into the ODE system of $\{A_k(t)\}$. By substituting $u = \sum_k A_k(t)\exp(it\partial_x^2)\delta_{ka}$ into (NLS) and noting that $i\partial_t \exp(it\partial_x^2)\delta_{ka} = -\partial_x^2\exp(it\partial_x^2)\delta_{ka}$, Lemma 3.3 yields

$$\sum_k i\frac{dA_k}{dt}\exp(it\partial_x^2)\delta_{ka} = \lambda|4\pi t|^{-(p-1)/2}\sum_k \widetilde{A}_k\exp(it\partial_x^2)\delta_{ka}.$$

Equating the terms on both hand sides, we arrive at the desired ODE system:

$$i\frac{dA_k}{dt} = \lambda|4\pi t|^{-(p-1)/2}\tilde{A}_k \tag{3.5}$$

with the initial condition $A_k(0) = \mu_k$. Now, showing the existence and uniqueness problems of (NLS) is equivalent to showing those of (3.5). To solve (3.5), let us consider the following integral equation.

$$\begin{aligned} \{A_k(t)\} &= \{\Phi_k(\{A_j(t)\})\} \\ &\equiv \{\mu_k\} - i\lambda\int_0^t |4\pi\tau|^{-(p-1)/2}\{\tilde{A}_k(\tau)\}\,d\tau. \end{aligned} \tag{3.6}$$

Then, we want to see the contraction mapping property of $\{\Phi_k\}$. The simple aplication of Parseval's identity derives the following.

Lemma 3.4 *Let $I = [0,T]$. Then, we have*

$$\|\{\tilde{A}_k\}\|_{L^\infty(I;\ell_1^2)} \le C\|\{A_k\}\|^p_{L^\infty(I;\ell_1^2)}, \tag{3.7}$$

$$\begin{aligned} &\|\{\tilde{A}_k^{(1)}\} - \{\tilde{A}_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)} \\ &\le C(\max_{j=1,2}\|\{A_k^{(j)}\}\|_{L^\infty(I;\ell_1^2)})^{p-1}\|\{A_k^{(1)}\} - \{A_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)}. \end{aligned} \tag{3.8}$$

Proof of Lemma 3.4. According to the description of $\tilde{A}_k$ as in Lemma 3.3 and the integration by parts, we see that

$$k\tilde{A}_k = (2\pi)^{-1}ie^{-i(ka)^2/4t}\langle\partial_\theta\mathcal{N}(\sum_j A_je^{-ij\theta}e^{i(ja)^2/4t}), e^{-ik\theta}\rangle_\theta.$$

Then, Parseval's identity and $\|\sum_j A_je^{-ij\theta+i(ja)^2/4t}\|_{L^\infty} \le C\|\{A_j\}\|_{\ell_1^2}$ yield

$$\begin{aligned} \|\{k\tilde{A}_k\}\|_{\ell_0^2} &= (2\pi)^{-1/2}\|\partial_\theta\mathcal{N}(\sum_j A_je^{-ij\theta}e^{i(ja)^2/4t})\|_{L^2} \\ &\le C\|\sum_j A_je^{-ij\theta}e^{i(ja)^2/4t}\|^{p-1}_{L^\infty}\|\sum_j jA_je^{-ij\theta}e^{i(ja)^2/4t}\|_{L^2} \\ &\le C\|\{A_j\}\|^p_{\ell_1^2}. \end{aligned}$$

Thus, we obtain (3.7). The proof for (3.8) follows similarly. Since there is a singularity at $u = 0$ of the nonlinearity $\mathcal{N}(u)$, we do not employ ℓ_1^2-norm to measure $\{A_k^{(1)}\} - \{A_k^{(2)}\}$. ❑

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof relies on the contraction mapping principle of $\{\Phi_k(\{A_j\})\}$. Let $\|\{\mu_k\}\|_{\ell_1^2} \le \rho_0$ and

$$\overline{B}_{2\rho_0} = \{\{A_k\} \in L^\infty([0,T];\ell_1^2);\ \|\{A_k\}\|_{L^\infty([0,T];\ell_1^2)} \le 2\rho_0\}$$

endowed with the metric in $L^\infty([0,T];\ell_0^2)$. Note that $\overline{B}_{2\rho_0}$ is closed in $L^\infty([0,T];\ell_0^2)$. Then, in virture of Lemma 3.4, we see that

$$\begin{aligned}
&\|\{\Phi_k(\{A_j\})\}\|_{L^\infty([0,T];\ell_1^2)} \le \rho_0 + CT^{(3-p)/2}(2\rho_0)^p,\\
&\|\{\Phi_k(\{A_j^{(1)}\})\} - \{\Phi_k(\{A_j^{(2)}\})\}\|_{L^\infty([0,T];\ell_0^2)}\\
&\le CT^{(3-p)/2}(2\rho_0)^{p-1}\|\{A_k^{(1)}\} - \{A_k^{(2)}\}\|_{L^\infty([0,T];\ell_0^2)}.
\end{aligned}$$

Thus, $\{\Phi_k(\{A_j\})\}$ is the contraction map on $\overline{B}_{2\rho_0}$ if T is sufficiently small. This implies that a solution to (3.6) exists in $L^\infty([0,T];\ell_1^2)$. Since $\int_0^t |4\pi\tau|^{-(p-1)/2}\{\tilde{A}_k\}\, d\tau$ belongs to $C([0,T];\ell_1^2)$ by Lebesgue's convergence theorem, the solution is ℓ_1^2-valued continuous function and so it belongs to $C^1((0,T];\ell_1^2)$. The uniqueness of $\{A_k(t)\}$ in $C(I;\ell_0^2)$ follows in the standard way. Hnece, Theorem 3.1 is obtained. □

To prove Theorem 3.2, we apply the a priori estimates described in the following.

