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Representation of successor-type proof-theoretically regular ordinals via limits

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Abstract

In this paper, we extend a result in [Ta04], that is, we show that every successor-type proof-theoretically regular ordinal has its own representation as a limit of a sequence consisting of certain canonical elements.

1 Introduction

In our previous paper [Ta04], we defined a set $\mathbf{Reg}(\mathcal{T}(M))$ based on $\mathcal{T}(M)$, which was a primitive recursive well-ordered set defined by M.Rathjen to establish the proof theoretic ordinal of **KRM**. We call elements of $\mathbf{Reg}(\mathcal{T}(M))$ "proof-theoretically regular ordinals based on $\mathcal{T}(M)$ (ptros)". In [Ta04], we also characterized some sort of ptros as proof-theoretical analogues of (hyper) inaccessible cardinals up to the least Mahlo cardinal. Since the characterization is based on $\mathbf{Reg}(\mathcal{T}(M))$ as an analogue of the set of regular cardinals up to the least Mahlo cardinal, it is significant to characterize ptros and find the relationship between $\mathbf{Reg}(\mathcal{T}(M))$ and the set of regular cardinals up to the least Mahlo cardinal. For these purpose, we are in the process of establishing a "canonical" fundamental sequence of each limit-type element of $\mathcal{T}(M)$. A coherent way to establish an appropriate fundamental sequence of each limit-type element of $\mathcal{T}(M)$ can be expected to be a coherent way to re-construct each element of $\mathcal{T}(M)$ as a more familiar concept, and hence, it turns out to provide a desirable characterization of ptros as proof-theoretical analogues of regular cardinals.

In this paper, we extend a result in [Ta04] (cf. Theorem 2.11 in this paper). The result gives a fundamental sequence of the least "successor-type" ptro $\psi_M^{\Omega_1}(\Omega_1)$, by which $\psi_M^{\Omega_1}(\Omega_1)$ can be characterized as the least fixed point of the function enumerating strongly critical ordinals. We here give a similar sequence $\{\gamma_n\}_{n\in\omega}$ of every successor-type ptro γ. Compared with the previous result in [Ta04], the proof of the property that $\gamma = \lim_{n\in\omega}\gamma_n$ needs some special attentions. Therefore, for (a certain type of) a given ordinal δ less than γ, we construct a labeled tree informing us the number $n \in \omega$ with $\delta < \gamma_n$.

In Section 2, we explain several definitions and results in [Ta04]. In Section 3, we show the extended version of the result above.

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2 Preliminaries

In this paper, M denotes the least Mahlo cardinal, and φ the veblin function. For more details, one can refer to [Bu92], [Ra98], [Ra99] or [Ta04].

Definition 2.1 (Rathjen98,99). For given ordinals α and β, we define a set $C^M(\alpha,\ \beta)$ called a *Skolem's hull* as well as functions χ^α and ψ_M^α called *collapsing functions*, as follows:
(M1) $\beta \cup \{0, M\} \subset C^M(\alpha, \beta)$;
(M2) $\gamma = \gamma_1 + \gamma_2$ & $\gamma_1, \gamma_2 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^M(\alpha, \beta)$;
(M3) $\gamma = \varphi\gamma_1\gamma_2$ & $\gamma_1, \gamma_2 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^M(\alpha, \beta)$;
(M4) $\gamma = \Omega_{\gamma_1}$ & $\gamma_1 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^I(\alpha, \beta)$;
(M5) $\gamma = \chi^\xi(\delta)$ & $\xi, \delta \in C^M(\alpha, \beta)$ & $\xi < \alpha$ & $\xi \in C^M(\xi, \gamma)$ & $\delta < M \Rightarrow \gamma \in C^M(\alpha, \beta)$
(M6) $\gamma = \psi_M^\xi(\kappa)$ & $\xi, \kappa \in C^M(\alpha, \beta)$ & $\xi < \alpha$ & $\xi \in C^M(\xi, \gamma) \Rightarrow \gamma \in C^M(\alpha, \beta)$;
$\chi^\alpha(\delta) \simeq$ the δ^{th} regular cardinal $\pi < M$ with $C^M(\alpha, \pi) \cap M = \pi$;
$\psi_M^\alpha(\kappa) \simeq \min\{\rho < \kappa :\ C^M(\alpha, \rho) \cap \kappa = \rho \wedge \kappa \in C^M(\alpha, \rho)\}$.

