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The complex Ginzburg-Landau equation on general domain

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded or "unbounded" domain with boundary $\partial\Omega$. This paper is concerned with the *smoothing effect* (i.e., the existence of unique global strong solutions for L^2-initial data) of the following initial-boundary value problem for the complex Ginzburg-Landau equation:

$$\text{(CGL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 \ \text{ in } \Omega \times \mathbb{R}_+, \\ u = 0 \ \text{ on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \ \ x \in \Omega. \end{cases}$$

Here $\lambda, \kappa \in \mathbb{R}_+ := (0, \infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $q \geq 2$ are constants, and u is a complex-valued unknown function. We assume for simplicity that Ω is of class C^2 and $\partial\Omega$ is bounded (or $\Omega = \mathbb{R}^N_+$) to characterize the domain of the Dirichlet Laplacian. There are many mathematical studies on the problem (CGL) (for the existence and uniqueness of solutions see, e.g., Temam [**9**], Yang [**10**] and Ginibre-Velo [**1**], [**2**]; for the large time behavior of solutions see, e.g., Hayashi-Kaikina-Naumkin [**3**]; for the inviscid limiting problem as $\lambda \downarrow 0$ and $\kappa \downarrow 0$ see, e.g., Machihara-Nakamura [**4**] and Ogawa-Yokota [**5**]).

In a previous paper [**6**, Theorem 1.3 with $p = 2$] we established the smoothing effect of (CGL) on the initial data without any restriction on $q \geq 2$ under the condition

$$\text{(1.1)} \qquad \frac{|\beta|}{\kappa} \leq \frac{2\sqrt{q-1}}{q-2}.$$

This condition implies that the mapping $u \mapsto (\kappa + i\beta)|u|^{q-2}u$ is accretive (see [**6**, Lemma 2.1]). Recently, we reported in [**7**, Theorem 1.1] that under the condition

$$\text{(1.2)} \qquad 2 \leq q \leq 2 + \frac{4}{N},$$

the smoothing effect of (CGL) on the initial data can be obtained even if condition (1.1) breaks down. However, it was additionally assumed in [**7**] that Ω is a "bounded" domain.

The purpose of this paper is to remove the boundedness assumption on Ω. For that purpose we develop an abstract theory formulated in terms of subdifferential operators in the same way as in [**6**] and [**7**]. However, we should remove the compactness condition which was effectively used in [**7**]. To this end we introduce a new type of condition using the Yosida approximation (see condition (**A5**) in Section 2).

Before stating our result, we define a strong solution to (CGL) as follows:

Definition 1.1. A function $u(\cdot) \in C([0,\infty); L^2(\Omega))$ is said to be a *strong solution* to (CGL) if $u(\cdot)$ has the following properties:

(a) $u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ a.a. $t > 0$;

(b) $u(\cdot)$ is locally absolutely continuous (so that strongly differentiable a.e.) on $\mathbb{R}_+$;

(c) $u(\cdot)$ satisfies the equation in (CGL) a.e. on $\mathbb{R}_+$ as well as the initial condition.

Now we state the main theorem in this paper.

Theorem 1.1. *Let Ω be a bounded or "unbounded" domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$. Assume that Ω is of class C^2 and $\partial\Omega$ is bounded (or $\Omega = \mathbb{R}_+^N$). Let $N \in \mathbb{N}$, $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $2 \le q \le 2 + 4/N$. Then for any $u_0 \in L^2(\Omega)$ there exists a unique global strong solution $u(\cdot) \in C([0,\infty); L^2(\Omega))$ to* (CGL) *such that*

$$u(\cdot) \in C_{\mathrm{loc}}^{0,1/2}(\mathbb{R}_+; L^2(\Omega)) \cap C(\mathbb{R}_+; H_0^1(\Omega)),$$
$$\frac{du}{dt}(\cdot), \Delta u(\cdot), |u|^{q-2}u \in L_{\mathrm{loc}}^2(\mathbb{R}_+; L^2(\Omega)),$$
$$\|u(t)\|_{L^2} \le e^{\gamma t}\|u_0\|_{L^2} \quad \forall\, t \ge 0,$$
$$\|u(t) - v(t)\|_{L^2} \le e^{K_1 t + K_2 e^{2\gamma_+ t}(\|u_0\|_{L^2} \vee \|v_0\|_{L^2})^2}\|u_0 - v_0\|_{L^2} \quad \forall\, t \ge 0,$$

where $v(\cdot)$ is a unique strong solution to (CGL) *with $v(0) = v_0 \in L^2(\Omega)$, $\gamma_+ := \max\{\gamma, 0\}$, and K_1 and K_2 are positive constants depending only on $\lambda, \kappa, \beta, \gamma, q, N$.*