Lemma 3.5 *Let $\{A_k(t)\}$ be the solution to (3.5) in $C([0,T];\ell_1^2) \cap C^1((0,T];\ell_1^2)$.*

(1) Then, we have

$$\frac{d\|\{A_k(t)\}\|_{\ell_0^2}^2}{dt} = \frac{Im\lambda}{\pi}(4\pi t)^{-(p-1)/2}\|v(t)\|_{L^{p+1}}^{p+1}, \tag{3.9}$$

where $v(t,\theta) = \sum_k A_k(t)e^{-ik\theta}e^{i(ka)^2/4t}$.

(2) In addition, if $Im\lambda \le 0$, then we have

$$\|\{kA_k(t)\}\|_{\ell_0^2} \le Ce^{2t}, \tag{3.10}$$

where the positive constant C does not depend on T.

Remark 3.5 The bound in (3.10) may be refined by sophisticating the estimates in the proof. We do not, however, concentrate ourselves to this kind of refinement.

Proof of Lemma 3.5. According to (3.5), we see that $v = v(t,\theta)$ satisfies the nonlinear equation like

$$i\partial_t v = -\frac{a^2}{4t^2}\partial_\theta^2 v + \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(v). \tag{3.11}$$

Of course, we require to check whether $\partial_t v$ and $\partial_\theta^2 v$ make a sense. This is justified by the mollification. In this proof, however, we do not consider this kind of matters since we want to avoid the complication of the proof. Let us remark that $\sqrt{2\pi}\|\{A_k(t)\}\|_{\ell_0^2} = \|v(t)\|_{L^2}$ and $\sqrt{2\pi}\|\{kA_k(t)\}\|_{\ell_0^2} = \|\partial_\theta v(t)\|_{L^2}$. Then, multiplying (3.11) with $\overline{v}$ and taking the imaginary part of integration, we obtain (3.9). On the other hand, multiplying (3.11) with $\overline{\partial_t v}$ and taking the real part of integration, we have

$$\begin{aligned} 0 &= -\frac{a^2}{4t^2}\frac{d}{dt}\|\partial_\theta v\|_{L^2}^2 + \frac{2\mathrm{Re}\lambda}{p+1}|4\pi t|^{-(p-1)/2}\frac{d}{dt}\|v\|_{L^{p+1}}^{p+1} \\ &\quad -2(\mathrm{Im}\lambda)|4\pi t|^{-(p-1)/2}\mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta. \end{aligned} \tag{3.12}$$

To estimate $\mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta$ in (3.12), let us multiply $\overline{\mathcal{N}(v)}$ on both hand sides of (3.11). Then, we see that

$$\begin{aligned} \mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta &= -\frac{a^2}{4t^2}\mathrm{Re}\langle\partial_\theta^2 v,\mathcal{N}(v)\rangle_\theta + (\mathrm{Re}\lambda)|4\pi t|^{-(p-1)/2}\|v\|_{L^{2p}}^{2p} \\ &\geq (\mathrm{Re}\lambda)|4\pi t|^{-(p-1)/2}\|v\|_{L^{2p}}^{2p}, \end{aligned} \tag{3.13}$$

since $\mathrm{Re}\langle\partial_\theta^2 v,\mathcal{N}(v)\rangle_\theta \leq 0$. Combining (3.12) and (3.13), we have

$$\frac{d}{dt}\|\partial_\theta v\|_{L^2}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\frac{d}{dt}\|v\|_{L^{p+1}}^{p+1} - K_2(\mathrm{Im}\lambda)(\mathrm{Re}\lambda)t^{3-p}\|v\|_{L^{2p}}^{2p} \leq 0, \tag{3.14}$$

where $K_1 = \dfrac{8}{(p+1)a^2(4\pi)^{(p-1)/2}}$ and $K_2 = \dfrac{8}{a^2(4\pi)^{p-1}}$. This is equivalent to

$$\frac{d}{dt}E(t) \leq \frac{(5-p)K_1\mathrm{Re}\lambda}{2}t^{(3-p)/2}\|v\|_{L^{p+1}}^{p+1}, \tag{3.15}$$

where

$$E(t) = \|\partial_\theta v\|_{L^2}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\|v\|_{L^{p+1}}^{p+1} - K_2(\mathrm{Im}\lambda)(\mathrm{Re}\lambda)\int_{t_0}^t \tau^{3-p}\|v(\tau)\|_{L^{2p}}^{2p}\,d\tau.$$

We first consider the case $\mathrm{Im}\lambda \leq 0$ and $\mathrm{Re}\lambda < 0$. By (3.15), we have $E(t) \leq$ (const.) for $t > t_0$, i.e.,

$$\|\partial_\theta v\|_{L^2}^2 \leq C_1 + C_2 t^{(5-p)/2}\|v\|_{L^{p+1}}^{p+1} + C_3\int_{t_0}^t \tau^{3-p}\|v(\tau)\|_{L^{2p}}^{2p}\,d\tau \tag{3.16}$$

for some positive constants C_1, C_2 and C_3. Applying the Gagliardo-Nirenberg inequalities:

$$\begin{aligned}\|v\|_{L^{p+1}}^{p+1} &\leq C\|v\|_{H^1}^{(p+1)\beta}\|v\|_{L^2}^{(p+1)(1-\beta)},\\ \|v\|_{L^{2p}}^{2p} &\leq C\|v\|_{H^1}^{2p\gamma}\|v\|_{L^2}^{2p(1-\gamma)}.\end{aligned}$$

where $1/(p+1) = \beta(1/2-1) + (1-\beta)/2$ and $1/(2p) = \gamma(1/2-1) + (1-\gamma)/2$, and using Young's inequality, we have

$$\begin{aligned}\|v(t)\|_{H^1}^2 &\leq C + Ct^{(5-p)/2}\|v(t)\|_{H^1}^{(p+1)\beta}\|v(t)\|_{L^2}^{(p+1)(1-\beta)}\\ &\quad + C\int_{t_0}^{t}\tau^{3-p}\|v(\tau)\|_{H^1}^{2p\gamma}\|v(\tau)\|_{L^2}^{2p(1-\gamma)}\,d\tau\\ &\leq C + Ct^{(5-p)/2}\|v(t)\|_{H^1}^{(p-1)/2} + C\int_{t_0}^{t}\tau^{3-p}\|v(\tau)\|_{H^1}^{p-1}\,d\tau\\ &\leq C(1+t)^3 + \frac{1}{2}\|v(t)\|_{H^1}^2 + \int_{t_0}^{t}\|v(\tau)\|_{H^1}^2\,d\tau.\end{aligned}\tag{3.17}$$

We here note that, since $\|v(t)\|_{L^2}$ has a finite bound in virture of (3.9), it is included in the positive constant C. Then, applying Gronwall's inequality to (3.17), we obtain (3.10).