Definition 2.2
(i) $\gamma =_{\mathrm{nf}} \alpha + \beta :\Leftrightarrow\ \gamma = \alpha + \beta$ & $\gamma > \alpha \geqq \beta$ & β is an additive principal number.
(ii) $\gamma =_{\mathrm{nf}} \varphi\alpha\beta :\Leftrightarrow\ \gamma = \varphi\alpha\beta$ & $\alpha, \beta < \gamma$.
(iii) $\gamma =_{\mathrm{nf}} \Omega_\alpha :\Leftrightarrow\ \gamma = \Omega_\alpha$ & $\alpha < \gamma$.
(iv) $\gamma =_{\mathrm{nf}} \psi_I^\alpha(\kappa) :\Leftrightarrow\ \gamma = \psi_I^\alpha(\kappa)$ & $\alpha \in C^I(\alpha, \gamma)$.
(v) $\gamma =_{\mathrm{nf}} \chi^\alpha(\beta) :\Leftrightarrow \gamma = \chi^\alpha(\beta)$ & $\beta < \gamma$ & $\alpha \in C^M(\alpha, \gamma)$.

Definition 2.3 (Rathjen95,98). We define a set $\mathcal{T}(M)$ called an *elementary ordinal representation system for* **KPM** and the *degree* $d(\alpha) < \omega$ of each element α of $\mathcal{T}(M)$, as follows:
(i) $0, M \in \mathcal{T}(M)$ & $d(0) = d(M) = 0$;
(ii) ($\gamma =_{\mathrm{nf}} \alpha + \beta$ & $\alpha, \beta \in \mathcal{T}(M)$)
$\Rightarrow$ ($\gamma \in \mathcal{T}(M)$ & $d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1$);
(iii) ($\gamma =_{\mathrm{nf}} \varphi\alpha\beta$ & $\alpha, \beta \in \mathcal{T}(M)$ & ($\gamma < M$ or $\alpha = 0$))
$\Rightarrow$ ($\gamma \in \mathcal{T}(M)$ & $d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1$);
(iv) ($\gamma =_{\mathrm{nf}} \Omega_\alpha < M$ & $\alpha > 0$ & $\alpha \in \mathcal{T}(M)$)
$\Rightarrow$ ($\gamma \in \mathcal{T}(M)$ & $d(\gamma) = d(\alpha) + 1$);
(v) ($\gamma =_{\mathrm{nf}} \chi^\xi(\alpha)$ & $\xi, \alpha \in \mathcal{T}(M)$)
$\Rightarrow$ ($\gamma \in \mathcal{T}(M)$ & $d(\gamma) = d(\alpha) + 1$);
(vi) ($\gamma =_{\mathrm{nf}} \psi_M^\alpha(\kappa)$ & $\kappa, \alpha \in \mathcal{T}(M)$)
$\Rightarrow$ ($\gamma \in \mathcal{T}(M)$ & $d(\gamma) = \max\{d(\kappa), d(\alpha)\} + 1$).

Theorem 2.4 (Rathjen91, Buchholz92). (1) Each element of $\mathcal{T}(M)$ has a unique representation with 0, M, $+$, φ, Ω, χ, ψ_M.
(2) $|\mathbf{KPM}| \leqq \psi_M^{\varepsilon_{M+1}}(\Omega_1) = \mathcal{T}(M) \cap \Omega_1$, where $|\mathbf{KPM}|$ denotes the proof theoretic ordinal of **KPM**.

Definition 2.5 An ordinal γ is called a *proof-theoretically regular ordinal based on* $\mathcal{T}(M)$ if γ is (expressed by) an element of $\mathcal{T}(M)$ having the form of $\psi_M^{\kappa}(\Omega_1)$ with $\kappa \in \mathbf{Reg}$, where $\mathbf{Reg}$ denotes the set of regular cardinals.

Definition 2.6 A ptro γ is called a *successor-type* ptro if γ has an element $\theta \in \mathcal{T}(M)$ satisfying that γ is the least ptro larger than θ.

