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Matrix theory from the viewpoint of cellular automata

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A cellular automaton can be regarded as a locally interacting transformation. The following relation for the parallel maps of cellular automata holds.

$$\begin{array}{ccc} \text{injectivity} & \Leftrightarrow & \text{reversibility} \\ \Downarrow & & \\ \text{surjectivity} & & \end{array}$$

Fig.1　Richardsons' relation

In particular, the following relation for the parallel maps with scope one always holds.

$$\begin{array}{ccc} \text{injectivity} & \Leftrightarrow & \text{reversibility} \\ \Updownarrow & & \\ \text{surjectivity} & & \end{array}$$

Fig.2 relation in case of scope one

Two relations above always hold if we take a finite set as a state set. However, if we take infinite rings as a state set, even in linear case, the relations above do not always hold.

Example:

(1) If we take a field as a state set, then the relations above hold.

(2) If we take a set of integers as a state set, then the relations do not hold.

A matrix A can be regarded as a linear local map with scope one. Consider the automaton $\langle Z, V, \{0\}, A\rangle$ where Z is a set of integers, called a cell space, V is a vector space, which denotes a state set of each cell, $\{0\}$ is a neighbor frame, A is a square matrix which denotes a linear local map with scope one over V.

The following proposition is well known in linear algebra.

Proposition 1. In a square matrix A, the following statements are all equivalent.
(1) A is injective.
(2) A is surjective.
(3) A is non-singular.

This proposition suggests that linear cellular automata over matrix rings may satisfy Richardson's relation.

So, for linear cellular automata over R, what is a condition for ring to always hold the relations.

Definition 1. Let R be a ring with identity. R is a linear cellular ring if any linear parallel maps over R always satisfies the following conditions (*).

$$\text{injectivity} \quad \Leftrightarrow \quad \text{reversibility} \qquad (*)$$

Let $M_n(R)$ be a n by n matrix ring over R. Then we have.
Theorem 1. If R be a commutative linear cellular ring with identity, then $M_n(R)$ is also a linear cellular ring which is non commutative.

Corollary 1. If F is a field, then $M_n(F)$ is also a linear cellular ring.

Corollary2. The parallel map on $V_n(C)^Z$ satisfies the following.

$$\begin{array}{ccc} \text{injectivity} & \Leftrightarrow & \text{surjectivity} \\ \Downarrow & & \\ \text{surjectivity} & & \end{array}$$

From theorem 1, corollary 1, the proposition mentioned before is obtained as a special case from the view point of cellular automata.

Corollary 3. In a square matrix A over C, the following statements are all equivalent.
(1) A is injective.
(2) A is surjective.
(3) A is non-singular.

We can generalize linear algebras from the view point of cellular automata.

Consider the following 1-dimensional cellular automaton $\langle Z, V_n(C), N, f \rangle$, where Z is the set of all integers, denoted a cell space, C is the set of all complex numbers, $V_n(C)$ is a vector space with n dimensions over C which denotes a state set, $N=\{-r,\cdots,0\cdots,r\} \subsetneq Z$, called a neighborhood, f is a linear local map over $V_n(C)$ denoted by $f=\sum_{j=-r}^{r} A_j x_j$, $A_j \in M_n(C)$.

The parallel map f_∞: $V_n(C)^Z \to V_n(C)^Z$ is usually defined.

Since f_∞ is a linear parallel map, we have the following correspondence.

$$f_\infty \leftrightarrow A(X) = \Sigma A_j X^j .$$

Similarly, we have

$$u=(\cdots,u_0,u_1,\cdots) \in V_n^{\ Z} \leftrightarrow u(X) = \Sigma u_i X^{-i} .$$

Then we have

$$f_\infty(u) \leftrightarrow A(X)u(X) .$$

Definition 1. The transposition and the complex conjugate transposition of a vector and a matrix are defined as follows.

$${}^t u(X) = \Sigma\, {}^t u_i X^i \ , \quad {}^t \overline{u}(X) = \Sigma\, {}^t \overline{u}_i X^i$$

$${}^t A(X) \overset{def}{=} \sum {}^t A_j X^{-j}, \quad {}^t \overline{A}(X) \overset{def}{=} \sum {}^t \overline{A}_j X^{-j},$$

The inner product of two vectors is defined in the following.

$$\begin{aligned} < u(X), v(X) > &= {}^t \overline{u}(X) v(X) \\ &= \sum {}^t \overline{u}_i v_j X^{i-j} \end{aligned}$$

Two vectors $u(X)$ and $v(X)$ are orthogonal if $< u(X), v(X) >= 0$

Remark 1.

$$< u(X), u(X) >= 0 \Leftrightarrow u(X) = 0$$

Definition 2.