We next consider the case $\mathrm{Im}\lambda \leq 0$ and $\mathrm{Re}\lambda \geq 0$. By (3.14), we have

$$\frac{d}{dt}\|\partial_\theta v(t)\|_{L^2}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\frac{d}{dt}\|v(t)\|_{L^{p+1}}^{p+1} \leq 0.$$

Let $F(t) = \|\partial_\theta v(t)\|_{L^2}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\|v(t)\|_{L^{p+1}}^{p+1}$. Then, from the above inequality, it follows that

$$\begin{aligned}\frac{d}{dt}F(t) &\leq \frac{5-p}{2}K_1(\mathrm{Re}\lambda)\|v(t)\|_{L^{p+1}}^{p+1}\\ &\leq \frac{5-p}{2}t^{-1}F(t).\end{aligned}$$

This implies that $F(t) \leq F(t_0)\left(\frac{t}{t_0}\right)^{(5-p)/2}$. Since $\|\partial_\theta v(t)\|_{L^2}^2 \leq F(t)$, there exists a positive constant C such that $\|v(t)\|_{H^1}^2 \leq C(1+t)^{(5-p)/2}$. Hence, we obtain (3.10) ❑

Proof of Theorem 3.2. If $\mathrm{Im}\lambda > 0$, then, Lemma 3.5 (3.9) and Hölder's inequality $\|v\|_{L^{p+1}}^{p+1} \geq (2\pi)^{-(p-1)/2}\|v\|_{L^2}^{p+1}$ give

$$\frac{d}{dt}\|v\|_{L^2}^2 \geq C\mathrm{Im}\lambda t^{-(p-1)/2}\|v\|_{L^2}^{p+1}.$$

This implies that $\|v(t)\|_{L^2} = \|\{A_k(t)\}\|_{\ell_0^2}$ blows up in positive finite time. On the other hand, if $\mathrm{Im}\lambda \leq 0$, then, Lemma 3.5 gives the a priori bound of $\|\{A_k(t)\}\|_{\ell_1^2}$ for any positive t. Hence, the local solution to (3.5) is continuated to the global one. ❑

4 The case $u(0,x) = \mu_{00}\delta_0 + \mu_{10}\delta_a + \mu_{01}\delta_b$ $(a/b \notin \mathbf{Q})$

In this section, we consider the case in which the initial data consists of triple δ-functions supported at $x = 0, a$ and b. If $a/b \in \mathbf{Q}$ ($\mathbf{Q}$ denotes the set of rational numbers), the location of δ-functions is the special one mentioned in Remark 3.2 and thus (NLS) is solvable as in Theorem 3.1 and 3.2. Therefore, our concern is to observe the case $a/b \notin \mathbf{Q}$. Before stating our main results, we introduce several new notations. We often use weighted sequence space $\ell^2_\alpha(\mathbf{Z}^2)$ endowed with the norm

$$\|\{A_{k_1,k_2}\}_{k_1,k_2\in\mathbf{Z}}\|_{\ell^2_\alpha} = \Big(\sum_{k_1,k_2\in\mathbf{Z}} (1+|k_1|+|k_2|)^{2\alpha}|A_{k_1,k_2}|\Big)^{1/2}.$$

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. The quantity $\|f\|_{L^q(\mathbf{T}^2)}$ denotes $\left(\int_{\mathbf{T}^2} |f(\theta_1,\theta_2)|^q \, d\theta_1 d\theta_2\right)^{1/q}$.

We next define the Besov space for periodic functions. For $s > 0$, $[s]$ denotes the greatest integer not exceeding s. Then, if s is not integer and $1 < q, r < \infty$, the Besov space $B^s_{q,r}(\mathbf{T}^2)$ is defined by

$$B^s_{q,r}(\mathbf{T}^2) = \{f \in L^q(\mathbf{T}^2);\ \|f\|_{B^s_{q,r}(\mathbf{T}^2)} < \infty\},$$

where

$$\begin{aligned}\|f\|_{B^s_{q,r}(\mathbf{T}^2)} &\equiv \|f\|_{L^q(\mathbf{T}^2)} + \|f\|_{\dot{B}^s_{q,r}} \\ &\equiv \|f\|_{L^q(\mathbf{T}^2)} + \left(\int_0^\infty \tau^{-rs-1} \sup_{|h|<\tau} \|d_h^{[s]+1} f\|^r_{L^q(\mathbf{T}^2)} \, d\tau\right)^{1/q}\end{aligned}$$

with $h = (h_1, h_2)$ and $d_h^N f(\theta_1,\theta_2) = \sum_{j=0}^{N} \begin{pmatrix} N \\ j \end{pmatrix} (-1)^k f(\theta_1 + jh_1, \theta_2 + jh_2)$. We remark that, if $0 \le \sigma \le 1$ and $1/q = \sigma/q_1 + (1-\sigma)/q_0$ with $1 \le q_1, q_0 \le \infty$, then the Gagliardo-Nirenberg type inequality $\|f\|_{\dot{B}^{\sigma s}_{q,r/\sigma}(\mathbf{T}^2)} \le C\|f\|^\sigma_{\dot{B}^s_{q_1,r}} \|f\|^{1-\sigma}_{L^{q_0}(\mathbf{T}^2)}$ follows from the above definition. We also note that $\|f\|_{B^s_{2,2}(\mathbf{T}^2)}$ is equivalent to

$$\|f\|_{H^s(\mathbf{T}^2)} \equiv \left(\sum_{k_1,k_2\in\mathbf{Z}} (1+|k_1|+|k_2|)^{2\alpha} |C_{k_1,k_2}|^2\right)^{1/2},$$

where C_{k_1,k_2} is the Fourier coefficient of f given by $(2\pi)^{-2}\int_{\mathbf{T}^2} f(\theta_1,\theta_2) e^{-i(k_1\theta_1 + k_2\theta_2)} \, d\theta_1 d\theta_2$. For more detail about Besov space, see [4].