Definition 2.7 An ordinal γ is called a *proof-theoretically inaccessible ordinal based on* $\mathcal{T}(M)$ if γ is an element of $\mathbf{Reg}(\mathcal{T}(M))$ as well as the supremum of $\mathbf{Reg}(\mathcal{T}(M)) \cap \gamma$, where $\mathbf{Reg}(\mathcal{T}(M))$ denotes the set of ptros based on $\mathcal{T}(M)$.

Theorem 2.8 (Takaki 04). All ptros are classified into the following two types:
(i) Successor-type ptros, which are of the form $\psi_M^{\Omega_{\alpha+1}}(\Omega_1)$ or $\psi_M^{\Omega_1}(\Omega_1)$;
(ii) Proof-theoretically inaccessible ordinals, which are of the form $\psi_M^{\chi^{\alpha}(\beta)}(\Omega_1)$ or $\psi_M^{M}(\Omega_1)$.

Definition 2.9 For each $n \in \omega$, we define Ψ_n by:

$$\Psi_n = \begin{cases} 0 \text{ if } n = 0; \\ \psi_M^{\Psi_{n-1}}(\Omega_1) \text{ if } n > 0. \end{cases}$$

Lemma 2.10 For each $n \in \omega$, $\Psi_n \in \mathcal{T}(M)$ and $\Psi_n < \Psi_{n+1}$.

The purpose of this paper is to extend the following theorem.

Theorem 2.11 (cf. Theorem 4 in [Ta04]). $\psi_M^{\Omega_1}(\Omega_1) = \lim_{n \in \omega} \Psi_n$.

3 Representation of successor-type ptros

Definition 3.1 Let α and β be elements of $\mathcal{T}(M)$. Then, for each $n \in \omega$, we define an ordinal $\Psi_n^{\beta}(\alpha)$, as follows:

$$\Psi_n^{\beta}(\alpha) = \begin{cases} \beta & \text{if } n = 0; \\ \psi_M^{\Psi_{n-1}^{\beta}(\alpha)}(\Omega_{\alpha+1}) & \text{otherwise.} \end{cases}$$

In particular, $\Psi_n(\alpha) := \Psi_n^0(\alpha)$

$\Psi_n^{\beta}(\alpha)$ also satisfies properties of Ψ_n.

Lemma 3.2 For each $\alpha, \beta \in \mathcal{T}(M)$, if

$$\beta < \psi_M^{\beta}(\Omega_{\alpha+1}) \qquad \text{and} \qquad \forall \xi \ \left(\alpha < \xi \ \Rightarrow \ \beta \in C^M(\beta, \xi) \right)$$

then, for each $n \in \omega$,

$$\Psi_n^{\beta}(\alpha) \in \mathcal{T}(M) \qquad \text{and} \qquad \Psi_n^{\beta}(\alpha) < \Psi_{n+1}^{\beta}(\alpha). \tag{1}$$

In particular, for each $\alpha \in \mathcal{T}(M)$ and $n < \omega$,

$$\Psi_n(\alpha) \in \mathcal{T}(M) \qquad \text{and} \qquad \Psi_n(\alpha) < \Psi_{n+1}(\alpha).$$

Proof. This lemma is shown by checking the properties in (1) as well as

$$\forall \xi \ (\alpha < \xi \Rightarrow \Psi_n^{\beta}(\alpha) \in C^M(\Psi_n^{\beta}(\alpha), \xi)),$$

by using induction on n. □

Now we give a representation of each successor-type ptro via $\Psi_n(\alpha)$ and the concept of limit.

Theorem 3.3 For each α with $\psi_M^{\Omega_{\alpha+1}}(\Omega_1) \in \mathcal{T}(M)$,

$$\psi_M^{\Omega_{\alpha+1}}(\Omega_1) = \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1). \tag{2}$$

Proof. Since in [Ta04] we dealt with the case where $\alpha = 0$, it suffices to show (2) in the case where $\alpha > 0$.

[1] One can show that $\psi_M^{\Omega_{\alpha+1}}(\Omega_1) \geqq \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1)$, by the following two claims.

Claim 1 (cf. Lemmas 9.(3) and 11 in [Ta04]). For each α and β, $\psi_M^{\beta}(\Omega_{\alpha+1})$ is defined and $\Omega_\alpha < \psi_M^{\beta}(\Omega_{\alpha+1}) < \Omega_{\alpha+1}$.