Remark 1.1. In this paper we ignore the accretivity of the nonlinear term under condition (1.1) effectively used in [**6**]. However, taking account of the usefulness of the accretivity, we can unify [**6**, Theorem 1.3 with $p = 2$] and Theorem 1.1 (see [**8**]).

2. Abstract theory

Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let S be a nonnegative selfadjoint operator with domain $D(S)$ in X. Let $\psi : X \to (-\infty, \infty]$ be a proper lower semi-continuous convex function, where "proper" means that $D(\psi) := \{u \in X;\ \psi(u) < \infty\} \ne \emptyset$. Then the subdifferential $\partial\psi(u)$ of ψ at $u \in D(\psi)$ is defined as the set $\{f \in X;\ \mathrm{Re}(f, v - u) \le \psi(v) - \psi(u)$ for every $v \in X\}$. Here we assume for simplicity that $\psi \ge 0$ and $\partial\psi$ is single-valued. As is well-known, S is also represented by a subdifferential: $S = \partial\varphi$, where φ is given by

$$\varphi(u) := \begin{cases} \dfrac{1}{2}\|S^{1/2}u\|^2 & \text{if } u \in D(\varphi) := D(S^{1/2}), \\ \infty & \text{otherwise.} \end{cases}$$

Then we consider the following abstract Cauchy problem in X:

$$\text{(ACP)} \qquad \begin{cases} \dfrac{du}{dt} + (\lambda + i\alpha)Su + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \\ u(0) = u_0, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$ are constants. To solve (ACP) we use the Moreau-Yosida approximation ψ_ε of ψ defined as

$$\psi_\varepsilon(v) := \min_{w\in X}\Big\{\psi(w) + \frac{1}{2\varepsilon}\|w - v\|^2\Big\}, \quad v \in X, \ \varepsilon > 0.$$

It is well-known that ψ_ε is Fréchet differentiable on X and the derivative $\psi_\varepsilon' = \partial(\psi_\varepsilon)$ coincides with the Yosida approximation $(\partial\psi)_\varepsilon$ of $\partial\psi$:

$$(\partial\psi)_\varepsilon := \frac{1}{\varepsilon}(1 - J_\varepsilon), \quad J_\varepsilon := (1 + \varepsilon\partial\psi)^{-1}, \quad \varepsilon > 0$$

(see Showalter [**11**, Proposition IV.1.8]), and so we can use the simplified notation $\partial\psi_\varepsilon$:

$$\partial\psi_\varepsilon := \partial(\psi_\varepsilon) = (\partial\psi)_\varepsilon.$$

We introduce the following five conditions on S and ψ; note that the compactness condition used in [**7**] is replaced with a new type of condition (**A5**).

(**A1**) $\exists q \in [2, \infty)$ such that $\psi(\zeta u) = |\zeta|^q \psi(u)$ for $u \in D(\psi)$ and $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta > 0$.

(**A2**) $D(S) \subset D(\partial\psi)$ and $\exists C_1 > 0$ such that $\|\partial\psi(u)\| \le C_1(\|u\| + \|Su\|)$ for $u \in D(S)$.

(**A3**) $\forall \eta > 0 \ \exists C_2 = C_2(\eta) > 0$ such that for $u \in D(S)$ and $\varepsilon > 0$,

$$|(Su, \partial\psi_\varepsilon(u))| \le \eta\|Su\|^2 + C_2\psi(J_\varepsilon u)^\theta \varphi(u),$$

where $\theta \in [0, 1]$ is a constant.