For the simplicity of description, we often use the brief notation $\{A_{k_1,k_2}\}$ in place of $\{A_{k_1,k_2}\}_{k_1,k_2\in\mathbf{Z}}$. Then, our first result is

Theorem 4.1 (local result) *Let $1 < \alpha < p$. Then, for some $T > 0$, there exists a unique solution to (NLS) described as*

$$u(t,x) = \sum_{k_1,k_2\in\mathbf{Z}} A_{k_1,k_2}(t)\exp(it\partial_x^2)\delta_{k_1a+k_2b}, \tag{4.1}$$

where the coefficient sequence $\{A_{k_1,k_2}(t)\} \in C([0,T];\ell^2_\alpha(\mathbf{Z}^2)) \cap C^1((0,T];\ell^2_\alpha(\mathbf{Z}^2))$ with $A_{k_1,k_2}(0) = \mu_{k_1,k_2}$ if $(k_1,k_2) = (0,0),(1,0),(0,1)$ and $A_{k_1,k_2}(0) = 0$ otherwise.

Remark 4.1. As mentioned in Remark 3.1, the solution in Theorem 4.1 causes the generation of new modes. The point remarkably different from Theorem 3.1 is that, for $t \neq 0$, $\exp(-it\partial_x^2)u$ looks like the point mass measures densely distributed on $\mathbf{R}$ since a/b is irrational. Reading the proof of Theorem 4.1, we see that it is possible to construct a solution even when the initial data consists of infintely many δ-functions given by $u(0,x) = \sum_{k_1,k_2\in\mathbf{Z}} \mu_{k_1,k_2}\delta_{k_1a+k_2b}$, where $\{\mu_{k_1,k_2}\} \in \ell^2_\alpha(\mathbf{Z}^2)$.

Similarly to Theorem 3.2, the sign of $\mathrm{Im}\lambda$ determines the global solvability of (NLS).

Theorem 4.2 (blowing up or global result) *(1) Let $Im\lambda > 0$. Then, the solution as in Theorem 4.1 blows up in positive finite time. Precisely speaking, the $\ell^2_0(\mathbf{Z}^2)$-norm of $\{A_{k_1,k_2}(t)\}$ tends to infinity at some positive time.*

(2) Let $Im\lambda \leq 0$ and, in addition, $|Re\lambda| \leq \dfrac{2\sqrt{p}}{p-1}|Im\lambda|$. Then, there exists a unique global solution as in Theorem 4.1. Futhermore,

$$\{A_{k_1,k_2}(t)\} \in C([0,\infty);\ell^2_\alpha(\mathbf{Z}^2)) \cap C^1((0,\infty);\ell^2_\alpha(\mathbf{Z}^2)).$$

Remark 4.2. As for the global result, it is still open whether the additional condition $|\mathrm{Re}\lambda| \leq \dfrac{2\sqrt{p}}{p-1}|\mathrm{Im}\lambda|$ is removed or not. In our proof, this condition will be applied to obtain the time global estimate of $\|\{A_{k_1,k_2}(t)\}\|_{\ell^2_1(\mathbf{Z}^2)}$. The key to derive this esimate is Liskevich-Perelmuter's inequality [16], i.e., if $\mathrm{Im}\lambda \leq 0$ and $|\mathrm{Re}\lambda| \leq \dfrac{2\sqrt{p}}{p-1}|\mathrm{Im}\lambda|$, then it follows that $\mathrm{Im}\left(\lambda(\mathcal{N}(v_1) - \mathcal{N}(v_2))\overline{(v_1 - v_2)}\right) \leq 0$.

The idea to prove Theorem 4.1 is quite analogous in the proof of Theorem 3.1. Namely, we reduce (NLS) into ODE system. To solve this ODE system, we use several lemmas given below.

Lemma 4.3 *Let $\alpha > 1$ and $\{A_{k_1,k_2}(t)\} \in C([0,T];\ell^2_\alpha(\mathbf{Z}^2))$. Then, we have*

$$\mathcal{N}\Big(\sum_{k_1,k_2\in\mathbf{Z}} A_{k_1,k_2}(t)U(t)\delta_{k_1a+k_2b}\Big) = |4\pi t|^{-(p-1)/2}\sum_{k_1,k_2\in\mathbf{Z}} \tilde{A}_{k_1,k_2}(t)U(t)\delta_{k_1a+k_2b}, \tag{4.2}$$

where $\tilde{A}_{k_1,k_2}(t) = (2\pi)^{-2}e^{-i(k_1a+k_2b)^2/4t}\langle \mathcal{N}(w), e^{-i(k_1\theta_1+k_2\theta_2)}\rangle_{\theta_1,\theta_2}$ *with*

$$w = w(t,\theta_1,\theta_2) = \sum_{k_1,k_2\in\mathbf{Z}} A_{k_1,k_2}(t)e^{i(k_1a+k_2b)^2/4t}e^{-i(k_1\theta_1+k_2\theta_2)}$$

and

$$\langle f, g\rangle_{\theta_1,\theta_2} = \int_{\mathbf{T}^2} f(\theta_1,\theta_2)\overline{g(\theta_1,\theta_2)}\, d\theta_1 d\theta_2.$$

Proof of Lemma 4.3. By using the factorization $\exp(it\partial_x^2)f = MD\mathcal{F}Mf$ as in the proof of Lemma 3.3, we see that

$$\begin{aligned}
(4.3)\quad & \mathcal{N}(\sum_{k_1,k_2} A_{k_1,k_2}(t)\exp(it\partial_x^2)\delta_{k_1a+k_2b}) \\
&= \mathcal{N}((2\pi)^{-1/2}MD\sum_{k_1,k_2} A_{k_1,k_2}(t)e^{-i(k_1ax+k_2bx)+i(k_1a+k_2b)^2/4t}) \\
&= |4\pi t|^{-(p-1)/2}(2\pi)^{-1/2}MD\mathcal{N}(\sum_{k_1,k_2} A_{k_1,k_2}(t)e^{-i(k_1ax+k_2bx)+i(k_1a+k_2b)^2/4t}).
\end{aligned}$$