$A(X)$ is unitary $\Leftrightarrow {}^t\overline{A}(X)A(X) = I$.

$A(X)$ is Hermitian $\Leftrightarrow$ ${}^t\overline{A}(X) = A(X)$.

$A(X)$ is idempotent $\Leftrightarrow$ $A(X)^2 = A(X)$.

$A(X)$ is projective $\Leftrightarrow$ Hermitian and idempotent.

$A(X)$ is normal $\Leftrightarrow$ ${}^t\overline{A}(X)A(X) = A(X){}^t\overline{A}(X)$.

Example :

Let $U = U_1 + iU_2$ be unitary with scope 1. By definition, ${}^t\overline{U}U = I$. Then we have ${}^tU_1U_2 - {}^tU_2U_1 = 0$, ${}^tU_1U_1 + {}^tU_2U_2 = I$. Let $A(X) = U_1 + U_2X$ be orthogonal matrix with scope 2. By definition ${}^tA(X)A(X) = I$. Then ${}^tU_1U_1 + {}^tU_2U_2 = I$, ${}^tU_1U_2 = {}^tU_2U_1 = 0$.
The orthogonal matrix with scope 2 is stronger than the unitary matrix with scope one.

The vector $v(X) \neq 0$ satisfying the equation $A(X)v(X) = \lambda(X)v(X)$ is called the eigenvector of $A(X)$ and call $\lambda(X)$ its eigenvalue.

In generalized linear algebras, we obtain the same results as ordinary ones.

Theorem 2. Hermitian matrix $A(X)$ has the following properties.
(1) For any vectors $u(X), v(X)$, $< A(X)u(X), v(X) > = < u(X), A(X)v(X) >$.
(2) ${}^t\overline{\lambda}(X) = \lambda(X)$.
In particular, in the theory with scope one, we have ${}^t\overline{\lambda} = \lambda$. Then λ is a real number.

Theorem 3. If $A(X)$ is unitary, the followings are true.
(1) For any vectors $u(X), v(X)$, $< A(X)u(X), A(X)v(X) > = < u(X), v(X) >$

(2) ${}^t\overline{\lambda}(X)\lambda(X) = 1$

In particular, in the theory with scope one, we have ${}^t\overline{\lambda}\lambda = 1$.

Theorem 4. $A(X)$ satisfies Cayley-Hamilton's Theorem.

Let $\Phi(\lambda) = | A(X) - \lambda(X)I |$. Then $\Phi(A(X)) = 0$.

Summary :

From the discussion above, we have the following correspondence.

Ordinary linear algebras (scope one) $\Leftrightarrow$ matrix over field F.

Generalized algebras (scope one) $\Leftrightarrow$ matrix over functional field $F[X]$.

Remark 2. If we consider 2-dimensional cell space, the resulting algebra is represented by a matrix over function field $F[X, Y]$.

Application to physics

We next consider the application of our theory to special relativity.

Special relativity is stated mathematically that the fundamental laws of physics are invariant by the orthogonal transformation in Minkowski space, so called Lorentz transformation.

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1, & -\beta \\ -\beta, & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \quad \beta = \frac{v}{c}$$

v: relative velocity between two different inertia systems .

c: velocity of light.

First consider the orthogonal matrix $T(\theta)$ in Euclidian space.

$$T(\theta) = \begin{pmatrix} \cos\theta, -\sin\theta \\ \sin\theta, \cos\theta \end{pmatrix}.$$

We can generalize $T(\theta)$ to $T_X(r, \theta)$ with scope 5 in the following way.

$$T_X(r, \theta) = P_X(r) \begin{pmatrix} \cos\theta, -\sin\theta \\ \sin\theta, \cos\theta \end{pmatrix} {}^t P_X(r)$$

where $P_X(r) = \dfrac{r}{1+r^2} \begin{pmatrix} 0, 1 \\ 0, r \end{pmatrix} X^{-1} + \dfrac{1}{1+r^2} \begin{pmatrix} 1, -r \\ r, 1 \end{pmatrix} + \dfrac{r}{1+r^2} \begin{pmatrix} r, 0 \\ -1, 0 \end{pmatrix} X, (-\infty \le r \le \infty)$.

$$P_X(0) = I \quad (\textit{Identity} \quad \textit{matrix}).$$

$${}^t P_X(r) P_X(r) = I.$$

Clearly $T_X(r, \theta)$ changes continuously from $T(\theta)$.