(**A4**) $\forall \eta > 0 \ \exists C_3 = C_3(\eta) > 0$ such that for $u, v \in D(\varphi) \cap D(\psi)$ and $\varepsilon > 0$,

$$|(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| \le \eta\varphi(u - v) + C_3\left(\frac{\psi(J_\varepsilon u) + \psi(J_\varepsilon v)}{2}\right)^\theta \|u - v\|^2,$$

where $\theta \in [0, 1]$ is the same constant as in (**A3**).

(**A5**) $\exists C_4 > 0$ such that for $u, v \in D(\partial\psi)$ and $\nu, \mu > 0$,

$$|(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)| \le C_4|\nu - \mu|\big(\sigma\|\partial\psi(u)\|^2 + \tau\|\partial\psi(v)\|^2\big),$$

where $\sigma, \tau > 0$ are constants satisfying $\sigma + \tau = 1$.

To state our abstract result we define a strong solution to (ACP) as follows:

Definition 2.1. A function $u(\cdot) \in C([0,\infty);X)$ is said to be a *strong solution* to (ACP) if $u(\cdot)$ has the following properties:

(a) $u(t) \in D(S) \cap D(\partial\psi)$ a.a. $t > 0$;

(b) $u(\cdot)$ is locally absolutely continuous (so that strongly differentiable a.e.) on $\mathbb{R}_+$;

(c) $u(\cdot)$ satisfies the equation in (ACP) a.e. on $\mathbb{R}_+$ as well as the initial condition.

Now we state the main result in this section.

Theorem 2.1. *Let $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume that conditions* **(A1)**–**(A5)** *are satisfied. Then for any $u_0 \in X$ there exists a unique strong solution $u(\cdot) \in C([0,\infty);X)$ to* (ACP). *Also, $u(\cdot)$ has the following properties:*

(a) *$u(\cdot) \in C^{0,1/2}_{\mathrm{loc}}(\mathbb{R}_+;X)$, with $\|u(t)\| \le e^{\gamma t}\|u_0\|$ $\forall\, t \ge 0$;*

(b) *$Su(\cdot)$, $\partial\psi(u(\cdot))$, $(du/dt)(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}_+;X)$;*

(c) *$\varphi(u(\cdot))$ and $\psi(u(\cdot))$ are locally absolutely continuous on $\mathbb{R}_+$.*

Furthermore, let $v(\cdot)$ be a unique strong solution to (ACP) *with $v(0) = v_0 \in X$. Then*

$$\|u(t) - v(t)\| \le e^{K_1 t + K_2 e^{2\gamma_+ t}(\|u_0\| \vee \|v_0\|)^2}\|u_0 - v_0\| \quad \forall\, t \ge 0, \tag{2.1}$$

where $K_1 := \gamma + (1-\theta)C_3\sqrt{\kappa^2+\beta^2}$ and $K_2 := \theta C_3\sqrt{\kappa^2+\beta^2}/(2q\kappa)$.

Now we shall prove Theorem 2.1. To this end we first take $u_0 \in D(\varphi) \cap D(\psi)$. In what follows we assume that $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{R}$ and conditions **(A1)**–**(A5)** are satisfied. Given $\varepsilon > 0$, we consider the following problem approximate to (ACP):

$$(\mathrm{ACP})_\varepsilon \qquad \begin{cases} \dfrac{du_\varepsilon}{dt} + (\lambda + i\alpha)Su_\varepsilon + (\kappa + i\beta)\partial\psi_\varepsilon(u_\varepsilon) - \gamma u_\varepsilon = 0, \quad t > 0, \\ u_\varepsilon(0) = u_0. \end{cases}$$

Since $\partial\psi_\varepsilon$ is Lipschitz continuous on X, it follows from [**6**, Proposition 3.1 (i)] that $(\mathrm{ACP})_\varepsilon$ has a unique strong solution $u_\varepsilon(\cdot) \in C([0,\infty);X)$ such that $u_\varepsilon(\cdot) \in C^{0,1/2}([0,T];X)$ and $(du_\varepsilon/dt)(\cdot)$, $Su_\varepsilon(\cdot) \in L^2(0,T;X)$ for every $T > 0$.

The following lemma was obtained in [**7**, Lemma 2.3] by using conditions **(A1)** and **(A3)** with $\eta := \lambda/(2\sqrt{\kappa^2+\beta^2})$.