Note that, to show the last equality in (4.3), we make use of the gauge invariance of the nonlinearity. Replacing ax (resp. bx) by θ_1 (resp. θ_2), we can regard

$$\mathcal{N}(\sum_{k_1,k_2} A_{k_1,k_2}(t)e^{-i(k_1\theta_1+k_2\theta_2)-i(k_1a+k_2b)^2/4t})$$

as a 2π-periodic function of θ_1 and θ_2. Therefore, by the Fourier series expansion,

$$\begin{aligned}
& \mathcal{N}(\sum_{k_1,k_2} A_{k_1,k_2}(t)e^{-i(k_1\theta_1+k_2\theta_2)+i(k_1a+k_2b)^2/4t}) \\
&= \sum_{k_1,k_2} C_{k_1,k_2}(t)e^{-i(k_1\theta_1+k_2\theta_2)} \\
&= \sum_{k_1,k_2} \tilde{A}_{k_1,k_2}(t)e^{i(k_1a+k_2b)^2/4t}e^{-i(k_1\theta_1+k_2\theta_2)} \\
&= (2\pi)^{1/2}\sum_{k_1,k_2} \tilde{A}_{k_1,k_2}(t)\mathcal{F}M\delta_{k_1a+k_2b},
\end{aligned}$$

where we let $C_{k_1,k_2}(t) = (2\pi)^{-1}\langle \mathcal{N}(w), e^{-i(k_1\theta_1+k_2\theta_2)}\rangle_{\theta_1,\theta_2}$ which is the Foirier coefficient of $\mathcal{N}(w)$ and rewrote $C_{k_1,k_2}(t) = \tilde{A}_{k_1,k_2}(t)e^{i(k_1a+k_2b)^2/4t}$. Plugging this into (4.3), we obtain Lemma 4.3. $\square$

Let us reduce (NLS) into ODE system. By substituting the infinite superposition of the linear solution $u(t,x) = \sum_{k_1,k_2} A_{k_1,k_2}(t)\exp(it\partial_x^2)\delta_{k_1a+k_2b}$ into (NLS) and noting that $i\partial_t \exp(it\partial_x^2)\delta_{k_1a+k_2b} = -\partial_x^2\exp(it\partial_x^2)\delta_{k_1a+k_2b}$, Lemma 4.3 yields

$$\sum_{k_1,k_2} i\frac{dA_{k_1,k_2}}{dt}\exp(it\partial_x^2)\delta_{k_1a+k_2b} = \lambda|4\pi t|^{-(p-1)/2}\sum_{k_1,k_2}\tilde{A}_{k_1,k_2}\exp(it\partial_x^2)\delta_{k_1a+k_2b}.$$

This implies that

$$\sum_{k_1,k_2} i\frac{dA_{k_1,k_2}}{dt}\delta_{k_1a+k_2b} = \lambda|4\pi t|^{-(p-1)/2}\sum_{k_1,k_2}\tilde{A}_{k_1,k_2}\delta_{k_1a+k_2b}. \tag{4.4}$$

Equating the terms on both hand sides of (4.4), we arrive at the following ODE system:

$$i\frac{dA_{k_1,k_2}}{dt} = \lambda|4\pi t|^{-(p-1)/2}\tilde{A}_{k_1,k_2}. \tag{4.5}$$

In fact, this identity holds by multiplying (4.4) with a test function supported around $x = k_1a + k_2b$ and by shrinking its support. To solve (4.5) with the initial condition $A_{k_1,k_2}(0) = \mu_{k_1,k_2}$, we translate it into the integral equation like

$$\begin{aligned}\{A_{k_1,k_2}(t)\} &= \{\Phi_{k_1,k_2}(\{A_{j_1,j_2}(t)\})\} \\ &\equiv \{\mu_{k_1,k_2}\} - i\lambda\int_0^t |4\pi\tau|^{-(p-1)/2}\{\tilde{A}_{k_1,k_2}(\tau)\}\,d\tau.\end{aligned} \tag{4.6}$$

This will be solved by contraction mapping argument. To this end, we need several lemmas concerning the nonlinear estimates.

Lemma 4.4 *Let* $1 < \alpha < p$ *and* $f = f(\theta_1,\theta_2) \in B^{\alpha}_{2,2}(\mathbf{T}^2)$. *Then, we have*

$$\||f|^{p-1}f\|_{B^{\alpha}_{2,2}(\mathbf{T}^2)} \le C\|f\|^{p-1}_{L^\infty(\mathbf{T}^2)}\|f\|_{B^{\alpha}_{2,2}(\mathbf{T}^2)}. \tag{4.7}$$

Proof of Lemma 4.4. This estimate is proved by refering to [7, 9]. □

Applying Lemma 4.4, we can estimate the sequence $\{\tilde{A}_{k_1,k_2}\}(=\{\tilde{A}_{k_1,k_2}(t)\})$ defined in Lemma 4.3.