Claim 2 (cf. Lemma 10 in [Ta04]). For each α_1, α_2 and π ($\in$ **Reg**), if $\psi_M^{\alpha_1}(\pi)$ and $\psi_M^{\alpha_2}(\pi)$ are defined and if $\alpha_1 \leqq \alpha_2$, then $\psi_M^{\alpha_1}(\pi) \leqq \psi_M^{\alpha_2}(\pi)$.

[2] In order to show that $\psi_M^{\Omega_{\alpha+1}}(\Omega_1) \leqq \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1)$, we show that, for each $\gamma < \psi_M^{\Omega_{\alpha+1}}(\Omega_1)$, there exists an $n \in \omega$ with $\gamma \leqq \psi_M^{\Psi_n(\alpha)}(\Omega_1)$, by using induction on $d(\gamma)$.

Since it is easy to check the property above in any case except the case where $\gamma = \psi_M^{\xi}(\pi)$ [1], we let $\gamma = \psi_M^{\xi}(\pi)$ in what follows.

For the given ξ (and α), we now define a labeled binary tree $T_2(\xi)$ (more precisely, $T_2(\xi, \alpha)$).

Definition 3.4 We define a labeled binary tree $T_2(\xi)$ to satisfy the following property (i).

(i) For each node $s \in T_2(\xi)$, we denote the label of s by l_s. Then, the label l_s of each node in $T_2(\xi)$ is an element of $\mathcal{T}(M)$ satisfying:

  (i.i) l_s is a subterm of ξ;

  (i.ii) $l_s \leqq \xi$;

  (i.iii) $l_s \in C^M(\xi, \psi_M^{\xi}(\Omega_1)$.

[1] More precisely, we should assume that $\gamma =_{\text{nf}} \psi_M^{\xi}(\pi)$. However, we use only the symbol "$=$" unless we need special attention.

(ii) We define each node of $T_2(\xi)$ and its label, by using recursion on the distance from the root of $T_2(\xi)$, as follows.

(ii.0) If $s \in T_2(\xi)$ is the root, then l_s is ξ.

Let s be a node of $T_2(\xi)$. Then, we define the successors (successor nodes) of s as well as their labels, according to the following conditions of l_s.

(ii.i) If $l_s = 0$, then s is a leaf, that is, s has no successor node.

(ii.ii) If $l_s = \delta + \eta$ or $l_s = \varphi\delta\eta$, then s has successors s_1 and s_2, and $l_{s_1} := \delta$, $l_{s_2} := \eta$.

(ii.iii) If $l_s = \Omega_\beta$ and $l_s = \chi^\delta(\eta)$, then s is a leaf.

(ii.iv) Let $l_s = \psi^\delta_M(\tau)$. In this case, $\tau \leqq \Omega_{\alpha+1}$ since $l_s \leqq \xi$.

(ii.iv.i) If $\tau < \Omega_{\alpha+1}$, then s is a leaf.

(ii.iv.ii) If $\tau = \Omega_{\alpha+1}$, then s has a successor s_1 and $l_{s_1} := \delta$.

Claim 3 $T_2(\xi)$ is well-defined to be a finite tree.

(Proof of Claim 3: In order to show that $T_2(\xi)$ is well-defined, we show that, for each node s of $T_2(\xi)$, l_s satisfies the properties (i.i)$\sim$(i.iii) above, by using induction on the distance from the root to s.

If s is the root, it is trivial since $l_s = \xi$.

We let $l_s = \psi^\delta_M(\Omega_{\alpha+1})$ and show that δ satisfies (i.i)$\sim$(i.iii), as follows. By induction hypothesis, l_s is a subterm of ξ, $l_s \leqq \xi$ and $l_s \in C^M(\xi, \gamma)$. Then, δ is also a subterm of ξ. On the other hand, $l_s > \Omega_1 > \gamma$. So, we have $\delta \in C^M(\xi, \gamma)$ and $\delta < \xi$ from Definition 2.1.(M5) and $l_s \in C^M(\xi, \gamma)$.

Any other case is similar to the case above.