Lemma 2.2. *Let $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ be the family of unique strong solutions to $(\mathrm{ACP})_\varepsilon$ with $u_0 \in D(\varphi) \cap D(\psi)$ as stated above. Then*

$$\|u_\varepsilon(t)\| \le e^{\gamma t}\|u_0\| \quad \forall\, t \ge 0, \tag{2.2}$$

$$2\lambda\int_0^t \varphi(u_\varepsilon(s))\,ds + q\kappa\int_0^t \psi(J_\varepsilon u_\varepsilon(s))\,ds \le \frac{1}{2}e^{2\gamma_+ t}\|u_0\|^2 \quad \forall\, t \ge 0, \tag{2.3}$$

$$\varphi(u_\varepsilon(t)) \le e^{K(t,\|u_0\|)}\varphi(u_0) \quad \forall\, t \ge 0, \tag{2.4}$$

$$\int_0^t \|Su_\varepsilon(s)\|^2\,ds \le \frac{2}{\lambda}e^{K(t,\|u_0\|)}\varphi(u_0) \quad \forall\, t \ge 0, \tag{2.5}$$

where $K(t,\|u_0\|) := k_1 t + k_2 e^{2\gamma_+ t}\|u_0\|^2$ and $k_1 := 2\gamma_+ + (1-\theta)C_2\sqrt{\kappa^2+\beta^2}$, $k_2 := \theta C_2\sqrt{\kappa^2+\beta^2}/(2q\kappa)$.

Next we shall state the following *key* lemma, in which a new type of condition (**A5**) plays an important role. For a proof see [**8**, Lemma 2.5].

Lemma 2.3. *Let $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ be the family of unique strong solutions to* $(\mathrm{ACP})_\varepsilon$ *with $u_0 \in D(\varphi) \cap D(\psi)$ as stated above. Then there exists a function $u(\cdot) \in C([0,\infty);X)$ such that $u(0) = u_0$ and*

$$u_\varepsilon(\cdot) \to u(\cdot) \ (\varepsilon \downarrow 0) \ \text{in } C([0,T];X) \ \forall\, T>0, \tag{2.6}$$

$$J_\varepsilon u_\varepsilon(\cdot) \to u(\cdot) \ (\varepsilon \downarrow 0) \ \text{in } L^2(0,T;X) \ \forall\, T>0. \tag{2.7}$$

Now we can prove the existence of strong solutions to (ACP) with "$u_0 \in D(\varphi) \cap D(\psi)$".

Lemma 2.4. *Let $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume that conditions* (**A1**) – (**A5**) *are satisfied. Then for any $u_0 \in D(\varphi) \cap D(\psi)$ there exists a unique strong solution $u(\cdot) \in C([0,\infty);X)$ to* (ACP) *such that*

(a) *$u(\cdot) \in C^{0,1/2}([0,T];X)$ $\forall\, T > 0$, with $\|u(t)\| \le e^{\gamma t}\|u_0\|$ $\forall\, t \ge 0$;*
(b) *$Su(\cdot)$, $\partial\psi(u(\cdot))$, $(du/dt)(\cdot) \in L^2(0,T;X)$ $\forall\, T > 0$;*
(c) *$\varphi(u(\cdot))$ and $\psi(u(\cdot))$ are absolutely continuous on $[0,T]$ $\forall\, T>0$, with*

$$2\lambda \int_0^t \varphi(u(s))\,ds + q\kappa \int_0^t \psi(u(s))\,ds \le \frac{1}{2} e^{2\gamma+t}\|u_0\|^2 \ \ \forall\, t \ge 0. \tag{2.8}$$

Furthermore, let $v(\cdot)$ be a unique strong solution to (ACP) *with $v(0) = v_0 \in D(\varphi) \cap D(\psi)$. Then*

$$\|u(t) - v(t)\| \le e^{K_1 t + K_2 e^{2\gamma+t}(\|u_0\| \vee \|v_0\|)^2}\|u_0 - v_0\| \ \ \forall\, t \ge 0, \tag{2.9}$$

where K_1 and K_2 are the same constants as in Theorem 2.1.