Corollary 4.5 *Let* $I = [0,T]$. *Then, we have*

$$\|\{\tilde{A}_{k_1,k_2}\}\|_{L^\infty(I;\ell^2_\alpha(\mathbf{Z}^2))} \le C\|\{A_{k_1,k_2}\}\|^p_{L^\infty(I;\ell^2_\alpha(\mathbf{Z}^2))}, \tag{4.8}$$

$$\begin{aligned}&\|\{\tilde{A}^{(1)}_{k_1,k_2}\} - \{\tilde{A}^{(2)}_{k_1,k_2}\}\|_{L^\infty(I;\ell^2_0(\mathbf{Z}^2))} \\ &\le C\left(\max_{j=1,2}\|\{A^{(j)}_{k_1,k_2}\}\|_{L^\infty(I;\ell^2_\alpha(\mathbf{Z}^2))}\right)^{p-1}\|\{A^{(1)}_{k_1,k_2}\} - \{A^{(2)}_{k_1,k_2}\}\|_{L^\infty(I;\ell^2_0(\mathbf{Z}^2))}.\end{aligned} \tag{4.9}$$

Proof of Corollary 4.5. By Parseval's identity,

$$\|\{\tilde{A}_{k_1,k_2}(t)\}\|_{\ell^2_\alpha(\mathbf{Z}^2)} = (2\pi)^{-1}\|\mathcal{N}(w(t))\|_{H^\alpha(\mathbf{T}^2)},$$

where $w(t) = w(t,\theta_1,\theta_1) = \sum_{k_1,k_2\in\mathbf{Z}} A_{k_1,k_2}(t)e^{i(k_1a+k_2b)^2/4t}e^{-i(k_1\theta_1+k_2\theta_2)}$. Applying Lemma 4.4, we have

$$\begin{aligned}\|\{\widetilde{A}_{k_1,k_2}(t)\}\|_{\ell^2_\alpha(\mathbf{Z}^2)} &\le C\|\mathcal{N}(w(t))\|_{B^\alpha_{2,2}(\mathbf{T}^2)}\\ &\le C\|w(t)\|^{p-1}_{L^\infty(\mathbf{T}^2)}\|w(t)\|_{B^\alpha_{2,2}(\mathbf{T}^2)}.\end{aligned}$$

Since $\|w(t)\|_{L^\infty(\mathbf{T}^2)} \le C\|w(t)\|_{H^\alpha(\mathbf{T}^2)} = 2\pi C\|\{A_{k_1,k_2}(t)\}\|_{\ell^2_\alpha(\mathbf{Z}^2)}$, we obtain (4.8). The proof for (4.9) more simply follows. Note that we can not replace $\|\{\widetilde{A}^{(1)}_{k_1,k_2}\}-\{\widetilde{A}^{(2)}_{k_1,k_2}\}\|_{L^\infty(I;\ell^2_0(\mathbf{Z}^2))}$ by the weighted ℓ^2-norm since the nonlinearity $\mathcal{N}(w)$ contains the singularity at $w=0$.
$\square$

Proof of Theorem 4.1. Let $I=[0,T]$, $\|\{\mu_{k_1,k_2}\}\|_{\ell^2_\alpha(\mathbf{Z}^2)} \le \rho_0$ and

$$\overline{B}_{2\rho_0} = \{\{A_{k_1,k_2}\} \in L^\infty(I;\ell^2_\alpha(\mathbf{Z}^2));\ \|\{A_{k_1,k_2}\}\|_{L^\infty(I;\ell^2_\alpha(\mathbf{Z}^2))} \le 2\rho_0\}.$$

Note that $\overline{B}_{2\rho_0}$ is closed in $L^\infty(I;\ell^2_0(\mathbf{Z}^2))$. We first show that $\{\Phi_{k_1,k_2}(\{A_{j_1,j_2}\})\}$ in (4.6) is the contraction map on $\overline{B}_{2\rho_0}$ with the metric of $L^\infty(I;\ell^2_0(\mathbf{Z}^2))$. By applying Corollary 4.5, it is easy to see that

$$\begin{aligned}&\|\{\Phi_{k_1,k_2}(\{A_{j_1,j_2}\})\}\|_{L^\infty(I;\ell^2_\alpha(\mathbf{Z}^2))} \le \rho_0 + CT^{(3-p)/2}(2\rho_0)^p,\\ &\|\{\Phi_{k_1,k_2}(\{A^{(1)}_{j_1,j_2}\}) - \{\Phi_{k_1,k_2}(\{A^{(2)}_{j_1,j_2}\})\|_{L^\infty(I;\ell^2_0(\mathbf{Z}^2))}\\ &\le CT^{(3-p)/2}(2\rho_0)^{p-1}\|\{A^{(1)}_{k_1,k_2}\} - \{A^{(2)}_{k_1,k_2}\}\|_{L^\infty(I;\ell^2_0(\mathbf{Z}^2))}.\end{aligned}$$

Thus, taking $T>0$ sufficiently small, we observe that $\{\Phi_{k_1,k_2}(\{A_{j_1,j_2}\})\}$ is the contraction map. This implies that a solution to (4.6) exists in $L^\infty(I;\ell^2_\alpha(\mathbf{Z}^2))$. Since $\int_0^t |4\pi\tau|^{-(p-1)/2}\{\widetilde{A}_{k_1,k_2}\}\,d\tau$ belongs to $C(I;\ell^2_\alpha(\mathbf{Z}^2))$ by Lebesgue's convergence theorem, the solution is $\ell^2_\alpha(\mathbf{Z}^2)$-valued continuous function and so it belongs to $C^1((0,T];\ell^2_\alpha(\mathbf{Z}^2))$. The uniqueness of $\{A_{k_1,k_2}(t)\}$ in $C(I;\ell^2_0(\mathbf{Z}^2))$ follows in the standard way. $\square$

Let us next prove Theorem 4.2. To continuate the local solution of the ODE system (4.5) to the global one, we need time global bound of $\|\{A_{k_1,k_2}(t)\}\|_{\ell^2_\alpha(\mathbf{Z}^2)}(\simeq \|w(t)\|_{B^\alpha_{2,2}(\mathbf{T}^2)})$. The estimate of $\|w(t)\|_{B^1_{2,2}(\mathbf{T}^2)}$ and the logalithmic Sobolev inequality due to Brezis-Gallouet [3] will present this bound.