Moreover, for each node $s \in T_2(\xi)$ and each successor s' of s, it holds that $d(s) > d(s')$. So, $T_2(\xi)$ is finite. $\square$)

Definition 3.5 (1) A node s of $T_2(\xi)$ $(=T_2(\xi, \alpha))$ is said to be *critical* when $l_s = \psi^\delta_M(\Omega_{\alpha+1})$ for some δ. CN denotes the set of critical nodes (of $T_2(\xi)$).
(2) For each path p of each subtree of $T_2(\xi)$, the number of critical nodes in p is called the *weight* of p. Moreover, for each subtree T of $T_2(\xi)$, the maximum number of weights of all paths of T is called the *weight* of T, and denoted by $\mathrm{wt}(T)$. Furthermore, for each node s of $T_2(\xi)$, the weight of the subtree of $T_2(\xi)$ with root s is called the *weight* of s, and denoted by $\mathrm{wt}(s)$.
(3) For each subtree T of $T_2(\xi)$, the maximum length of all paths of T is called the *height* of T. Moreover, for each node s of $T_2(\xi)$, the height of the subtree of $T_2(\xi)$ with root s is called the *depth* of s, and denoted by $\mathrm{dp}(s)$.

Claim 4 For each node s of $T_2(\xi)$, it holds that $l_s < \Psi_{\mathrm{wt}(s)+1}(\alpha)$.

(Proof of Claim 4: We show the claim by induction on the depth of s.
(i) If s is a leaf, then $l_s \leqq \Omega_\alpha$. So, since $\Omega_\alpha < \Psi_n(\alpha)$ for each $n > 0$, we have $l_s < \Psi_1(\alpha)$.

(ii) Assume that s is not any leaf. Then, $l_s =_{\text{nf}} \delta + \eta$, $l_s =_{\text{nf}} \varphi\delta\eta$, or $l_s =_{\text{nf}} \psi_M^{\delta}(\Omega_{\alpha+1})$.

Let $l_s =_{\text{nf}} \psi_M^{\delta}(\Omega_{\alpha+1})$. Then, $l_s \in \text{CN}$ and s has one successor s_1 with $l_{s_1} = \delta$. Since $\text{wt}(s_1) = \text{wt}(s) - 1$ and $\text{dp}(s_1) < \text{dp}(s)$, the induction hypothesis implies that $l_{s_1} < \Psi_{\text{wt}(s)}(\alpha)$. On the other hand, since $l_s \in \mathcal{T}(M)$ and $\Psi_{\text{wt}(s)+1}(\alpha) \in \mathcal{T}(M)$, we have $l_s < \Psi_{\text{wt}(s)+1}(\alpha)$ (cf. Lemma 16 in [Ta04]).

Any other case is similar to or easier than the case above. □)

By Claim 4, we have $\xi < \Psi_{\text{wt}(T_2(\xi))+1}(\alpha)$, and hence, by Claim 2,

$$\gamma \leqq \psi_M^{\Psi_{\text{wt}(T_2(\xi))+1}(\alpha)}(\Omega_1).$$

So, the proof of Theorem 3.3 is completed. □

We can also expect that each $\psi_M^{\Psi_n(\alpha)}(\Omega_1)$ has itself as its reglar expression, that is, $\psi_M^{\Psi_n(\alpha)}(\Omega_1) \in \mathcal{T}(M)$. Unfortunately, we have not yet completed the proof of the property. However, it is not hard to show this property for each α less than a certain ordinal. For example, one can easily show the following proposition.

Proposition 3.6 For each $\alpha \in \mathcal{T}(M)$ and $n \in \omega$, if $\alpha \in C^M(\Psi_n(\alpha), \psi_M^{\Psi_n(\alpha)}(\Omega_1))$, then

$$\psi_M^{\Psi_n(\alpha)}(\Omega_1) \in \mathcal{T}(M) \quad \text{and} \quad \psi_M^{\Psi_n(\alpha)}(\Omega_1) < \psi_M^{\Psi_{n+1}(\alpha)}(\Omega_1).$$

By Theorem 3.3 and Proposition 3.6, each successor-type ptro $\psi_M^{\Omega_{\alpha+1}}(\Omega_1)$ has a *fundamental* sequence $\{\psi_M^{\Psi_n(\alpha)}(\Omega_1)\}_{n\in\omega}$ if $\alpha \in C^M(\Psi_n(\alpha), \psi_M^{\Psi_n(\alpha)}(\Omega_1))$.

Reference

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