Proof. Let $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ be the family as stated above. Let $T > 0$. Then it follows from (2.5) that $\{Su_\varepsilon(\cdot)\}_{\varepsilon>0}$ is bounded in $L^2(0,T;X)$. As noted in the proof of Lemma 2.3, $\{\partial\psi_\varepsilon(u_\varepsilon(\cdot))\}_{\varepsilon>0}$ is bounded in $L^2(0,T;X)$ and so is $\{(du_\varepsilon/dt)(\cdot))\}_{\varepsilon>0}$ in view of the equation in $(\mathrm{ACP})_\varepsilon$. Since S, $\partial\psi$ and d/dt are demiclosed as operators in $L^2(0,T;X)$, we see from Lemma 2.3 that

$$Su_\varepsilon(\cdot) \to Su(\cdot), \quad \partial\psi_\varepsilon(u_\varepsilon(\cdot)) = \partial\psi(J_\varepsilon u_\varepsilon(\cdot)) \to \partial\psi(u(\cdot))$$

and $(du_\varepsilon/dt)(\cdot) \to (du/dt)(\cdot)$ $(n \to \infty)$ weakly in $L^2(0,T;X)$ and $u(\cdot)$ satisfies properties (a) and (b). Therefore we can conclude that $u(\cdot)$ is a strong solution to (ACP). Property (c) is derived from (a) and (b). Letting $\varepsilon \downarrow 0$ in (2.3) and using (2.6), we obtain (2.8).

To prove (2.9) we use the limiting case of condition (**A5**): $\forall\, \eta > 0\ \exists\, C_3 = C_3(\eta) > 0$ such that for $u, v \in D(\partial\varphi) \cap D(\partial\psi)$,

$$|(\partial\psi(u) - \partial\psi(v), u - v)| \le \eta\varphi(u-v) + C_3\left(\frac{\psi(u)+\psi(v)}{2}\right)^\theta \|u-v\|^2; \tag{2.10}$$

note that for $u \in D(\partial\psi)$, $\partial\psi_\varepsilon(u) \to \partial\psi(u)$ $(\varepsilon \downarrow 0)$ in X. Now let $u(\cdot)$ and $v(\cdot)$ be strong solutions to (ACP) with $u(0) = u_0$ and $v(0) = v_0$, respectively. As in the proof of Lemma 2.3, it follows from (2.10) that

$$
\begin{aligned}
(2.11)\qquad & \frac{1}{2}\frac{d}{dt}\|u-v\|^2 \\
& \le \gamma\|u-v\|^2 - 2\lambda\varphi(u-v) + \sqrt{\kappa^2+\beta^2}\,|(\partial\psi(u)-\partial\psi(v), u-v)| \\
& \le \left\{\gamma + \tilde{C}_3\left(\frac{\psi(u)+\psi(v)}{2}\right)^{\theta}\right\}\|u-v\|^2 \\
& \le \Psi(u,v)\|u-v\|^2,
\end{aligned}
$$

where $\tilde{C}_3 := C_3\sqrt{\kappa^2+\beta^2}$ and $\Psi(u,v)$ is given by

$$\Psi(u,v) := \gamma + \tilde{C}_3\left\{(1-\theta) + \theta\left(\frac{\psi(u)+\psi(v)}{2}\right)\right\} = K_1 + K_2 q\kappa\big(\psi(u)+\psi(v)\big)$$

(K_1 and K_2 are the same constants as in Theorem 2.1). Here (2.8) implies that

$$\int_0^t \Psi(u(s),v(s))\,ds \le K_1 t + K_2 e^{2\gamma+t}(\|u_0\| \vee \|v_0\|)^2.$$

Therefore we can obtain (2.9) by integration of (2.11). □

To prove Theorem 2.1 we need the following lemma (cf. [**7**, Lemma 2.4]).

Lemma 2.5. *Let $u(\cdot)$ be a strong solution to* (ACP) *with $u(0) = u_0 \in D(\varphi)\cap D(\psi)$ as in* Lemma 2.4 *constructed under conditions* (**A1**)$-$(**A5**). *Then*

$$(2.12)\qquad t\varphi(u(t)) + \frac{\lambda}{2}\int_0^t s\|Su(s)\|^2\,ds \le \frac{1}{4\lambda}e^{K(t,\|u_0\|)+2\gamma+t}\|u_0\|^2 \quad \forall\, t \ge 0,$$

where $K(t,\|u_0\|)$ is the same as in Lemma 2.2.