Lemma 4.6 *Let $\{A_{k_1,k_2}\}$ be the solution to (4.5) in $C([0,T];\ell^2_\alpha(\mathbf{Z}^2))\cap C^1((0,T];\ell^2_\alpha(\mathbf{Z}^2))$.*

(1) Then, we have

$$\frac{d}{dt}\|\{A_{k_1,k_2}(t)\}\|^2_{\ell^2_\alpha(\mathbf{Z}^2)} = \frac{Im\lambda}{2\pi^2|4\pi t|^{(p-1)/2}}\|w(t)\|^{p+1}_{L^{p+1}(\mathbf{T}^2)}, \tag{4.10}$$

where $w(t) = w(t,\theta_1,\theta_2) = \sum_{k_1,k_2} A_{k_1,k_2}(t)e^{i(k_1a+k_2b)^2/4t}e^{-i(k_1\theta_1+k_2\theta_2)}$.

(2) If $Im\lambda \leq 0$ *and* $|Re\lambda| \leq \dfrac{2\sqrt{p}}{p-1}|Im\lambda|$*, then we have*

$$\|\{A_{k_1,k_2}(t)\}\|_{\ell_1^2(\mathbf{Z}^2)} \leq C, \tag{4.11}$$

where the positive constant C *does not depend on* T.

Proof of Lemma 4.6. According to the ODE system (4.5), $w(t,\theta_1,\theta_2)$ satisfies

$$i\partial_t w = -(4t^2)^{-1}(a\partial_{\theta_1} + b\partial_{\theta_2})^2 w + \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(w). \tag{4.12}$$

Multiplying $\overline{w}$ on both hand sides of (4.12), we have

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbf{T}^2)} = 2\mathrm{Im}\lambda|4\pi t|^{(p-1)/2}\|w(t)\|^{p+1}_{L^{p+1}(\mathbf{T}^2)}.$$

Note that $\|w(t)\|^2_{L^2(\mathbf{T}^2)} = (2\pi)^2\|\{A_{k_1,k_2}(t)\}\|_{\ell_0^2(\mathbf{Z}^2)}$ by Parseval's identity. Then, we obtain (4.10). Let us next prove (4.11). We here remark that, since the second order differentiation on the right hand side of (4.12) is degenerated, the $\ell_1^2(\mathbf{Z}^2)$-estimate of $\{A_{k_1,k_2}(t)\}$, (or equivalently $H^1(\mathbf{T}^2)$-estimate of $w(t)$) is not derived analogously in the proof of Lemma 3.5 (2). However, we establish the desired estimate by taking advantage of the nonlinear dissipation. Applying ∂_{θ_j} $(j = 1,2)$ to (4.12) and taking the imaginary part after multiplying $\overline{\partial_{\theta_j} w}$, we have

$$\frac{d}{dt}\|\partial_{\theta_j} w(t)\|^2_{L^2(\mathbf{T}^2)} = |4\pi t|^{-(p-1)/2}\cdot 2\mathrm{Im}\left(\lambda\langle\partial_{\theta_j}\mathcal{N}(w(t)), \partial_{\theta_j} w(t)\rangle_{\theta_1,\theta_2}\right).$$

Note that, if $\mathrm{Im}\lambda \leq 0$ and $|\mathrm{Re}| \leq \dfrac{2\sqrt{p}}{p-1}\mathrm{Im}\lambda$, Liskevich-Perelmuter's inequality [16] gives

$$\mathrm{Im}\left(\lambda\langle\partial_{\theta_j}\mathcal{N}(w(t)), \partial_{\theta_j} w(t)\rangle_{\theta_1,\theta_2}\right) \leq 0.$$

Accordingly, $\|\partial_{\theta_j} w(t)\|_{L^2(\mathbf{T}^2)} \leq \|\partial_{\theta_j} w(t_0)\|_{L^2(\mathbf{T}^2)}$ for $t > t_0$, which implies that

$$\|\{k_j A_{k_1,k_2}(t)\}\|_{\ell_1^2(\mathbf{Z}^2)} \leq \|\{k_j A_{k_1,k_2}(t_0)\}\|_{\ell_1^2(\mathbf{Z}^2)}.$$

By taking $t_0 > 0$ sufficiently small, the local existence argument as in the proof of Theorem 4.1 gives $\|\{k_j A_{k_1,k_2}(t)\}\|_{\ell_1^2(\mathbf{Z}^2)} \leq 2\rho_0$ if $0 < t < t_0$. Hence, we obtain (4.11). ☐

Lemma 4.7 (Brezis-Gallouet) *Let* $\alpha > 1$*. Then, there exists some positive constant* C_α *depending only on* α *such that*

$$\|f\|_{L^\infty(\mathbf{T}^2)} \leq C_\alpha\left(1 + \|f\|_{H^1(\mathbf{T}^2)}\sqrt{\log(1 + \|f\|_{H^\alpha(\mathbf{T}^2)})}\right). \tag{4.13}$$

Proof of Lemma 4.7. We refer to [3]. □

We are now in the position to prove Theorem 4.2.

Proof of Theorem 4.2. We first prove Theorem 4.2 (1). By Hölder's inequality, $\|w(t)\|_{L^{p+1}(\mathbf{T}^2)}^{p+1} \geq (2\pi)^{-(p-1)}\|w(t)\|_{L^2(\mathbf{T}^2)}^{p+1}$. Then, by Lemma 4.6 (4.10), we have

$$\frac{d}{dt}\|\{A_{k_1,k_2}(t)\}\|_{\ell_0^2}^2 \geq C\mathrm{Im}\lambda t^{-(p-1)/2}\|\{A_{k_1,k_2}(t)\}\|_{\ell_0^2}^{p+1}.$$

Solving this differential inequality, we observe the blowing-up of ℓ_0^2-norm. We next prove Theorem 4.2 (2). Making use of (4.12) and Lemma 4.4, we see that

$$\begin{aligned} \frac{d}{dt}\|w(t)\|_{H^\alpha(\mathbf{T}^2)}^2 &\leq C|t|^{-(p-1)/2}\|\mathcal{N}(w(t))\|_{H^\alpha(\mathbf{T}^2)}\|w(t)\|_{H^\alpha(\mathbf{T}^2)} \\ &\leq C|t|^{-(p-1)/2}\|w(t)\|_{L^\infty(\mathbf{T}^2)}^{p-1}\|w(t)\|_{H^\alpha(\mathbf{T}^2)}^2 \end{aligned} \tag{4.14}$$