Properties of $T(\theta)$ and $T_X(r,\theta)$

$T(\theta)$ has the following properties.

(1) It has no interaction.

(2) It has 1-parameter.

(3) ${}^tT(\theta)T(\theta)=I$.

(4) $ds^2=dx^2+dy^2$ is preserved.

$T_X(r,\theta)$ has the following properties.

(1) It has interaction.

(2) It has 2-parameters.

(3) ${}^tT_X(r,\theta)T_X(r,\theta)=I$

(4) $ds^2=\sum_{i\in Z} dx_i^2+dy_i^2$ is preserved.

Clearly, $T_X(r,\theta)$ is weaker than $T(\theta)$.

We next consider the orthogonal matrix $L(\beta)$ in Minkowski space.

$$L(\beta)=\frac{1}{\sqrt{1-\beta^2}}\begin{pmatrix}1, & -\beta\\ -\beta, & 1\end{pmatrix},\ \beta=\frac{v}{c},$$ v is a relative velocity between two different inertia systems

and c is the velocity of light.

We can generalize $L(\beta)$ to $L_X(\beta,\gamma)$ in the following way.

$L_X(\beta,r)=U_X(r)\ L(\beta)\ {}^tU_X(r)$, where

$$U_X(\gamma)=\frac{-\gamma}{1-r^2}\begin{pmatrix}0,1\\0,\gamma\end{pmatrix}X^{-1}+\frac{1}{1-\gamma^2}\begin{pmatrix}1,\gamma\\ \gamma,1\end{pmatrix}+\frac{-\gamma}{1-\gamma^2}\begin{pmatrix}\gamma,0\\1,0\end{pmatrix}X,\ (0\le\ |\gamma|\ \le 1)$$

$U_X(0)=I\quad (\mathit{Identity}\quad \mathit{matrix})$.

${}^tU_X(\gamma)\Lambda U_X(\gamma)=\Lambda,\ \Lambda=\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}$.

Clearly $L_X(\beta,\gamma)$ changes continuously from $L(\beta)$.

Properties of $L(\beta)$ and $L_X(\beta,\gamma)$.

$L(\beta)$ has the following properties.

(1) It has no interaction.

(2) It has 1-parameter.

(3) ${}^tL\Lambda L=\Lambda$, $\Lambda=\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}$.

(4) $ds^2=c^2dt^2-dx^2$ is preserved.

$L_X(\beta,\gamma)$ has the following properties.

(1) It has interaction.

(2) It has 2-parameters

(3) ${}^tL(X)\Lambda L(X)=\Lambda$, $\Lambda=\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}$.

(4) $ds^2=\sum_{i\in Z} c^2dt_i^2-dx_i^2$ is preserved.

Clearly, $L_X(\beta,\gamma)$ is weaker than $L(\beta)$.

Physical meanings of $L_X(\beta,\gamma)$.

γ has no-dimension physically, therefore it must express the ratio. We put

$\gamma = \frac{\ell_0}{\ell}$, where ℓ_0 is a minimal size of length and ℓ is the length of target object.

$\gamma = \frac{\ell_0}{\ell}$ *is invariant* by the following Lorentz contraction:

$$\ell_0 \to \ell_0\sqrt{1-\beta^2}\ , \quad \ell \to \ell\sqrt{1-\beta^2}\ .$$

$L_X(\beta,\gamma) \to L(\beta)$ if $\ell \to \infty\,(\gamma \to 0)$ and $L_X(\beta,\gamma)$ is gradually shifted away from $L(\beta)$ as ℓ tends ℓ_0.

Example.

When we consider the dynamics of an apple, the size ℓ of it is about $0.1(meter)$.

Since ℓ_0 is very small, $\gamma = \frac{\ell_0}{\ell} \approx 0$. Then $L(\beta)$ is applied.

Proposal:

How about applying $L_X(\beta,\gamma)$ not $L(\beta)$ in the super micro space?
How about generalizing special relativity in the following?
" The fundamental laws of physics are invariant under the Lorentz transformation with interaction. "

The item to verify in the experiments.

We need check whether there exists a minimal size of space –time in the experiments.

ℓ_0 : *minimal size of space.*
t_0 : *minimal size of time.*
$\ell_0 = 2\pi c t_0$ $\quad c$: *velocity of light*

Conclusion :

Matrix theory and special relativity is obtained as a special case of our theory.

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