Proof. We use the limiting case of condition (**A3**): $\forall \eta > 0\ \exists\, C_2 = C_2(\eta) > 0$ such that for $u \in D(S)\cap D(\partial\psi)$,

$$(2.13)\qquad |(Su, \partial\psi(u))| \le \eta\|Su\|^2 + C_2\psi(u)^\theta\varphi(u),$$

where $\theta \in [0,1]$ is the same constant as before; note that for $u \in D(\partial\psi)$, $\partial\psi_\varepsilon(u) \to \partial\psi(u)$ $(\varepsilon \downarrow 0)$ in X and $\psi(J_\varepsilon u) \le \psi_\varepsilon(u) \le \psi(u)$. As in the proof of [**7**, Lemma 2.3], we see from (2.13) that

$$(2.14)\qquad \frac{d}{ds}\left[\exp\left(-\int_0^s k(r)\,dr\right)\varphi(u(s))\right] + \frac{\lambda}{2}\exp\left(-\int_0^s k(r)\,dr\right)\|Su(s)\|^2 \le 0,$$

where $k(r) := k_1 + 2k_2 q\kappa\psi(u(r)) \ge 0$, and

$$(2.15)\qquad 0 \le \int_s^t k(r)\,dr \le \int_0^t k(r)\,dr \le K(t,\|u_0\|) \ \ \forall\, s \in [0,t].$$

Multiplying the both sides of (2.14) by $s \in [0,t]$ and integrating it on $[0,t]$ yield

$$t\varphi(u(t)) + \frac{\lambda}{2}\int_0^t s\cdot \exp\left(\int_s^t k(r)\,dr\right)\|Su(s)\|^2\,ds \le \int_0^t \exp\left(\int_s^t k(r)\,dr\right)\varphi(u(s))\,ds$$
$$\le \exp\left(\int_0^t k(r)\,dr\right)\int_0^t \varphi(u(s))\,ds.$$

Therefore (2.12) follows from (2.8) and (2.15). □

Once Lemmas 2.4 and 2.5 are established, we can prove Theorem 2.1 in the same way as in the proof of [**6**, Theorem 5.2] (see also [**7**]).

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by applying Theorem 2.1 to (CGL). Let $X := L^2(\Omega)$ with inner product $(\cdot,\cdot)_{L^2}$ and norm $\|\cdot\|_{L^2}$. Let $2 \le q \le 2 + 4/N$. Then we define the nonnegative selfadjoint operator S in X and the proper lower semi-continuous convex function ψ on X as follows:

$$Su := -\Delta u \quad \text{for } u \in D(S) := H^2(\Omega)\cap H_0^1(\Omega),$$
$$\psi(u) := \begin{cases} \dfrac{1}{q}\|u\|_{L^q}^q & \text{if } u \in D(\psi) := L^2(\Omega)\cap L^q(\Omega),\\ \infty & \text{otherwise.}\end{cases}$$

As is well-known, the subdifferential of ψ is given by

$$\partial\psi(u) = |u|^{q-2}u \quad \text{for } u \in D(\partial\psi) = L^2(\Omega)\cap L^{2(q-1)}(\Omega).$$

Therefore we can regard (CGL) as one of (ACP)s.

To apply Theorem 2.1 it suffices to show that all the conditions $(\mathbf{A1})-(\mathbf{A5})$ introduced in Section 2 are satisfied. Here we consider only the new type of condition $(\mathbf{A5})$. For the verification of other conditions $(\mathbf{A1})-(\mathbf{A4})$ see [**7**]. We begin with the strong differentiability of the resolvent with respect to approximating parameter ε.