Then, Lemma 4.6 (4.11) and Lemma 4.7 yield

$$\begin{aligned} \|w(t)\|_{L^\infty(\mathbf{T}^2)} &\leq C(1+\|w(t)\|_{H^1(\mathbf{T}^2)}\sqrt{\log(1+\|w(t)\|_{H^\alpha(\mathbf{T}^2)})}) \\ &\leq C\sqrt{\log(2+\|w(t)\|_{H^\alpha(\mathbf{T}^2)})} \end{aligned} \tag{4.15}$$

Plugging (4.15) into (4.14), we have the following differential inequality

$$\begin{aligned} \frac{d}{dt}\|w(t)\|_{H^\alpha(\mathbf{T}^2)} &\leq C|t|^{-(p-1)/2}\left(\log(2+\|w(t)\|_{H^\alpha(\mathbf{T}^2)})\right)^{(p-1)/2}\|w(t)\|_{H^\alpha(\mathbf{T}^2)} \\ &\leq C|t|^{-(p-1)/2}\left(\log(2+\|w(t)\|_{H^\alpha(\mathbf{T}^2)})\right)^{(p-1)/2}(2+\|w(t)\|_{H^\alpha(\mathbf{T}^2)}). \end{aligned}$$

From this inequality, it follows that

$$\frac{d}{dt}\log(2+\|w(t)\|_{H^\alpha(\mathbf{T}^2)}) \leq C|t|^{-(p-1)/2}(\log(2+\|w(t)\|_{H^\alpha}))^{(p-1)/2}.$$

Thus, $\|\{A_{k_1,k_2}(t)\}\|_{\ell_\alpha^2(\mathbf{Z}^2)} = (2\pi)^{-1}\|w(t)\|_{H^\alpha(\mathbf{T}^2)} \leq Ce^t < \infty$ for $t \in [0,T]$ with the positive constant C independent of T. Hence, the local solution $\{A_{k_1,k_2}(t)\}$ is continuated into the global one. □

References

[1] T. Abe and N. Okazawa, "*Global solvability of the complex Ginzburg-Landau equation with distribution-valued initial data, II: 1-dimensional Dirichlet problem*", the 28th conference of evolution euqation (2002, December) at Chuo University.

[2] H. Brezis and A. Friedman, "*Nonlinear parabolic equations involving measures as initial data*", J. Math. Pures Appl. **62**(1983), 73–97.

[3] H. Brezis and T. Gallouet, "*Nonlinear Schrödinger evolution equations*", Nonlinear Anal. TMA **4**(1980), 677–681.

[4] J. Bergh and J. Löfström, "Interpolation spaces, An introduction", Springer Verlag, 1976.

[5] T. Cazenave, "An Introduction to Nonlinear Schrödinger Equations," Textos de Métodos Matemáticos **22**, Instituto de Matemática, Rio de Janeiro, 1989.

[6] T. Cazenave and F.B. Weissler, "*The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*", Nonlenear Anal. TMA, **14**(1990), 807–836.

[7] J. Ginibre, T. Ozawa and G. Velo, "*On the existence of the wave operators for a class of nonlinear Schrödinger equations*", Ann. Inst. H. Poincaré Phys. Théor. **60**(1994), 211–239.

[8] J. Ginibre and G. Velo, "*On a class of nonlinear Schrö dinger equations*", J. Funct. Anal. **32**(1979), 1–71.

[9] J. Ginibre and G. Velo, "*The global Cauchy problem for the nonlinear Klein-Gordon equation*", Math. Z. **189**(1985), 487–505.

[10] H. Hasimoto, "*A soliton on a vortex filament*", J. Fluid Mechanics **51**(1972), 477–485.

[11] N. Hayashi, "*Classical solutions of nonlinear Schrödinger equations,*" Manuscripta Math. **55**(1986), 171–190.

[12] T. Kato, "*On nonlinear Schrödinger equations*", Ann. Inst. H. Poincaré Phys. Théor. **46**(1986), 113–129.

[13] T. Kato, "*Nonlinear Schrödinger equations,*" in "Schrödinger Operators" (H. Holden, A. Jensen, Eds.), Lecture Notes in Phys. **345**, Springer, Berlin-Heidelberg- New York, 1989.

[14] T. Kato, "*On Schrödinger equations. II. H^s-solutions and unconditional well-posedness,*" J. d'Anal. Math. **67** (1995), 281–306.

[15] C. Kenig, G. Ponce and L. Vega, "*On the ill-posedness of some canonical dispersive equations*", Duke Math. J. **106**(2001), 627–633.

[16] V.A. Liskevich and M.A. Perelmuter, "*Analyticity of submakovian semigroups*", Proc. Amer. Math. Soc. **123** (1995), 1097–1104.

[17] R.M. Miura, "*Korteweg-de Vries equation and generalizations, I: A remarkable explicit nonlinear transfoemation*", J. Math. Phys. **9**(1968), 1202–1204.

[18] M. Nakamura and T. Ozawa, "*Nonlinear Schrösinger equations in the Sobolev space of critical order*," J. Funct. Anal. **155**(1998), 364–380.

[19] H. Pecher, "*Solutions of semilinear Schrödinger equations in H^s*," Ann. Inst. Henri Poincaré, Phys. Théor. **67**(1997), 259–296.

[20] R. S. Strichartz, *Restriction of Fourier transform to quadratic surfaces and decay of wave equation*, Duke Math. J. **44**(1977), 705–714.

[21] Y. Tsutumi, "*L^2 solutions for nonlinear Scrödinger equations and nonlinear groups*", Funk. Ekva. **30**(1987), 115–125.

[22] Y. Tsutsumi, "*Global strong solutions for nonlinear Schrödinger equations*," Nonlinear Anal. TMA **11**(1987), 1143–1154.

[23] Y. Tsutsumi, "*The Cauchy problem for the Korteweg-de Vries equation with measure as initial data*", SIAM J. Math. Anal. **20**(1989), 582–588.

[24] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. **110**(1987), 415–426.