Lemma 3.1. *Let $f \in D(\partial\psi)$. For $\varepsilon \in [0,\infty)$ and $x \in \Omega$ put*

$$u_\varepsilon(x) := \begin{cases} (1+\varepsilon\partial\psi)^{-1}f(x) & (\varepsilon > 0),\\ f(x) & (\varepsilon = 0).\end{cases} \tag{3.1}$$

Then $u_\varepsilon \in C^1([0,E];L^2(\Omega))\ \forall\, E > 0$ (as a function of ε), with

$$\frac{\partial u_\varepsilon}{\partial\varepsilon} = \begin{cases} -\dfrac{1}{1+\varepsilon(q-1)|u_\varepsilon|^{q-2}}\partial\psi_\varepsilon(f) & (\varepsilon > 0),\\ -\partial\psi(f) & (\varepsilon = 0).\end{cases} \tag{3.2}$$

Proof. Using the inverse function theorem, we can show that $u_\varepsilon \in C^1([0,E];L^2(\Omega))$ for every $E>0$ (for the proof see [**8**, Proposition 3.4]). Here we derive only (3.2). To this end let $f \in D(\partial\psi)$ and $\varepsilon > 0$. Then it follows from (3.1) that

$$u_\varepsilon(x) + \varepsilon |u_\varepsilon(x)|^{q-2} u_\varepsilon(x) = f(x). \tag{3.3}$$

Writing as

$$u_\varepsilon(x) = v_\varepsilon(x) + i w_\varepsilon(x), \quad f(x) = g(x) + ih(x),$$

we see that (3.3) is equivalent to

$$\begin{cases} v_\varepsilon(x) + \varepsilon(v_\varepsilon(x)^2 + w_\varepsilon(x)^2)^{(q-2)/2} v_\varepsilon(x) = g(x), \\ w_\varepsilon(x) + \varepsilon(v_\varepsilon(x)^2 + w_\varepsilon(x)^2)^{(q-2)/2} w_\varepsilon(x) = h(x). \end{cases}$$

Differentiating the both sides with respect to ε yields

$$\begin{cases} \dfrac{\partial v_\varepsilon}{\partial \varepsilon} + |u_\varepsilon|^{q-2} v_\varepsilon + \varepsilon(q-2)|u_\varepsilon|^{q-4}\Big(v_\varepsilon \dfrac{\partial v_\varepsilon}{\partial \varepsilon} + w_\varepsilon \dfrac{\partial w_\varepsilon}{\partial \varepsilon}\Big) v_\varepsilon + \varepsilon |u_\varepsilon|^{q-2} \dfrac{\partial v_\varepsilon}{\partial \varepsilon} = 0, \\ \dfrac{\partial w_\varepsilon}{\partial \varepsilon} + |u_\varepsilon|^{q-2} w_\varepsilon + \varepsilon(q-2)|u_\varepsilon|^{q-4}\Big(v_\varepsilon \dfrac{\partial v_\varepsilon}{\partial \varepsilon} + w_\varepsilon \dfrac{\partial w_\varepsilon}{\partial \varepsilon}\Big) w_\varepsilon + \varepsilon |u_\varepsilon|^{q-2} \dfrac{\partial w_\varepsilon}{\partial \varepsilon} = 0. \end{cases}$$

Solving this system of equations with respect to $\partial v_\varepsilon/\partial\varepsilon$ and $\partial w_\varepsilon/\partial\varepsilon$, we have

$$\begin{cases} \dfrac{\partial v_\varepsilon}{\partial \varepsilon} = -\dfrac{1}{1+\varepsilon(q-1)|u_\varepsilon|^{q-2}} |u_\varepsilon|^{q-2} v_\varepsilon, \\ \dfrac{\partial w_\varepsilon}{\partial \varepsilon} = -\dfrac{1}{1+\varepsilon(q-1)|u_\varepsilon|^{q-2}} |u_\varepsilon|^{q-2} w_\varepsilon. \end{cases}$$

This implies that

$$\frac{\partial u_\varepsilon}{\partial \varepsilon} = -\frac{1}{1+\varepsilon(q-1)|u_\varepsilon|^{q-2}} \partial\psi(u_\varepsilon), \quad \varepsilon > 0.$$

Since $\partial\psi(u_\varepsilon) = \partial\psi_\varepsilon(f)$, we obtain (3.2) with $\varepsilon > 0$. In addition, it follows that

$$\|\varepsilon^{-1}(u_\varepsilon - f) + \partial\psi(f)\|_{L^2} = \|\partial\psi_\varepsilon(f) - \partial\psi(f)\|_{L^2} \to 0 \ \ (\varepsilon \downarrow 0).$$

This shows that $(\partial u_\varepsilon/\partial\varepsilon)|_{\varepsilon=0} = -\partial\psi(f)$ and hence (3.2) is true at $\varepsilon = 0$. □

As a consequence of Lemma 3.1 we have

Lemma 3.2. *Let $q \ge 2$. Then for $u, v \in D(\partial\psi)$ and $\nu, \mu > 0$,*

$$|(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)_{L^2}| \le (q-1)|\nu - \mu| \left[\frac{2q-3}{2(q-1)} \|\partial\psi(u)\|_{L^2}^2 + \frac{1}{2(q-1)} \|\partial\psi(v)\|_{L^2}^2\right].$$

Proof. The computation is almost the same as in [**8**, Lemma 3.7]. Let $u \in D(\partial\psi) = L^2(\Omega) \cap L^{2(q-1)}(\Omega)$. For $\varepsilon \in [0,\infty)$ and $x \in \Omega$ put

$$u_\varepsilon(x) := \begin{cases} (1+\varepsilon\partial\psi)^{-1}u(x) & (\varepsilon > 0), \\ u(x) & (\varepsilon = 0). \end{cases}$$

Then Lemma 3.1 implies that $u_\varepsilon \in C^1([0,E];L^2(\Omega))$ for every $E > 0$. Since $\partial\psi_\varepsilon(u) = \varepsilon^{-1}(u-u_\varepsilon)$ for $\varepsilon > 0$, it follows from (3.2) that

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}[\partial\psi_\varepsilon(u)] &= -\frac{1}{\varepsilon^2}(u-u_\varepsilon) - \frac{1}{\varepsilon}\cdot\frac{\partial u_\varepsilon}{\partial\varepsilon} \\ &= -\frac{1}{\varepsilon}\Big[\partial\psi_\varepsilon(u) + \frac{\partial u_\varepsilon}{\partial\varepsilon}\Big] \\ &= -\frac{(q-1)|u_\varepsilon|^{q-2}}{1+(q-1)\varepsilon|u_\varepsilon|^{q-2}}\partial\psi_\varepsilon(u) \\ &= -\frac{(q-1)|u_\varepsilon|^{2(q-2)}u_\varepsilon}{1+(q-1)\varepsilon|u_\varepsilon|^{q-2}}, \quad \varepsilon > 0. \end{aligned}$$

Since $|u_\varepsilon| \le |u|$, we obtain

$$\Big|\frac{\partial}{\partial\varepsilon}[\partial\psi_\varepsilon(u)]\Big| \le (q-1)|u_\varepsilon|^{2q-3} \le (q-1)|u|^{2q-3}, \quad \varepsilon > 0.$$

Therefore we see that for $\nu, \mu > 0$,

$$|\partial\psi_\nu(u) - \partial\psi_\mu(u)| = \Big|\int_\mu^\nu \frac{\partial}{\partial\varepsilon}[\partial\psi_\varepsilon(u)]\,d\varepsilon\Big| \le (q-1)|\nu-\mu|\cdot|u|^{2q-3},$$

and hence

$$(3.4) \qquad |(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)_{L^2}| \le (q-1)|\nu-\mu|\int_\Omega |u|^{2q-3}|v|\,dx.$$

It follows from Hölder's inequality and Young's inequality that

$$\int_\Omega |u|^{2q-3}|v|\,dx \le \|u\|_{L^{2(q-1)}}^{2q-3}\|v\|_{L^{2(q-1)}} \le \Big(\frac{2q-3}{2(q-1)}\|u\|_{L^{2(q-1)}}^{2(q-1)} + \frac{1}{2(q-1)}\|v\|_{L^{2(q-1)}}^{2(q-1)}\Big).$$

Applying this inequality to the right-hand side of (3.4), we can obtain the desired inequality because of $\|u\|_{L^{2(q-1)}}^{2(q-1)} = \|\partial\psi(u)\|_{L^2}^2$. □

Lemma 3.2 shows that condition (**A5**) is satisfied with

$$\sigma := \frac{2q-3}{2(q-1)}, \quad \tau := \frac{1}{2(q-1)}.$$

Therefore Theorem 2.1 applies to give the assertion of Theorem 1.1.

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